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CONTACT SHAPE OPTIMIZATION BASED ON THE RECIPROCAL VARIATIONAL FORMULATION

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Abstract. The paper deals with a class of optimal shape design problems for elastic bodies unilaterally supported by a rigid foundation. Cost and constraint functionals defining the problem depend on contact stresses, i.e. their control is of primal interest. To this end, the so-called reciprocal variational formulation of contact problems making it possible to approximate directly the contact stresses is used. The existence and approximation results are established. The sensitivity analysis is carried out.

Keywords: shape optimization, contact problems, reciprocal variational formulation, sensitivity analysis

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Introduction

The paper deals with shape optimization of structures which are in mutual contact. In contrast to classical problems in which the governing relation describing the state of the mechanical system is given by equations, in contact optimization problems the state relation is represented by variational inequalities. Just this fact makes the whole matter more involved. It is known that such problems are in general non-smooth, i.e. the mapping: *control variable* $\longrightarrow$ *state* is not continuously differentiable, or better this mapping (under appropriate assumptions) is only directionally differentiable (see [12]). This phenomenon has to be taken into account when solving optimization problems numerically and explains why classical gradient type minimization methods may fail. One of typical problems arising in practice can be formulated as follows: how to design contact surfaces in order to get properly distributed contact stresses: for instance to avoid the contact stress concentration. A natural question arises, namely how to choose a cost functional by means of which one can control the behaviour of contact stresses. One possibility, frequently used by engineers, is to minimize the maximum of the contact pressure. Unfortunately this choice of the cost functional excludes rigorous mathematical analysis, since there is no reason to expect such high regularity of the solution of the state problem. [1] introduced the total potential energy evaluated at the equilibrium state as a possible candidate for the cost functional, yielding the constant distribution of the contact pressure. This phenomenon was numerically verified in [6] and later mathematically justified in [10]. Interpreting the optimality conditions it was shown that under the usual hypothesis valid in the linear elasticity, the contact pressure is 'almost' constant. This phenomenon is due to the constant volume constraint imposed on the elements of the admissible family of domains. Another very nice property is that the cost functional assumed to be a function of the shape is once continuously differentiable. Thus classical optimization methods based on gradient informations can be used. However, there are some drawbacks:

(a) if there is no volume constraint then one cannot expect the constant distribution of contact stresses;

(b) the control of the contact pressure is *passive*: we do not have any influence on its magnitude. The value of the contact pressure is related to the Lagrange multiplier associated with the constant volume constraint.

In order to control actively contact stresses, a least square approach seems to be very natural: one tries to adjust the contact zone in such a way that the resulting contact pressure is as close as possible to a given distribution. To avoid the difficulties with the regularity of the solution mentioned above, one has to be careful with the choice of a norm when defining the least square functional. In [5] and [7] the dual norm of functionals over the trace space was chosen. Since the numerical treatment of such a norm is difficult, the authors used its equivalent expression in the form of the classical $H^1(\Omega)$-norm of the solution of an auxiliary problem. This choice of the cost functional has the following advantages:

(c) the active control of contact stresses is possible;

(d) in special cases, the optimization problem is smooth.

In both examples of the cost functionals presented, the following discrepancy appears: in order to control quantities defined on the boundary, one has to solve problems in the whole domain Ω. A natural question arises, namely if it would be possible to use another variational formulation, which is adequate to the situation, i.e. the formulation expressed in terms of the contact pressure. Such a formulation exists and is known as the *reciprocal variational formulation*. It has been studied by [9] in the frictionless case and by [8] in problems involving friction.

The present paper gives mathematical analysis of a class of optimal shape design problems for deformable bodies unilaterally supported by a rigid foundation by using the reciprocal variational formulation of state problems. Such approach can be used at any time when a cost functional or functionals defining technological constraints depend on contact stresses. Besides, the reciprocal variational formulation seems to be one of the most efficient methods for the numerical realization of state problems.

The paper is organized as follows: in Section 1, the reciprocal variational formulation of contact problems with a given friction is briefly recalled. In Section 2 a class of optimal shape design problems with functionals depending on contact stresses and using the reciprocal formulation is defined and the existence of optimal shapes is established. Section 3 is devoted to the approximation of the continuous problem. It is proved that under appropriate assumptions, the discrete problem and the continuous one are close on subsequences. Finally, in Section 4 the sensitivity analysis in finite dimension is carried out.

1. Reciprocal variational formulation of contact problems

First we introduce notation and several definitions of functional spaces which will be used in what follows.

Let $\hat{\Omega} = (a,b) \times (0,\gamma)$, $0 < a < b$, $\gamma > 0$ be a rectangle, the boundary $\partial\hat{\Omega}$ of which is decomposed as follows:

$$\overline{\partial\hat{\Omega}} = \overline{\hat{\Gamma}} \cup \overline{\hat{\Gamma}}_u \cup \overline{\hat{\Gamma}}_P, \tag{1.1}$$

where $\hat{\Gamma} = (a,b) \times \{0\}$ and $\hat{\Gamma}_u$ is non-empty and open in $\partial\hat{\Omega}$. Denote by $F_\alpha\colon\ \mathbb{R}^2 \mapsto \mathbb{R}^2$ a mapping defined by

$$\begin{aligned} F_\alpha(\hat{x}_1,\hat{x}_2) = (x_1,x_2) \quad &\text{with} \\ x_1 = \hat{x}_1&, \\ x_2 = \frac{\gamma-\alpha(\hat{x}_1)}{\gamma}&\hat{x}_2 + \alpha(\hat{x}_1), \end{aligned}$$

where $\alpha\colon\ [a,b] \mapsto \mathbb{R}^1$ is a *non-negative*, *Lipschitz continuous* function in $[a,b]$. The image of $F_\alpha(\hat{\Omega})$, denoted by $\Omega(\alpha)$, is given by

$$\Omega(\alpha) = \left\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \in (a,b),\ \alpha(x_1) < x_2 < \gamma\right\}.$$

In accordance with (1.1), the boundary of $\Omega(\alpha)$ is decomposed as follows:

$$\overline{\partial\Omega}(\alpha) = \overline{\Gamma}(\alpha) \cup \overline{\Gamma}_u(\alpha) \cup \overline{\Gamma}_P(\alpha),$$

where $\Gamma(\alpha) = F_\alpha(\hat{\Gamma})$, $\Gamma_u(\alpha) = F_\alpha(\hat{\Gamma}_u)$, $\Gamma_P(\alpha) = F_\alpha(\hat{\Gamma}_P)$. In particular, $\Gamma(\alpha)$ is the graph of α:

$$\Gamma(\alpha) = \{(x_1, x_2) \mid x_1 \in (a,b),\ x_2 = \alpha(x_1)\}\,.$$

Let

$$\begin{aligned} V(\alpha) &= \big\{v \in H^1\big(\Omega(\alpha)\big) \mid v = 0 \text{ on } \Gamma_u(\alpha)\big\}\,,\\ V(\hat{\Omega}) &= \big\{v \in H^1(\hat{\Omega}) \mid v = 0 \text{ on } \hat{\Gamma}_u\big\},\\ \mathbb{V}(\alpha) &= V(\alpha) \times V(\alpha), \quad \mathbb{V}(\hat{\Omega}) = V(\hat{\Omega}) \times V(\hat{\Omega}) \end{aligned}$$

be the Sobolev spaces of functions defined in $\Omega(\alpha)$ and $\hat{\Omega}$, respectively.

It is readily seen that

$$v \in \mathbb{V}(\alpha) \quad \text{iff} \quad v \circ F_\alpha \in \mathbb{V}(\hat{\Omega}). \tag{1.2}$$

From now on we shall suppose that the function α characterizing the mapping F_α belongs to an admissible set $\mathscr{U}_{ad}$ defined as follows:

$$\mathscr{U}_{ad} = \big\{\alpha \in C^{0,1}([a,b]) \mid 0 \leqslant \alpha \leqslant C_0,\ |\,\mathrm{d}\alpha/\,\mathrm{d}x_1| \leqslant C_1 \text{ a.e. in } (a,b)\big\}\,,$$

i.e. $\mathscr{U}_{ad}$ contains functions which are *uniformly bounded and equi-Lipschitz continuous* in $[a,b]$, C_0, C_1 are given positive constants. One can easily verify that there exist positive constants c_1, c_2 which do not depend on $\alpha \in \mathscr{U}_{ad}$, such that

$$c_1\|v\|_{1,\Omega(\alpha)} \leqslant \|v \circ F_\alpha\|_{1,\hat{\Omega}} \leqslant c_2\|v\|_{1,\Omega(\alpha)} \tag{1.3}$$

holds for any $v \in \mathbb{V}(\alpha)$ and any $\alpha \in \mathscr{U}_{ad}$.

Denote by

$$\begin{aligned} H^{1/2}\big(\Gamma(\alpha)\big) &\equiv \operatorname{trace}_{\Gamma(\alpha)}\ V(\alpha),\\ H^{1/2}(\hat{\Gamma}) &\equiv \operatorname{trace}_{\hat{\Gamma}}\ V(\hat{\Omega}),\\ \mathbb{H}^{1/2}\big(\Gamma(\alpha)\big) &\equiv H^{1/2}\big(\Gamma(\alpha)\big) \times H^{1/2}\big(\Gamma(\alpha)\big),\\ \mathbb{H}^{1/2}(\hat{\Gamma}) &\equiv H^{1/2}(\hat{\Gamma}) \times H^{1/2}(\hat{\Gamma}) \end{aligned}$$

the trace spaces on $\Gamma(\alpha)$ and $\hat{\Gamma}$, respectively. In view of (1.2) one has

$$\varphi \in \mathbb{H}^{1/2}\big(\Gamma(\alpha)\big) \quad \text{iff} \quad \varphi \circ F_\alpha \in \mathbb{H}^{1/2}(\hat{\Gamma}),\ \alpha \in \mathscr{U}_{ad}. \tag{1.4}$$

By $H^{-1/2}(\hat{\Gamma})$ we denote the dual space to $H^{1/2}(\hat{\Gamma})$ with the duality pairing $\langle\ ,\ \rangle$, and by $\mathbb{H}^{-1/2}(\hat{\Gamma}) \equiv H^{-1/2}(\hat{\Gamma}) \times H^{-1/2}(\hat{\Gamma})$ the dual space over $\mathbb{H}^{1/2}(\hat{\Gamma})$. The duality pairing between $\mathbb{H}^{-1/2}(\hat{\Gamma})$ and $\mathbb{H}^{1/2}(\hat{\Gamma})$ will be denoted by $\langle\ ,\ \rangle$ again. If $\varphi \in H^{1/2}\big(\Gamma(\alpha)\big)$ and $\mu \in H^{-1/2}(\hat{\Gamma})$ then in view of (1.4) the value $\langle \mu, \varphi \circ F_\alpha \rangle$ is well defined for any $\alpha \in \mathscr{U}_{ad}$. *From now on, if* $\varphi \in H^{1/2}\big(\Gamma(\alpha)\big)$ *and* $\mu \in H^{-1/2}(\hat{\Gamma})$ *then the value* $\langle \mu, \varphi \rangle_\alpha$ *is defined by*

$$\langle \mu, \varphi \rangle_\alpha \overset{\text{def}}{\equiv} \langle \mu, \varphi \circ F_\alpha \rangle.$$

Now we introduce norms in $\mathbb{H}^{1/2}\big(\Gamma(\alpha)\big)$ and $\mathbb{H}^{1/2}(\hat{\Gamma})$. Let $\varphi \in \mathbb{H}^{1/2}\big(\Gamma(\alpha)\big)$ with $\alpha \in \mathscr{U}_{ad}$. We define

$$\|\varphi\|_{1/2,\alpha} \equiv \inf |\!|\!| v |\!|\!|_{1,\Omega(\alpha)},$$

where inf is taken over all functions $v \in \mathbb{V}(\alpha)$ satisfying $v = \varphi$ on $\Gamma(\alpha)$ in the sense of traces and

$$|\!|\!| v |\!|\!|^2_{1,\Omega(\alpha)} = \big(\Lambda\varepsilon(v), \varepsilon(v)\big)_{0,\Omega(\alpha)}.$$

The symbol $\varepsilon(v) = \big\{\varepsilon_{ij}(v)\big\}^2_{i,j=1}$ stands for the symmetric tensor of small deformations with $\varepsilon_{ij}(v) = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$ and Λ is a linear symmetric mapping from the space of 2×2 symmetric matrices into itself:

$$\sigma = \Lambda\varepsilon \quad \text{iff} \quad \sigma_{ij} = c_{ijkl}\varepsilon_{kl},$$

where the elasticity coefficients $c_{ijkl} \in L^\infty(\hat{\Omega})$ satisfy the following *symmetry and ellipticity* conditions:

$$c_{ijkl} = c_{jikl} = c_{klij} \quad \text{a.e. in } \hat{\Omega}, \tag{1.5}$$

$$\begin{aligned} \exists \alpha = \text{const.} > 0\colon\ c_{ijkl}\xi_{ij}\xi_{kl} \geqslant \alpha\xi_{ij}\xi_{ij} \quad &\text{a.e. in } \hat{\Omega} \\ &\forall \xi_{ij} = \xi_{ji} \in \mathbb{R}^1. \end{aligned} \tag{1.6}$$

It is easy to check that

$$\|\varphi\|_{1/2,\alpha} = |\!|\!| v(\varphi) |\!|\!|_{1,\Omega(\alpha)},$$

where $v(\varphi) \in \mathbb{V}(\alpha)$ is the unique solution to the following linear elasticity problem:

$$\begin{cases} \big(\Lambda\varepsilon\big(v(\varphi)\big), \varepsilon(\psi)\big)_{0,\Omega(\alpha)} = 0 & \forall \psi \in \mathbb{V}_0(\alpha) \\ v(\varphi) = \varphi \text{ on } \Gamma(\alpha), \end{cases} \tag{1.7}$$

where

$$\mathbb{V}_0(\alpha) = \big\{v \in \mathbb{V}(\alpha) \mid v = 0 \text{ on } \Gamma(\alpha)\big\}.$$

The norm in $\mathbb{H}^{1/2}(\hat\Gamma)$ is defined in a similar way. Notice that because of (1.3), (1.4) and the definition of the fractional norm one also has

$$c_1\|\varphi\|_{1/2,\alpha} \leqslant \|\varphi\circ F_\alpha\|_{1/2,\hat\Gamma} \leqslant c_2\|\varphi\|_{1/2,\alpha} \tag{1.8}$$

with the same c_1, c_2 as in (1.3) and, in particular, not depending on $\alpha\in\mathscr{U}_{ad}$.

Next we shall introduce norms in $\mathbb{H}^{-1/2}(\hat\Gamma)$ by setting

$$\|\mu\|_{-1/2,\alpha} \overset{\text{def}}{\equiv} |\!|\!| v |\!|\!|_{1,\Omega(\alpha)}, \quad \mu\in\mathbb{H}^{-1/2}(\hat\Gamma), \tag{1.9}$$

where $v\equiv v(\mu)\in\mathbb{V}(\alpha)$ is the unique solution to the problem

$$\big(\Lambda\varepsilon(v),\varepsilon(\psi)\big)_{0,\Omega(\alpha)} = \langle\mu,\psi\rangle_\alpha \quad \forall\psi\in\mathbb{V}(\alpha). \tag{1.10}$$

Recall that $\langle\mu,\psi\rangle_\alpha\equiv\langle\mu_1,\varphi_1\circ F_\alpha\rangle+\langle\mu_2,\varphi_2\circ F_\alpha\rangle$, where $\varphi_i\equiv\operatorname{trace}_{\Gamma(\alpha)}\psi_i,\ i=1,2.$ It is readily seen that (1.9) defines a norm in $\mathbb{H}^{-1/2}(\hat\Gamma)$. Let $\|\ \|_{-1/2,\hat\Gamma}$ stand for the *classical* dual norm in $\mathbb{H}^{-1/2}(\hat\Gamma)$:

$$\|\mu\|_{-1/2,\hat\Gamma} = \sup_{\psi\neq 0}\frac{\langle\mu,\psi\rangle}{\|\psi\|_{1/2,\hat\Gamma}}.$$

We shall show that

$$\frac{1}{c_2}\|\mu\|_{-1/2,\alpha} \leqslant \|\mu\|_{-1/2,\hat\Gamma} \leqslant \frac{1}{c_1}\|\mu\|_{-1/2,\alpha} \tag{1.11}$$

holds for any $\mu\in\mathbb{H}^{-1/2}(\hat\Gamma)$ and any $\alpha\in\mathscr{U}_{ad}$ with the same constants c_1, c_2 as in (1.8). Indeed, from (1.8), (1.9) and (1.10) one has

$$\begin{aligned}
\|\mu\|^2_{-1/2,\alpha} = |\!|\!| v(\mu)|\!|\!|^2_{1,\Omega(\alpha)} &= \langle\mu, v(\mu)\rangle_\alpha \\
&\leqslant \|\mu\|_{-1/2,\hat\Gamma}\|v(\mu)\circ F_\alpha\|_{1/2,\hat\Gamma} \\
&\leqslant c_2\|\mu\|_{-1/2,\hat\Gamma}\|v(\mu)\|_{1/2,\alpha} \\
&\leqslant c_2\|\mu\|_{-1/2,\hat\Gamma}|\!|\!| v(\mu)|\!|\!|_{1,\Omega(\alpha)} \\
&= c_2\|\mu\|_{-1/2,\hat\Gamma}\|\mu\|_{-1/2,\alpha},
\end{aligned}$$

which proves the first inequality in (1.11). Now let $\psi\in\mathbb{H}^{1/2}(\hat\Gamma)$. Then there is a function $v\in\mathbb{V}(\hat\Omega)$ such that $v=\psi$ on $\hat\Gamma$. Denote $w=v\circ F_\alpha^{-1}\in\mathbb{V}(\alpha)$. Then it follows from (1.10) that

$$\begin{aligned}
\langle\mu,\psi\rangle = \langle\mu, w\circ F_\alpha\rangle &= \big(\Lambda\varepsilon\big(v(\mu)\big),\varepsilon(w)\big)_{0,\Omega(\alpha)} \\
&\leqslant |\!|\!| v(\mu)|\!|\!|_{1,\Omega(\alpha)}|\!|\!| w|\!|\!|_{1,\Omega(\alpha)}
\end{aligned} \tag{1.12}$$

holds for any function $w \in \mathbb{V}(\alpha)$ such that $w \circ F_\alpha = \psi$ on $\hat{\Gamma}$. Thus

$$\begin{aligned} \langle \mu, \psi \rangle &\leqslant \|\mu\|_{-1/2,\alpha} \inf \|w\|_{1,\Omega(\alpha)} \\ &= \|\mu\|_{-1/2,\alpha} \|\psi \circ F_\alpha^{-1}\|_{1/2,\alpha} \\ &\leqslant \frac{1}{c_1} \|\mu\|_{-1/2,\alpha} \|\psi\|_{1/2,\hat{\Gamma}}, \end{aligned}$$

where inf is taken over all functions $w \in \mathbb{V}(\alpha)$ such that $w \circ F_\alpha = \psi$ on $\hat{\Gamma}$. This implies the second inequality in (1.11).

Now we pass to the mathematical formulation of contact problems with a given friction. For more details on this subject we refer to [4].

Let a plane deformable body be represented by the domain $\Omega(\alpha)$ for some $\alpha \in \mathscr{U}_{ad}$. Recall that $\Omega(\alpha) = F_\alpha(\hat{\Omega})$. The decomposition of $\partial\Omega(\alpha)$ into $\Gamma_u(\alpha)$, $\Gamma(\alpha)$ and $\Gamma_P(\alpha)$ has been already described above. The body is subjected to body forces F, to surface tractions P on $\Gamma_P(\alpha)$ and supported by the rigid half-plane $\mathbb{S} \equiv \mathbb{R}^2_- = \{(x_1, x_2) \mid x_2 \leqslant 0\}$ from below. On the contact part of $\partial\Omega(\alpha)$, represented by the portion $\Gamma(\alpha)$, the unilateral and friction conditions will be prescribed. We start with the *primal formulation* of the problem.

Denote by

$$\mathscr{J}_\alpha(v) = \frac{1}{2}\big(\Lambda\varepsilon(v), \varepsilon(v)\big)_{0,\Omega(\alpha)} + j_\alpha(v) - L_\alpha(v)$$

the *total potential energy functional*, where

$$\begin{aligned} \big(\Lambda\varepsilon(v), \varepsilon(w)\big)_{0,\Omega(\alpha)} &\equiv \int_{\Omega(\alpha)} c_{ijkl}\varepsilon_{ij}(v)\varepsilon_{kl}(w)\,\mathrm{d}x, \\ j_\alpha(v) &\equiv \int_a^b g\big|v_1\big(x_1, \alpha(x_1)\big)\big|\,\mathrm{d}x_1 = (g, |v_1 \circ F_\alpha|)_{0,\hat{\Gamma}}, \\ L_\alpha(v) &\equiv \int_{\Omega(\alpha)} F_i v_i\,\mathrm{d}x + \int_{\Gamma_P(\alpha)} P_i v_i\,\mathrm{d}s. \end{aligned}$$

The meaning of the symbols is the following:

- c_{ijkl} are the components of the linear Hooke's law, satisfying (1.5) and (1.6);
- $g \in L^2(\hat{\Gamma})$ is a non-negative function defined in $\hat{\Gamma} = (a,b) \times \{0\}$;
- $F \in \big(L^2(\hat{\Omega})\big)^2$, $P \in \big(L^2(\partial\hat{\Omega})\big)^2$ are the given body forces and surface tractions, respectively.

In the definition of L_α, the restrictions of F_i, P_i onto $\Omega(\alpha)$ and $\Gamma_P(\alpha)$, respectively, are used.

Let $K(\alpha) \subset \mathbb{V}(\alpha)$ be the set of *kinematically admissible* functions defined as follows:

$$K(\alpha) = \big\{v = (v_1, v_2) \in \mathbb{V}(\alpha) \mid v_2\big(x_1, \alpha(x_1)\big) \geqslant -\alpha(x_1) \text{ a.e. in } (a,b)\big\}.$$

By the *primal variational formulation* to the Signorini problem with a given friction we mean the following minimization problem:

$$(\mathscr{P}(\alpha)) \qquad \begin{cases} \text{Find } u \in K(\alpha) \text{ such that} \\ \mathscr{J}_\alpha(u) \leqslant \mathscr{J}_\alpha(v) \quad \forall v \in K(\alpha) \end{cases}$$

or equivalently

$$(\mathscr{P}(\alpha))' \qquad \begin{cases} \text{Find } u \in K(\alpha) \text{ such that} \\ \big(\Lambda\varepsilon(u), \varepsilon(v-u)\big)_{0,\Omega(\alpha)} + j_\alpha(v) - j_\alpha(u) \geqslant L_\alpha(v-u) \\ \hfill \forall v \in K(\alpha). \end{cases}$$

It is well known that under our conditions $(\mathscr{P}(\alpha))$ has a unique solution u.

In order to release the kinematical constraint defining $K(\alpha)$ and to regularize the non-smooth term j_α, the duality approach will be used. To this end we introduce the following convex subsets of $H^{-1/2}(\hat{\Gamma})$:

$$\begin{aligned} \Lambda_1 &= \{\mu_1 \in L^2(\hat{\Gamma}) \mid |\mu_1| \leqslant 1 \text{ a.e. in } \hat{\Gamma}\}, \\ \Lambda_2 &= \{\mu_2 \in H^{-1/2}(\hat{\Gamma}) \mid \mu_2 \geqslant 0\}, \\ \mathbf{\Lambda} &\equiv \Lambda_1 \times \Lambda_2. \end{aligned}$$

The ordering $\geqslant$ is defined in the standard way:

$$\mu_2 \geqslant 0 \quad \text{iff} \quad \langle \mu_2, \varphi \rangle \geqslant 0 \quad \forall \varphi \in H^{1/2}(\hat{\Gamma}), \quad \varphi \geqslant 0.$$

Remark 1.1. In what follows we shall suppose the function g is positive a.e. in (a,b). If g were equal to zero in a set with a positive Lebesgue measure, the functions from Λ_1 would be defined on $\hat{\Gamma} \setminus \operatorname{supp} g$.

Next we make the following assumption concerning functions α from $\mathscr{U}_{ad}$. We shall suppose that for any $\alpha \in \mathscr{U}_{ad}$ there exists a function $\tilde{\alpha} \in V(\alpha)$ such that $\tilde{\alpha} = \alpha$ on $\Gamma(\alpha)$. This assumption slightly restricts the choice of $\mathscr{U}_{ad}$ in the following sense: if $\overline{\Gamma}_u(\alpha) \cap \overline{\Gamma}(\alpha) \neq \emptyset$ then α has to be equal to zero at the points of this intersection.

Using the Lagrange multiplier technique we have

$$\inf_{v \in K(\alpha)} \mathscr{J}_\alpha(v) = \inf_{v \in \mathbb{V}(\alpha)} \sup_{\mu \in \mathbf{\Lambda}} \mathscr{L}_\alpha(v, \mu),$$

where $\mathscr{L}_\alpha \colon \mathbb{V}(\alpha) \times \mathbf{\Lambda} \to \mathbb{R}^1$ is the Lagrangean defined as follows:

$$\mathscr{L}_\alpha(v, \mu) \equiv \mathscr{J}_\alpha(v) - \langle \mu_g, v \rangle_\alpha - \langle \mu_2, \alpha \rangle$$

with the following meaning of notation:

$$\begin{aligned}\langle \mu_g, v\rangle_\alpha &\stackrel{\mathrm{def}}{\equiv} (g\mu_1, v_1)_{0,\alpha} + \langle \mu_2, v_2\rangle_\alpha,\\ \mu &= (\mu_1, \mu_2) \in \mathbf{\Lambda},\\ v &= (v_1, v_2) \in \mathbb{V}(\alpha),\end{aligned}$$

where

$$(g\mu_1, v_1)_{0,\alpha} \equiv \int_a^b g\mu_1(x_1) v_1(x_1, \alpha(x_1))\,\mathrm{d}x_1 = (g\mu_1, v_1 \circ F_\alpha)_{0,\hat{\Gamma}}.$$

By a *mixed variational* formulation of $\big(\mathscr{P}(\alpha)\big)$ we mean the problem of finding a saddle-point of $\mathscr{L}_\alpha$ on $\mathbb{V}(\alpha) \times \mathbf{\Lambda}$:

$$\big(\mathscr{M}(\alpha)\big) \qquad \begin{cases} \text{Find } (w, \lambda) \in \mathbb{V}(\alpha) \times \mathbf{\Lambda} \text{ such that} \\ \mathscr{L}_\alpha(w, \mu) \leqslant \mathscr{L}_\alpha(w, \lambda) \leqslant \mathscr{L}_\alpha(v, \lambda) \quad \forall v \in \mathbb{V}(\alpha),\ \forall \mu \in \mathbf{\Lambda} \end{cases}$$

or equivalently

$$\big(\mathscr{M}(\alpha)\big) \qquad \begin{cases} \text{Find } (w, \lambda) \in \mathbb{V}(\alpha) \times \mathbf{\Lambda} \text{ such that} \\ \big(\Lambda\varepsilon(w), \varepsilon(v)\big)_{0,\Omega(\alpha)} = L_\alpha(v) + \langle \lambda_g, v\rangle_\alpha \quad \forall v \in \mathbb{V}(\alpha) \\ \langle (\mu - \lambda)_g, w\rangle_\alpha + \langle \mu_2 - \lambda_2, \alpha\rangle \geqslant 0 \quad \forall \mu \in \mathbf{\Lambda}. \end{cases}$$

It is well-know that if (w, λ) is a saddle-point of $\mathscr{L}_\alpha$ on $\mathbb{V}(\alpha) \times \mathbf{\Lambda}$ then

$$\mathscr{L}_\alpha(w, \lambda) = \min_{v \in \mathbb{V}(\alpha)} \sup_{\mu \in \mathbf{\Lambda}} \mathscr{L}_\alpha(v, \mu) = \max_{\mu \in \mathbf{\Lambda}} \inf_{v \in \mathbb{V}(\alpha)} \mathscr{L}_\alpha(v, \mu). \tag{1.13}$$

The so-called *reciprocal variational formulation* is based on the elimination of the displacement field $v \in \mathbb{V}(\alpha)$ by using the second equality in (1.13):

$$\mathscr{L}_\alpha(w, \lambda) = \max_{\mu \in \mathbf{\Lambda}} \tilde{\mathscr{S}}_\alpha(\mu),$$

where

$$\tilde{\mathscr{S}}_\alpha(\mu) = \inf_{v \in \mathbb{V}(\alpha)} \mathscr{L}_\alpha(v, \mu).$$

It is also well-known (see [3]) that the reciprocal variational formulation

$$\big(\mathscr{R}(\alpha)\big) \qquad \begin{cases} \text{Find } \tilde{\lambda} \in \mathbf{\Lambda} \text{ such that} \\ \tilde{\mathscr{S}}_\alpha(\tilde{\lambda}) = \max_{\mu \in \mathbf{\Lambda}} \tilde{\mathscr{S}}_\alpha(\mu) \end{cases}$$

has a solution provided there exists a saddle-point of $\mathscr{L}_\alpha$ on $\mathbb{V}(\alpha) \times \mathbf{\Lambda}$. Moreover $\tilde{\lambda} = \lambda$, where λ is the second component of the corresponding saddle-point.

The formulation $\big(\mathscr{R}(\alpha)\big)$ was already studied in [8]. We briefly recall how to derive the explicit form of $\tilde{\mathscr{S}}_\alpha$ and we will show directly that $\big(\mathscr{R}(\alpha)\big)$ has a unique solution.

Let $\mu \in \mathbf{\Lambda}$ be fixed. Then one has

$$\inf_{v \in \mathbb{V}(\alpha)} \mathscr{L}_\alpha(v,\mu) = \mathscr{L}_\alpha(u(\mu),\mu) = -\frac{1}{2} L_\alpha\big(u(\mu)\big) - \frac{1}{2}\langle \mu_g, u(\mu)\rangle_\alpha - \langle \mu_2, \alpha\rangle, \tag{1.14}$$

where $u(\mu) \in \mathbb{V}(\alpha)$ is the unique solution to

$$\big(\Lambda\varepsilon\big(u(\mu)\big), \varepsilon(v)\big)_{0,\Omega(\alpha)} = L_\alpha(v) + \langle \mu_g, v\rangle_\alpha \quad \forall v \in \mathbb{V}(\alpha). \tag{1.15}$$

In view of linearity of (1.15), one can split the solution $u \equiv u(\mu)$ and write $u = q + z$ with $q, z \in \mathbb{V}(\alpha)$ being the solutions to the linear elasticity problems

$$\big(\Lambda\varepsilon(q), \varepsilon(v)\big)_{0,\Omega(\alpha)} = L_\alpha(v) \quad \forall v \in \mathbb{V}(\alpha), \tag{1.16}$$

$$\big(\Lambda\varepsilon(z), \varepsilon(v)\big)_{0,\Omega(\alpha)} = \langle \mu_g, v\rangle_\alpha \quad \forall v \in \mathbb{V}(\alpha), \tag{1.17}$$

i.e. q, z are the displacement fields induced by the given forces F, P and the contact tractions μ, respectively. From (1.14) one has

$$\begin{aligned}\inf_{v \in \mathbb{V}(\alpha)} \mathscr{L}_\alpha(v,\mu) = & -\frac{1}{2} L_\alpha(q) - \frac{1}{2} L_\alpha(z) - \frac{1}{2}\langle \mu_g, q\rangle_\alpha \\ & - \frac{1}{2}\langle \mu_g, z\rangle_\alpha - \langle \mu_2, \alpha\rangle.\end{aligned} \tag{1.18}$$

Taking into account (1.16), (1.17) as well as the symmetry of Hooke's law we get

$$L_\alpha(z) = \big(\Lambda\varepsilon(q), \varepsilon(z)\big)_{0,\Omega(\alpha)} = \langle \mu_g, q\rangle_\alpha.$$

Using this equality in (1.18) we finally arrive at

$$\begin{aligned}\tilde{\mathscr{S}}_\alpha(\mu) = \inf_{v \in \mathbb{V}(\alpha)} \mathscr{L}_\alpha(v,\mu) = & -\frac{1}{2} L_\alpha(q) - \langle \mu_g, q\rangle_\alpha \\ & - \frac{1}{2}\langle \mu_g, z\rangle_\alpha - \langle \mu_2, \alpha\rangle.\end{aligned} \tag{1.19}$$

Let $G_\alpha\colon \mathbb{V}'(\alpha) \mapsto \mathbb{V}(\alpha)$ be Green's operator corresponding to our linear elasticity problem, i.e.

$$G_\alpha(f) = u(f) \in \mathbb{V}(\alpha), \;\; f \in \mathbb{V}'(\alpha),$$

where $u(f)$ is the unique solution to

$$\big(\Lambda\varepsilon\big(u(f)\big), \varepsilon(v)\big)_{0,\Omega(\alpha)} = [f, v]_\alpha \quad \forall v \in \mathbb{V}(\alpha),$$

where $[\ ,\ ]_\alpha$ stands for the duality pairing between $\mathbb{V}'(\alpha)$ and $\mathbb{V}(\alpha)$. Then the solutions of (1.16) and (1.17) can be written as

$$q = G_\alpha(L_\alpha),\ \ z = G_\alpha(\mu_g).$$

Using this notation in (1.19) we finally obtain

$$\tilde{\mathscr{S}}_\alpha(\mu) = -\frac{1}{2}b_\alpha(\mu,\mu) + \mathscr{F}_\alpha(\mu) - \frac{1}{2}L_\alpha\bigl(G_\alpha(L_\alpha)\bigr), \tag{1.20}$$

where $b_\alpha\colon \mathbf{\Lambda}\times\mathbf{\Lambda}\mapsto\mathbb{R}^1$, $\mathscr{F}_\alpha\colon \mathbf{\Lambda}\mapsto\mathbb{R}^1$ are a bilinear form and a linear form, respectively, defined as follows:

$$\begin{aligned} b_\alpha(\mu,\nu) &\equiv \langle\mu_g, G_\alpha(\nu_g)\rangle_\alpha, \quad \mu,\,\nu\in\mathbf{\Lambda};\\ \mathscr{F}_\alpha(\mu) &\equiv\ -\langle\mu_g, G_\alpha(L_\alpha)\rangle_\alpha - \langle\mu_2,\alpha\rangle, \quad \mu\in\mathbf{\Lambda}. \end{aligned}$$

Since the last term in (1.20) does not depend on μ one can neglect it and pass to a more convenient form of the reciprocal energy functional:

$$\mathscr{S}_\alpha(\mu) \equiv -\tilde{\mathscr{S}}_\alpha(\mu) - \frac{1}{2}L_\alpha\bigl(G_\alpha(L_\alpha)\bigr) = \frac{1}{2}b_\alpha(\mu,\mu) - \mathscr{F}_\alpha(\mu).$$

The equivalent expression for $\bigl(\mathscr{R}(\alpha)\bigr)$ now reads as follows:

$$\bigl(\mathscr{R}(\alpha)\bigr) \qquad \begin{cases} \quad\text{Find } \lambda\in\mathbf{\Lambda} \text{ such that}\\ \mathscr{S}_\alpha(\lambda)\leqslant\mathscr{S}_\alpha(\mu) \quad \forall\mu\in\mathbf{\Lambda}.\end{cases}$$

Before we prove that $\bigl(\mathscr{R}(\alpha)\bigr)$ has a unique solution we need the following auxiliary result:

Lemma 1.1. *One has*

$$b_\alpha(\mu,\mu) = |\!|\!| u(\mu_g)|\!|\!|^2_{1,\Omega(\alpha)} = \|\mu_g\|^2_{-1/2,\alpha},$$

where $u(\mu_g)\in\mathbb{V}(\alpha)$ is the unique solution to

$$\bigl(\Lambda\varepsilon\bigl(u(\mu_g)\bigr),\varepsilon(v)\bigr)_{0,\Omega(\alpha)} = \langle\mu_g, v\rangle_\alpha \quad \forall v\in\mathbb{V}(\alpha). \tag{1.21}$$

P r o o f. Inserting $v := u(\mu_g)$ into (1.21) one has

$$\langle\mu_g, u(\mu_g)\rangle_\alpha = |\!|\!| u(\mu_g)|\!|\!|^2_{1,\Omega(\alpha)} = \|\mu_g\|^2_{-1/2,\alpha}$$

as follows from (1.9). On the other hand,

$$\langle \mu_g, u(\mu_g)\rangle_\alpha = \langle \mu_g, G_\alpha(\mu_g)\rangle_\alpha = b_\alpha(\mu,\mu)$$

if we take into account the definition of b_α.

Since

$$\lim_{\|\mu\|_{-1/2,\alpha}\to\infty} \mathscr{S}_\alpha(\mu) = +\infty,$$

$\mathscr{S}_\alpha$ is coercive on $\mathbf{\Lambda}$. Moreover, $\mathscr{S}_\alpha$ is strictly convex and weakly lower semicontinuous on $\mathbf{\Lambda}$. Consequently, $(\mathscr{R}(\alpha))$ has a unique solution $\lambda \equiv \lambda(\alpha)$. The mixed variational formulation $(\mathscr{M}(\alpha))$ has a unique solution (w,λ) as well, as follows from [3], Proposition 2.4. The first component w solves $(\mathscr{P}(\alpha))$, i.e. $w = u$, while λ solves $(\mathscr{R}(\alpha))$. □

Interpretation of λ

Let $(u,\lambda) \in K(\alpha)\times\mathbf{\Lambda}$ be the solution of $(\mathscr{M}(\alpha))$. Then it follows from $(\mathscr{M}(\alpha))_2$ that

$$\big(\Lambda\varepsilon(u), \varepsilon(v)\big)_{0,\Omega(\alpha)} = L_\alpha(v) + \langle \lambda_g, v\rangle_\alpha \quad \forall v \in \mathbb{V}(\alpha). \tag{1.22}$$

From Green's formula applied to the left-hand side of (1.22) we have

$$\langle T_1, v_1\rangle_{\partial\Omega} + \langle T_2, v_2\rangle_{\partial\Omega} = \langle \lambda_g, v\rangle_\alpha \quad \forall v \in \mathbb{V}(\alpha),$$

where the symbol $\langle\, ,\, \rangle_{\partial\Omega}$ stands for the duality pairing between $H^{-1/2}\big(\partial\Omega(\alpha)\big)$ and $H^{1/2}\big(\partial\Omega(\alpha)\big) = \mathrm{trace}_{\partial\Omega(\alpha)} V(\alpha)$. If $T_1, T_2 \in L^2\big(\partial\Omega(\alpha)\big)$ then

$$\begin{aligned} &T_1\sqrt{1+(\alpha')^2} = g\lambda_1,\\ &T_2\sqrt{1+(\alpha')^2} = \lambda_2 \qquad \text{in } (a,b). \end{aligned}$$

2. Contact shape optimization based on the reciprocal variational formulation $(\mathscr{R}(\alpha))$

Until now, the shape of $\Omega(\alpha)$, determined by the function $\alpha \in \mathscr{U}_{ad}$, has been fixed. Next, the functions α will be considered to be *design variables*, variations of which lead to a configuration with a-priori given properties. In many problems arising in practice, the distribution of contact stresses along the contact part is of primal interest. For this reason, optimal shape design problems with cost functionals or functionals defining constraints in which contact stresses appear in the argument are important. Since the reciprocal variational formulation enables us to compute contact stresses directly, it is natural to use it in such a type of problems.

In this section we will study the following class of optimal shape design problems:

$$(\mathbb{P}) \qquad \begin{cases} \qquad \text{Minimize } E_0\big(\alpha, \lambda(\alpha)\big) \\ \text{subject to } E_j\big(\alpha, \lambda(\alpha)\big) \leqslant 0 \quad j = 1, 2, \ldots, s, \ \alpha \in \mathscr{U}_{ad}, \end{cases}$$

where $\lambda(\alpha) \in \mathbf{\Lambda}$ is the solution to $\big(\mathscr{R}(\alpha)\big)$ and $\big\{E_j\big\}_{j=0}^{s}$ are given functionals defined in $\mathscr{U}_{ad} \times \mathbf{\Lambda}$.

Before we prove the existence of at least one solution to $\big(\mathbb{P}\big)$ we have to specify what we mean by the convergence of the contact stresses with respect to boundary variations.

Definition 2.1. Let $\alpha_n \rightrightarrows \alpha$ (uniformly) in $[a, b]$, $\alpha_n, \alpha \in \mathscr{U}_{ad}$ and let $\lambda^{(n)} \equiv \lambda(\alpha_n) \in \mathbf{\Lambda}$ be solutions to $\big(\mathscr{R}(\alpha_n)\big)$. We write

$$\lambda^{(n)} \rightharpoonup \lambda \quad \text{in } \mathbb{H}^{-1/2}(\hat{\Gamma})$$

iff

$$\langle \lambda_g^{(n)}, \psi \rangle_{\alpha_n} \longrightarrow \langle \lambda_g, \psi \rangle_{\alpha}, \quad n \to \infty, \tag{2.1}$$

holds for any function $\psi \in \mathbb{H}^1(\hat{\Omega})$ such that ψ vanishes in a neighbourhood of $\Gamma_u(\alpha) \subset \partial\hat{\Omega}$.

Remark 2.1. Let us recall that (2.1) has to be understood in the following sense:

$$\big(g\lambda_1^{(n)}, \varphi_{1n}\big)_{0,\hat{\Gamma}} + \langle \lambda_2^{(n)}, \varphi_{2n} \rangle \longrightarrow \big(g\lambda_1, \varphi_1\big)_{0,\hat{\Gamma}} + \langle \lambda_2, \varphi_2 \rangle, \quad n \to \infty,$$

where $\lambda^{(n)} = \big(\lambda_1^{(n)}, \lambda_2^{(n)}\big)$, $\lambda = \big(\lambda_1, \lambda_2\big)$, $\psi = \big(\psi_1, \psi_2\big)$ and $\varphi_{in} \equiv \big(\operatorname{trace}_{\Gamma(\alpha_n)} \psi\big) \circ F_{\alpha_n}$, $\varphi_i \equiv \big(\operatorname{trace}_{\Gamma(\alpha)} \psi\big) \circ F_\alpha$, $\quad i = 1, 2$.

To guarantee the existence of solutions to $\big(\mathbb{P}\big)$, we need the following *lower semi-continuity* property of E_j, $j = 0, \ldots, s$:

$$(\mathbb{A}) \qquad \begin{cases} \left.\begin{array}{l} \alpha_n \rightrightarrows \alpha \quad \text{in } [a, b], \ \alpha_n, \alpha \in \mathscr{U}_{ad}, \\ \lambda(\alpha_n) \rightharpoonup \lambda(\alpha) \ \text{(in the sense of Definition 2.1)} \end{array}\right\} \Longrightarrow \\ \Longrightarrow \displaystyle\lim_{n\to\infty} \inf E_j\big(\alpha_n, \lambda(\alpha_n)\big) \geqslant E_j\big(\alpha, \lambda(\alpha)\big), \ j = 0, \ldots, s. \end{cases}$$

Finally, denote by $\tilde{\mathscr{U}}_{ad}$ the set of all *admissible pairs*, i.e.

$$\tilde{\mathscr{U}}_{ad} = \big\{\big(\alpha, \lambda(\alpha)\big) \mid \alpha \in \mathscr{U}_{ad}, \ \lambda(\alpha) \text{ solves } \big(\mathscr{R}(\alpha)\big), \ E_j\big(\alpha, \lambda(\alpha)\big) \leqslant 0, \\ j = 1, \ldots, s\big\}.$$

The main result of this section is

Theorem 2.1. *Let* $(\mathbb{A})$ *be satisfied and* $\tilde{\mathscr{U}}_{ad} \neq \emptyset$. *Then* $(\mathbb{P})$ *has at least one solution.*

Before we prove this theorem, we establish the following auxiliary result.

Lemma 2.1. *Let* $\alpha_n \rightrightarrows \alpha$ *in* $[a,b]$, $\alpha_n, \alpha \in \mathscr{U}_{ad}$ *and let* $\lambda^{(n)} \equiv \lambda(\alpha_n) \in \mathbf{\Lambda}$ *be solutions to* $(\mathscr{R}(\alpha_n))$. *Then there exist a subsequence of* $\{\lambda^{(n)}\}$ *(denoted in the same way as the original sequence) and an element* $\lambda \in \mathbf{\Lambda}$ *such that*

$$\lambda^{(n)} \rightharpoonup \lambda \quad \text{in } \mathbb{H}^{-1/2}(\hat{\Gamma}), \quad n \to \infty$$

and $\lambda \equiv \lambda(\alpha)$ *solves* $(\mathscr{R}(\alpha))$.

Proof. Let $u_n \in K(\alpha_n)$, $\lambda^{(n)} \in \mathbf{\Lambda}$ be the solutions to $(\mathscr{P}(\alpha_n))$, $(\mathscr{R}(\alpha_n))$, respectively. Then the couple $(u_n, \lambda^{(n)})$ solves $(\mathscr{M}(\alpha_n))$. Let $\psi \in \mathbb{H}^1(\hat{\Omega})$ be a given function, vanishing in a neighbourhood of $\Gamma_u(\alpha)$. Since $\alpha_n \rightrightarrows \alpha$ in $[a,b]$ we have also that $\Gamma_u(\alpha_n) \rightrightarrows \Gamma_u(\alpha)$ and consequently, the restriction ψ on $\Omega(\alpha_n)$ belongs to $\mathbb{V}(\alpha_n)$ provided n is sufficiently large. Such a function can be used as a test function in $(\mathscr{M}(\alpha_n))$:

$$(2.2) \qquad \big(\Lambda\varepsilon(u_n), \varepsilon(\psi)\big)_{0,\Omega(\alpha_n)} = L_{\alpha_n}(\psi) + \langle \lambda_g^{(n)}, \psi \rangle_{\alpha_n}.$$

It is known (see Lemma 7.2 in [7]) that there exist a subsequence of $\{u_n\}$ (denoted by the same symbol) and a function $\hat{u} \in \mathbb{H}^1(\hat{\Omega})$ such that

$$\tilde{u}_n \rightharpoonup \hat{u} \quad \text{(weakly) in } \mathbb{H}^1(\hat{\Omega}),$$

where the symbol "$\sim$" stands for the uniform extension of functions from the domain of their definition onto $\hat{\Omega}$, and $u \equiv \hat{u}|_{\Omega(\alpha)}$ solves $(\mathscr{P}(\alpha))$.

Now (2.2) implies

$$(2.3) \qquad \begin{aligned} \lim_{n\to\infty} \langle \lambda_g^{(n)}, \psi \rangle_{\alpha_n} &= \lim_{n\to\infty} \big\{ \big(\Lambda\varepsilon(u_n), \varepsilon(\psi)\big)_{0,\Omega(\alpha_n)} - L_{\alpha_n}(\psi) \big\} \\ &= \big(\Lambda\varepsilon(u), \varepsilon(\psi)\big)_{0,\Omega(\alpha)} - L_\alpha(\psi) \end{aligned}$$

again by virtue of the results from Section 7.3 in [7]. Since $u \in K(\alpha)$ is the solution to $(\mathscr{P}(\alpha))$ and $\lambda(\alpha) \in \mathbf{\Lambda}$ is the unique solution of $(\mathscr{R}(\alpha))$ (such a unique solution exists for any $\alpha \in \mathscr{U}_{ad}$), the couple $(u, \lambda(\alpha))$ solves $(\mathscr{M}(\alpha))$. Hence

$$\big(\Lambda\varepsilon(u), \varepsilon(\psi)\big)_{0,\Omega(\alpha)} = L_\alpha(\psi) + \langle \lambda_g, \psi \rangle_\alpha.$$

Comparing this with (2.3) we arrive at the assertion of Lemma. $\square$

P r o o f o f T h e o r e m 2.1. Let $\{\alpha_n\}$, $\alpha_n \in \tilde{\mathscr{U}}_{ad}$ be a minimizing sequence:

$$q = \inf_{\alpha \in \tilde{\mathscr{U}}_{ad}} E_0\bigl(\alpha, \lambda(\alpha)\bigr) = \lim_{n\to\infty} E_0\bigl(\alpha_n, \lambda(\alpha_n)\bigr).$$

In view of the Ascoli-Arzelà theorem there is a subsequence of $\{\alpha_n\}$ (denoted by the same symbol) and an element $\alpha^* \in \mathscr{U}_{ad}$ such that

$$\alpha_n \rightrightarrows \alpha^* \quad \text{in } [a,b]$$

and at the same time

$$\lambda^{(n)} \rightharpoonup \lambda(\alpha^*) \quad \text{in } \mathbb{H}^{-1/2}(\hat{\Gamma})$$

in the sense of Definition 2.1 with $\lambda(\alpha^*)$ being the solution to $\bigl(\mathscr{R}(\alpha^*)\bigr)$. It follows from $(\mathbb{A})$ that

$$\lim_{n\to\infty} \inf E_j\bigl(\alpha_n, \lambda(\alpha_n)\bigr) \geqslant E_j\bigl(\alpha^*, \lambda(\alpha^*)\bigr),\ j = 1,\ldots,s$$

so that

$$E_j\bigl(\alpha^*, \lambda(\alpha^*)\bigr) \leqslant 0 \quad \forall j = 1,\ldots,s,$$

i.e. $\alpha^* \in \tilde{\mathscr{U}}_{ad}$. Since also

$$q = \lim_{n\to\infty} E_0\bigl(\alpha_n, \lambda(\alpha_n)\bigr) \geqslant E_0\bigl(\alpha^*, \lambda(\alpha^*)\bigr) \geqslant q$$

we see that α^* solves $(\mathbb{P})$. □

Next, we present two examples of optimal shape design problems with functionals satisfying $(\mathbb{A})$.

E x a m p l e 2.1. Set

$$\begin{aligned} E_0\bigl(\alpha, \lambda(\alpha)\bigr) &= \operatorname{meas} \Omega(\alpha),\\ E_1\bigl(\alpha, \lambda(\alpha)\bigr) &= (g\lambda_1(\alpha), \varphi)_{0,\hat{\Gamma}} + \langle \lambda_2(\alpha), \varphi\rangle - d, \end{aligned}$$

where $d \in \mathbb{R}^1$ and $\varphi \in C_0^\infty([a,b])$ are given. Problem $(\mathbb{P})$ with such a choice of the functionals may be interpreted as the weight minimization under the additional constraint, namely the average of the contact stresses does not exceed an apriori given value d on a portion of $\Gamma(\alpha)$ determined by supp φ. From the previous analysis we see that $(\mathbb{A})$ is satisfied in this case.

E x a m p l e 2.2. Set $s = 0$ and

$$E_0\big(\alpha, \lambda(\alpha)\big) = \frac{1}{2}\|\lambda_g(\alpha)\|^2_{-1/2,\alpha}.$$

We also slightly modify the definition of $\mathscr{U}_{ad}$:

$$\mathscr{U}_{ad} = \left\{\alpha \in C^{0,1}([a,b]) \mid 0 \leqslant \alpha \leqslant C_0,\ |\,\mathrm{d}\alpha/\,\mathrm{d}x_1| \leqslant C_1,\quad \int_a^b \alpha(x_1)\,\mathrm{d}x_1 = C_2\right\},$$

where C_0, C_1 and C_2 are chosen in such a way that $\mathscr{U}_{ad} \neq 0$. Shape optimization with respect to E_0 corresponds to the minimization of energy of contact stresses provided *the weight of structures is prescribed.* Recall that

$$E_0\big(\alpha, \lambda(\alpha)\big) = \frac{1}{2}\|\lambda_g(\alpha)\|^2_{-1/2,\alpha} = \frac{1}{2}|\!|\!| v(\lambda_g)|\!|\!|^2_{1,\Omega(\alpha)} = \frac{1}{2}\langle\lambda_g, v(\lambda_g)\rangle_\alpha,$$

where $v(\lambda_g) \in \mathbb{V}(\alpha)$ is the solution to

$$\begin{aligned}(2.4)\qquad \big(\Lambda\varepsilon\big(v(\lambda_g)\big), \varepsilon(\psi)\big)_{0,\Omega(\alpha)} = \langle\lambda_g, \psi\rangle_\alpha = \big(\Lambda\varepsilon\big(u(\alpha)\big), \varepsilon(\psi)\big)_{0,\Omega(\alpha)}&\\ - L_\alpha(\psi)\quad \forall\psi \in \mathbb{V}(\alpha),&\end{aligned}$$

where $u(\alpha) \in K(\alpha)$ solves $\big(\mathscr{P}(\alpha)\big)$. We shall show that E_0 satisfies $\big(\mathbb{A}\big)$. Indeed, let $\alpha_n \rightrightarrows \alpha$ in $[a,b]$, $\alpha_n, \alpha \in \mathscr{U}_{ad}$. Then one can find a subsequence of $\{u_n\}$ $\big($solutions of $\big(\mathscr{P}(\alpha_n)\big)\big)$ and a function $\hat{u} \in \mathbb{H}^1(\hat{\Omega})$ such that

$$(2.5)\qquad \tilde{u}_n \rightharpoonup \hat{u}\quad \text{in } \mathbb{H}^1(\hat{\Omega}),$$

where the meaning of "$\sim$" is the same as in the proof of Lemma 2.1. Moreover, $u \equiv \hat{u}\big|_{\Omega(\alpha)}$ solves $\big(\mathscr{P}(\alpha)\big)$. Let us consider (2.4) on each $\Omega(\alpha_n)$:

$$(2.6)\qquad \big(\Lambda\varepsilon(v_n), \varepsilon(\psi)\big)_{0,\Omega(\alpha_n)} = \langle\lambda_g^{(n)}, \psi\rangle_{\alpha_n} = \big(\Lambda\varepsilon(u_n), \varepsilon(\psi)\big)_{0,\Omega(\alpha_n)} - L_{\alpha_n}(\psi)$$

holds for any $\psi \in \mathbb{H}^1(\hat{\Omega})$ vanishing in a neighbourhood of $\Gamma_u(\alpha)$ and for n sufficiently large. Passing to the limit on the right-hand side of (2.6) we obtain

$$(2.7)\qquad \lim_{n\to\infty}\big\{\big(\Lambda\varepsilon(u_n), \varepsilon(\psi)\big)_{0,\Omega(\alpha_n)} - L_{\alpha_n}(\psi)\big\} = \big(\Lambda\varepsilon(u), \varepsilon(\psi)\big)_{0,\Omega(\alpha)} - L_\alpha(\psi).$$

From (2.6) and the fact that the constant c in Korn's inequality

$$\big(\varepsilon_{ij}(q), \varepsilon_{ij}(q)\big)_{0,\Omega(\alpha)} \geqslant c\|q\|^2_{1,\Omega(\alpha)}\quad \forall q \in \mathbb{V}(\alpha),$$

can be chosen independently of $\alpha \in \mathscr{U}_{ad}$ we see that also the sequence $\{\Vert v_n \Vert_{1,\Omega(\alpha_n)}\}$ is bounded. Using the same technique as in [7] one can find a subsequence of $\{v_n\}$ (still denoted by the same symbol) such that

$$\tilde{v}_n \rightharpoonup \hat{v} \quad \text{in } \mathbb{H}^1(\hat{\Omega}) \tag{2.8}$$

and

$$\lim_{n\to\infty} \big(\Lambda\varepsilon(v_n), \varepsilon(\psi)\big)_{0,\Omega(\alpha_n)} = \big(\Lambda\varepsilon(\hat{v}), \varepsilon(\psi)\big)_{0,\Omega(\alpha)}$$

holds for any function $\psi \in \mathbb{H}^1(\hat{\Omega})$ with the above mentioned property. From this, (2.6) and (2.7) we see that the function $v \equiv \hat{v}\big|_{\Omega(\alpha)} \in \mathbb{V}(\alpha)$ satisfies

$$\begin{aligned} \big(\Lambda\varepsilon(v), \varepsilon(\psi)\big)_{0,\Omega(\alpha)} &= \big(\Lambda\varepsilon\big(u(\alpha)\big), \varepsilon(\psi)\big)_{0,\Omega(\alpha)} - L_\alpha(\psi) \\ &= \langle \lambda_g(\alpha), \psi\rangle_\alpha, \end{aligned} \tag{2.9}$$

where $\lambda(\alpha) \in \mathbf{\Lambda}$ is the solution of the reciprocal formulation $\big(\mathscr{R}(\alpha)\big)$. From (2.9) we get also

$$\Vert v \Vert_{1,\Omega(\alpha)} = \|\lambda_g(\alpha)\|_{-1/2,\alpha}.$$

It is readily seen that E_0 satisfies the assumption $\big(\mathbb{A}\big)$. Indeed,

$$\begin{aligned} \lim_{n\to\infty} \inf E_0\big(\alpha_n, \lambda(\alpha_n)\big) &= \lim_{n\to\infty} \inf \frac{1}{2}\Vert v_n \Vert^2_{1,\Omega(\alpha_n)} \\ &\geqslant \frac{1}{2}\Vert v \Vert^2_{1,\Omega(\alpha)} = E_0\big(\alpha, \lambda(\alpha)\big) \end{aligned}$$

where we have made use of the fact that $\alpha_n \rightrightarrows \alpha$ in $[a,b]$ and (2.8).

R e m a r k 2.2. To see why the reciprocal variational formulation is more advantageous in many situations, let us consider the functional E_0 from Example 2.2. We already know that

$$E_0\big(\alpha, \lambda(\alpha)\big) \equiv \frac{1}{2}\|\lambda_g(\alpha)\|^2_{-1/2,\alpha} = \frac{1}{2}\Vert v \Vert^2_{1,\Omega(\alpha)},$$

where $v \in \mathbb{V}(\alpha)$ is the solution of

$$\big(\Lambda\varepsilon(v), \varepsilon(\psi)\big)_{0,\Omega(\alpha)} = \big(\Lambda\varepsilon\big(u(\alpha)\big), \varepsilon(\psi)\big)_{0,\Omega(\alpha)} - L_\alpha(\psi) \quad \forall \psi \in \mathbb{V}(\alpha). \tag{2.10}$$

In [5] the evaluation of E_0 by means of the function v satisfying (2.10) was used. However, before v is computed, one has to solve $\big(\mathscr{P}(\alpha)\big)$ in order to get $u(\alpha)$ necessary for the evaluation of the right-hand side in (2.10). On the other hand, taking into account that

$$E_0\big(\alpha, \lambda(\alpha)\big) \equiv \frac{1}{2}\|\lambda_g(\alpha)\|^2_{-1/2,\alpha} = \frac{1}{2}\langle \lambda_g(\alpha), v\rangle_\alpha,$$

we see that only the boundary data $\lambda_g(\alpha)$ and $\operatorname{trace}_{\Gamma(\alpha)} v$ are needed. The contact stress $\lambda_g(\alpha)$ is directly available from the reciprocal variational formulation $\big(\mathscr{R}(\alpha)\big)$, while

$$\operatorname{trace}_{\Gamma(\alpha)} v = \operatorname{trace}_{\Gamma(\alpha)} u - \operatorname{trace}_{\Gamma(\alpha)} w,$$

where $u \in K(\alpha)$ solves $\big(\mathscr{P}(\alpha)\big)$ and $w \in \mathbb{V}(\alpha)$ is the solution of (1.16). Let us note that $\operatorname{trace}_{\Gamma(\alpha)} u$ is usually available as a by-product, when conjugate gradient type methods are used for the numerical realization of $\big(\mathscr{R}(\alpha)\big)$ since $\operatorname{trace}_{\Gamma(\alpha)} u$ corresponds to the Lagrange multiplier associated with the constraint $\lambda \in \mathbf{\Lambda}$.

3. Approximation of $(\mathbb{P})$

The aim of this section is to describe the approximation of $(\mathbb{P})$. We shall define appropriate discretizations of $(\mathbb{P})$ which are close on subsequences to the original continuous setting.

First we start with the approximation of the set $\mathscr{U}_{ad}$, characterizing the admissible shapes.

Let $\{D_h\}$, $h \to 0+$ be a family of partitions of $[a,b]$, the norms of which tend to zero. Let $D_h\colon\ a = x_1^0 < x_1^1 < \ldots < x_1^D = b$ be the set of nodes of D_h and define

$$\mathscr{U}_{ad}^h = \big\{\alpha_h \in C([a,b]) \mid \alpha_h \text{ piecewise linear over } D_h\big\} \cap \mathscr{U}_{ad}.$$

Finally, let $\{D_H\}$, $H \to 0+$ be another family of partitions of $[a,b]$, *generally different* from $\{D_h\}$. With any $\{D_H\}$ the following sets will be associated:

$$\begin{aligned}
L_H &= \big\{\mu^H \in L^2(\hat\Gamma) \mid \mu^H \text{ piecewise constant over } D_H\big\},\\
\Lambda_1^H &= \big\{\mu_1^H \in L_H \mid |\mu_1^H| \leqslant 1 \quad \text{a.e. in } (a,b)\big\},\\
\Lambda_2^H &= \big\{\mu_2^H \in L_H \mid \mu_2^H \geqslant 0 \quad \text{a.e. in } (a,b)\big\},\\
\mathbf{\Lambda}_H &= \Lambda_1^H \times \Lambda_2^H.
\end{aligned}$$

For a given $\alpha_h \in \mathscr{U}_{ad}^h$, the approximation of the reciprocal energy functional is defined as follows:

$$\mathscr{S}_{\alpha_h}(\mu_H) = \frac{1}{2} b_{\alpha_h}(\mu_H, \mu_H) - \mathscr{F}_{\alpha_h}(\mu_H), \quad \mu_H \in \mathbf{\Lambda}_H, \tag{3.1}$$

where

$$\begin{aligned}
b_{\alpha_h}(\mu_H, \nu_H) &\overset{\text{def}}{\equiv} \big(g\mu_1^H, (\mathscr{G}_{\alpha_h}(\nu_H))_1\big)_{0,\alpha_h} + \big(\mu_2^H, (\mathscr{G}_{\alpha_h}(\nu_H))_2\big)_{0,\alpha_h}\\
\mathscr{F}_{\alpha_h}(\mu_H) &\overset{\text{def}}{\equiv} -\big(g\mu_1^H, (\mathscr{G}_{\alpha_h}(L_{\alpha_h}))_1\big)_{0,\alpha_h} - \big(\mu_2^H, (\mathscr{G}_{\alpha_h}(L_{\alpha_h}))_2\big)_{0,\alpha_h}\\
&\quad - (\mu_2^H, \alpha_h)_{0,\hat\Gamma},
\end{aligned}$$

where $\mathscr{G}_{\alpha_h}$ denotes an approximation of Green's mapping G_α and $\big(\mathscr{G}_{\alpha_h}(.)\big)_j$, $j = 1, 2$ is the j-th component of the corresponding approximated displacement field in $\Omega(\alpha_h)$. Here we use the similar convention as we did in the continuous setting: the scalar product $(\mu^H, v)_{0,\alpha_h}$, where $\mu^H \in L_H$, $v \in V(\alpha_h)$ has to be read as $(\mu^H, \varphi_h)_{0,\hat{\Gamma}}$, where $\varphi_h \equiv \big(\mathrm{trace}_{\Gamma(\alpha_h)}\, v\big) \circ F_{\alpha_h}$. Below we describe one possible way of constructing $\mathscr{G}_{\alpha_h}$ starting from a finite element approximation of the mixed formulation $\big(\mathscr{M}(\alpha)\big)$.

Since $\Omega(\alpha_h)$ is a *polygonal domain* for any $\alpha_h \in \mathscr{U}^h_{ad}$, one can construct its triangulation denoted by $\mathscr{T}(h, \alpha_h)$. Next we shall consider only such families $\{\mathscr{T}(h, \alpha_h)\}$ which are *topologically equivalent* and *uniformly regular* with respect to $h \to 0+$, $\alpha_h \in \mathscr{U}^h_{ad}$. This means:

(j) *for any $h > 0$ fixed, the position of the nodes belonging to $\mathscr{T}(h, \alpha_h)$ depends continuously on the variations of $\alpha_h \in \mathscr{U}^h_{ad}$;*

(jj) *for any $h > 0$, the number of the nodes from $\mathscr{T}(h, \alpha_h)$ is the same for all $\alpha_h \in \mathscr{U}^h_{ad}$ and the nodes have the same neighbours;*

(jjj) *there is $\vartheta_0 > 0$ such that*

$$\vartheta(h, \alpha_h) \geqslant \vartheta_0 \quad \forall h > 0,\ \alpha_h \in \mathscr{U}^h_{ad},$$

where $\vartheta(h, \alpha_h)$ is the minimal interior angle of all triangles from $\mathscr{T}(h, \alpha_h)$.

Finally, the only contact nodes, i.e. the nodes where the unilateral condition is prescribed, are those with the coordinates $A_j \equiv \big(x_1^j, \alpha_h(x_1^j)\big)$, $\alpha_h \in \mathscr{U}^h_{ad}$, such that $A_j \in \overline{\Gamma}(\alpha_h) \setminus \overline{\Gamma}_u(\alpha_h)$. The domain $\Omega(\alpha_h)$ with the triangulation $\mathscr{T}(h, \alpha_h)$ will be denoted by Ω_h in what follows.

With any $\alpha_h \in \mathscr{U}^h_{ad}$ and any $\mathscr{T}(h, \alpha_h)$, the following sets of piecewise-linear functions defined in Ω_h will be associated:

$$\begin{aligned}
V_h(\alpha_h) &= \big\{v_h \in C(\overline{\Omega}_h) \mid v_h\big|_T \in P_1(T) \quad \forall T \in \mathscr{T}(h, \alpha_h),\\
&\qquad v_h = 0 \text{ on } \Gamma_u(\alpha_h)\big\};\\
\mathbb{V}_h(\alpha_h) &= V_h(\alpha_h) \times V_h(\alpha_h);\\
K_h(\alpha_h) &= \big\{v_h = (v_1^h, v_2^h) \in \mathbb{V}_h(\alpha_h) \mid v_2^h\big(x_1^j, \alpha_h(x_1^j)\big) \geqslant -\alpha_h(x_1^j),\ \forall j\\
&\qquad \text{such that } A_j \in \overline{\Gamma}(\alpha_h) \setminus \overline{\Gamma}_u(\alpha_h)\big\}.
\end{aligned}$$

The *approximation* of the mixed formulation $\big(\mathscr{M}(\alpha)\big)$ now reads as follows:

$$\big(\mathscr{M}(\alpha_h)\big)_h^H \qquad \begin{cases} \text{Find } (u_h, \lambda_H) \in \mathbb{V}_h(\alpha_h) \times \mathbf{\Lambda}_H \text{ such that}\\ \mathscr{L}_{\alpha_h}(u_h, \mu_H) \leqslant \mathscr{L}_{\alpha_h}(u_h, \lambda_H) \leqslant \mathscr{L}_{\alpha_h}(v_h, \lambda_H)\\ \qquad \forall v_h \in \mathbb{V}_h(\alpha_h), \quad \forall \mu_H \in \mathbf{\Lambda}_H, \end{cases}$$

where

$$\mathscr{L}_{\alpha_h}(v_h, \mu_H) \equiv \mathscr{J}_{\alpha_h}(v_h) - (g\mu_1^H, v_1^h)_{0,\alpha_h} - (\mu_2^H, v_2^h)_{0,\alpha_h} - (\mu_2^H, \alpha_h)_{0,\hat{\Gamma}}$$

with $v_h \in \mathbb{V}_h(\alpha_h),\ \mu_H \in \mathbf{\Lambda}_H$.

An equivalent expression of $\big(\mathscr{M}(\alpha_h)\big)_h^H$ is given by:

$$\big(\mathscr{M}(\alpha_h)\big)_h^H \qquad \begin{cases} \text{Find } (u_h, \lambda_H) \in \mathbb{V}_h(\alpha_h) \times \mathbf{\Lambda}_H \text{ such that} \\ \big(\Lambda\varepsilon(u_h), \varepsilon(v_h)\big)_{0,\Omega_h} = L_{\alpha_h}(v_h) + (g\lambda_1^H, v_1^h)_{0,\alpha_h} \\ \qquad +(\lambda_2^H, v_2^h)_{0,\alpha_h} \quad \forall v_h \in \mathbb{V}_h(\alpha_h), \\ \big(g(\mu_1^H - \lambda_1^H), u_1^h\big)_{0,\alpha_h} + \big(\mu_2^H - \lambda_2^H, u_2^h\big)_{0,\alpha_h} \\ \qquad +\big(\mu_2^H - \lambda_2^H, \alpha_h\big)_{0,\hat{\Gamma}} \geqslant 0 \quad \forall \mu_H \in \mathbf{\Lambda}_H. \end{cases}$$

It is well-known (see [4]) that if the *stability condition*

$$(g\mu_1^H, v_1^h)_{0,\alpha_h} + (\mu_2^H, v_2^h)_{0,\alpha_h} = 0 \quad \forall v_h \in \mathbb{V}_h(\alpha_h) \Longrightarrow \mu_H \equiv 0 \quad \text{in } \hat{\Gamma} \tag{3.2}$$

holds, then $\big(\mathscr{M}(\alpha_h)\big)_h^H$ has a unique solution (u_h, λ_H).

Remark 3.1 (*the validity of* (3.2)). The stability condition (3.2) holds for example if g is a piecewise constant function over the partition D_H used for the construction of $\mathbf{\Lambda}_H$ and, moreover, if the ratio H/h is *sufficiently large* (see [4]). In other words, the partition D_H has to be coarser then the triangulation $\mathscr{T}(h, \alpha_h)$ defining $\mathbb{V}_h(\alpha_h)$. If g is not piecewise constant over D_H then it can be approximated by its projection to L_H. *Next we shall suppose that* (3.2) *is valid for any* $\alpha_h \in \mathscr{U}_{ad}^h$.

Using exactly the same approach as we did in Section 1, we can derive the discrete version of $\big(\mathscr{R}(\alpha)\big)$. Indeed, if $(u_h, \lambda_H) \in \mathbb{V}_h(\alpha_h) \times \mathbf{\Lambda}_H$ is the solution of $\big(\mathscr{M}(\alpha_h)\big)_h^H$ then

$$\mathscr{L}_{\alpha_h}(u_h, \lambda_H) = \min_{\mathbb{V}_h(\alpha_h)} \sup_{\mathbf{\Lambda}_H} \mathscr{L}_{\alpha_h}(v_h, \mu_H) = \max_{\mathbf{\Lambda}_H} \inf_{\mathbb{V}_h(\alpha_h)} \mathscr{L}_{\alpha_h}(v_h, \mu_H). \tag{3.3}$$

Eliminating the displacement field $v_h \in \mathbb{V}_h(\alpha_h)$ one obtains the formulation in terms of $\mu_H \in \mathbf{\Lambda}_H$. The approximation $\mathscr{G}_{\alpha_h}$ of Green's operator G_α is now defined as follows:

$$\begin{cases} \mathscr{G}_{\alpha_h}\colon\ \mathbb{V}_h'(\alpha_h) \mapsto \mathbb{V}_h(\alpha_h), \\ \mathscr{G}_{\alpha_h}(f_h) = u_h; \quad f_h \in \mathbb{V}_h'(\alpha_h), \end{cases} \tag{3.4}$$

where $u_h \in \mathbb{V}_h(\alpha_h)$ is the unique solution of

$$\big(\Lambda\varepsilon(u_h), \varepsilon(v_h)\big)_{0,\Omega_h} = f_h(v_h) \quad \forall v_h \in \mathbb{V}_h(\alpha_h). \tag{3.5}$$

In what follows, the approximation $\mathscr{G}_{\alpha_h}$ used in the definition of $\mathscr{S}_{\alpha_h}$ will be given by (3.4).

Let $(u_h, \lambda_H) \in \mathbb{V}_h(\alpha_h) \times \mathbf{\Lambda}_H$ be the solution of $\big(\mathscr{M}(\alpha_h)\big)_h^H$. Then, using the classical duality theory we obtain that $\lambda_H \in \mathbf{\Lambda}_H$ is the solution of the approximated reciprocal variational formulation

$$\big(\mathscr{R}(\alpha_h)\big)_H \qquad \begin{cases} \quad \text{Find } \lambda_H \in \mathbf{\Lambda}_H \text{ such that} \\ \mathscr{S}_{\alpha_h}(\lambda_H) \leqslant \mathscr{S}_{\alpha_h}(\mu_H) \quad \forall \mu_H \in \mathbf{\Lambda}_H, \end{cases}$$

while the first component u_h solves the unilateral boundary value problem

$$\big(\mathscr{P}(\alpha_h)\big)_h^H \qquad \begin{cases} \quad \text{Find } u_h \in K_h^H(\alpha_h) \text{ such that} \\ \big(\Lambda\varepsilon(u_h), \varepsilon(v_h - u_h)\big)_{0,\Omega_h} + j_{\alpha_h}^H(v_h) - j_{\alpha_h}^H(u_h) \\ \qquad\qquad \geqslant L_{\alpha_h}(v_h - u_h) \quad \forall v_h \in K_h^H(\alpha_h), \end{cases}$$

where

$$\begin{aligned} K_h^H(\alpha_h) &= \big\{ v_h \in \mathbb{V}_h(\alpha_h) \mid (v_2^h, \mu_2^H)_{0,\alpha_h} + (\alpha_h, \mu_2^H)_{0,\hat{\Gamma}} \geqslant 0 \quad \forall \mu_2^H \in \Lambda_2^H \big\}, \\ j_{\alpha_h}^H(v_h) &= \sup_{\mu_1^H \in \Lambda_1^H} \big\{ -(g\mu_1^H, v_1^h)_{0,\alpha_h} \big\}. \end{aligned}$$

R e m a r k 3.2. The convex set $K_h^H(\alpha_h)$ is the *external approximation* of $K(\alpha_h)$. The unilateral condition on $\Gamma(\alpha_h)$ for $v_h \in K_h^H(\alpha_h)$ is satisfied in the following weak sense:

$$\int_{\overline{x}_1^{i-1}}^{\overline{x}_1^i} v_h \circ F_{\alpha_h} \, \mathrm{d}x_1 \geqslant - \int_{\overline{x}_1^{i-1}}^{\overline{x}_1^i} \alpha_h \, \mathrm{d}x_1, \quad i = 1, \dots, m(H), \tag{3.6}$$

where $a \equiv \overline{x}_1^0 < \overline{x}_1^1 < \ldots < \overline{x}_1^{m(H)} \equiv b$ are the nodes of D_H, i.e. the unilateral condition is satisfied in the sense of the integral mean value on any interval $[\overline{x}_1^{i-1}, \overline{x}_1^i]$. The sublinear term j_α characterizing the friction is approximated by $j_{\alpha_h}^H$.

In what follows we will study how the solution (u_h, λ_H) of $\big(\mathscr{M}(\alpha_h)\big)_h^H$ depends on the variations of α_h, h and H. To this end we will use an alternative formulation of $\big(\mathscr{M}(\alpha_h)\big)_h^H$ by satisfying the unilateral constraint (3.6) a-priori, i.e. the formulation in terms of u_h, λ_1^H only:

$$\big(\tilde{\mathscr{M}}(\alpha_h)\big)_h^H \qquad \begin{cases} \quad \text{Find } (u_h, \lambda_1^H) \in K_h^H(\alpha_h) \times \Lambda_1^H \text{ such that} \\ \big(\Lambda\varepsilon(u_h), \varepsilon(v_h - u_h)\big)_{0,\Omega_h} \geqslant L_{\alpha_h}(v_h - u_h) \\ \qquad\qquad + (g\lambda_1^H, v_1^h - u_1^h)_{0,\alpha_h} \quad \forall v_h \in K_h^H(\alpha_h), \\ \big(g(\mu_1^H - \lambda_1^H), u_1^h\big)_{0,\alpha_h} \geqslant 0 \quad \forall \mu_1^H \in \Lambda_1^H. \end{cases}$$

In what follows we will suppose that the mesh sizes h, H characterizing $\{\mathscr{T}(h,\alpha_h)\}$, $\{D_H\}$, respectively satisfy

$$h \to 0+ \quad \text{iff} \quad H \to 0+\,.$$

In order to prove the counterpart of Lemma 2.1 we shall slightly modify Definition 2.1.

Definition 3.1. Let $\alpha_h \rightrightarrows \alpha$ in $[a,b]$, $\alpha_h \in \mathscr{U}_{ad}^h$, $\alpha \in \mathscr{U}_{ad}$ and let $\lambda_H(\alpha_h) \in \mathbf{\Lambda}_H$ be a solution of $\big(\mathscr{R}(\alpha_h)\big)_H$. We write

$$\lambda_H(\alpha_h) \rightharpoonup \lambda \quad \text{in } \mathbb{H}^{-1/2}(\hat{\Gamma}), \quad h, H \to 0+$$

iff

$$(g\lambda_1^H, \psi_1^h)_{0,\alpha_h} + (\lambda_2^H, \psi_2^h)_{0,\alpha_h} \longrightarrow (g\lambda_1, \psi_1)_{0,\alpha} + \langle \lambda_2, \psi_2 \rangle_\alpha$$

holds for any sequence $\{\psi_h\}$, $\psi_h = (\psi_1^h, \psi_2^h) \in \mathbb{V}_h(\alpha_h)$ such that

$$\|\psi_h - \psi\|_{1,\Omega(\alpha_h)} \to 0 \quad \text{as} \quad h \to 0+,$$

where $\psi \in \mathbb{H}^1(\hat{\Omega})$ has the same property as in Definition 2.1.

Recall that

$$(g\lambda_1^H, \psi_1^h)_{0,\alpha_h} \equiv (g\lambda_1^H, \varphi_1^h)_{0,\hat{\Gamma}},$$

where $\varphi_1^h \equiv \big(\mathrm{trace}_{\Gamma(\alpha_h)}\, \psi_1^h\big) \circ F_{\alpha_h}$. Similarly for the other terms.

Now we prove the following continuity result:

Lemma 3.1. *Let $\alpha_h \rightrightarrows \alpha$ in $[a,b]$, $\alpha_h \in \mathscr{U}_{ad}^h$, $\alpha \in \mathscr{U}_{ad}$ and let (u_h, λ_H) be the unique solution of $\big(\mathscr{M}(\alpha_h)\big)_h^H$. Then there exist a subsequence of $\{(u_h, \lambda_H)\}$ (denoted by the same symbol) and elements $\hat{u} \in \mathbb{H}^1(\hat{\Omega})$, $\hat{\lambda} \in \mathbf{\Lambda}$ such that*

$$\begin{aligned} \tilde{u}_h &\rightharpoonup \hat{u} \quad \textit{in } \mathbb{H}^1(\hat{\Omega}), \quad h \to 0+; \\ \lambda_H &\rightharpoonup \hat{\lambda} \quad \textit{in } \mathbb{H}^{-1/2}(\hat{\Gamma}), \quad H \to 0+ \end{aligned}$$

(in the sense of Definition 3.1). Moreover, the function $u \equiv \hat{u}\big|_{\Omega(\alpha)}$ solves $\big(\mathscr{P}(\alpha)\big)$ and $\hat{\lambda} \equiv \lambda(\alpha)$ solves the corresponding reciprocal variational formulation $\big(\mathscr{R}(\alpha)\big)$.

Proof. It will be done in several steps.

(i) Using the fact that the constant of Korn's inequality can be chosen independently of $\alpha_h \in \mathscr{U}_{ad}^h$ (see Example 2.2) we see that there exists a constant $c > 0$ such that

$$\|u_h\|_{1,\Omega(\alpha_h)} \leqslant c$$

so that the sequence of the uniform extensions $\tilde{u}_h$ of u_h from $\Omega(\alpha_h)$ onto $\hat{\Omega}$ is bounded as well:

$$\|\tilde{u}_h\|_{1,\hat{\Omega}} \leqslant c.$$

Therefore there exist a subsequence of $\{\tilde{u}_h\}$ (denoted by the same symbol) and a function $\hat{u} \in \mathbb{H}^1(\hat{\Omega})$ such that

$$\tilde{u}_h \rightharpoonup \hat{u} \quad \text{in } \mathbb{H}^1(\hat{\Omega}), \quad h \to 0+.$$

(*ii*) Here we show that $u \equiv \hat{u}\big|_{\Omega(\alpha)}$ belongs to $K(\alpha)$. The fact that $u = 0$ on $\Gamma_u(\alpha)$ is obvious. To show that u satisfies the unilateral condition on $\Gamma(\alpha)$ it is sufficient to verify that

$$(u_2, \mu_2)_{0,\alpha} + (\alpha, \mu_2)_{0,\hat{\Gamma}} \geqslant 0$$

holds for any $\mu_2 \geqslant 0$, $\mu_2 \in L^2(\hat{\Gamma})$. Let such a μ_2 be given. Then there exists a sequence $\{\mu_2^H\}$, $\mu_2^H \in \Lambda_2^H$ such that

$$\mu_2^H \to \mu_2 \quad \text{in } L^2(\hat{\Gamma}), \quad H \to 0+.$$

Using exactly the same technique as in Lemma 1.1 in [7] one can prove that

$$\begin{cases} (u_2^h, \mu_2^H)_{0,\alpha_h} \longrightarrow (u_2, \mu_2)_{0,\alpha}; \\ (\alpha_h, \mu_2^H)_{0,\hat{\Gamma}} \longrightarrow (\alpha, \mu_2)_{0,\hat{\Gamma}}, \quad h, H \to 0+. \end{cases} \tag{3.7}$$

Since the sum of the terms on the left-hand side of (3.7) is non-negative in view of the fact that $u_h \in K_h^H(\alpha_h)$, the limit sum is non-negative as well. Thus $u \in K(\alpha)$.

(*iii*) Now we prove that u solves $(\mathscr{P}(\alpha))$. Let $\hat{v} \in \mathbb{H}^1(\hat{\Omega})$ be such that $v \equiv \hat{v}\big|_{\Omega(\alpha)}$ belongs to $K(\alpha)$. Then there exists a subsequence of $\{\alpha_h\}$ (still denoted by the same symbol) and a sequence $\{\overline{v}_h\}$, $\overline{v}_h \in K_h(\alpha_h)$ such that

$$\|\overline{v}_h - \hat{v}\|_{1,\Omega(\alpha_h)} \to 0, \quad h \to 0+. \tag{3.8}$$

The construction of such a sequence is described in Lemmas 7.3 and 7.4 in [7]. Since $K_h(\alpha_h) \subset K_h^H(\alpha_h)$, the function $\overline{v}_h$ can be used as a test function in $\big(\tilde{\mathscr{M}}(\alpha_h)\big)_h^H$.

Let $\overline{\mu}_1 \in \Lambda_1$ be given. Then there exists a sequence $\{\overline{\mu}_1^H\}$, $\overline{\mu}_1^H \in \Lambda_1^H$ such that

$$\overline{\mu}_1^H \to \overline{\mu}_1 \quad \text{in } L^2(\hat{\Gamma}), \quad H \to 0+. \tag{3.9}$$

Since $\{\lambda_1^H\}$ is bounded, one may assume that there exists $\lambda_1 \in \Lambda_1$ such that

$$\lambda_1^H \rightharpoonup \lambda_1 \quad \text{in } L^2(\hat{\Gamma}), \quad H \to 0+\,. \tag{3.10}$$

Using exactly the same technique as in [7] Section 7.4 we have

$$\begin{aligned}
\lim_{h\to 0+} \sup\big(\Lambda\varepsilon(u_h), \varepsilon(\overline{v}_h - u_h)\big)_{0,\Omega_h} &\leqslant \big(\Lambda\varepsilon(u), \varepsilon(\hat{v} - u)\big)_{0,\Omega(\alpha)};\\
\lim_{h\to 0+} L_{\alpha_h}(\overline{v}_h - u_h) &= L_\alpha(\hat{v} - u);\\
\lim_{h,H\to 0+} \big(g\lambda_1^H, \overline{v}_1^h - u_1^h\big)_{0,\alpha_h} &= \big(g\lambda_1, \hat{v}_1 - u_1\big)_{0,\alpha};\\
\lim_{h,H\to 0+} \big(g(\overline{\mu}_1^H - \lambda_1^H), u_1^h\big)_{0,\alpha_h} &= \big(g(\overline{\mu}_1 - \lambda_1), u_1\big)_{0,\alpha},
\end{aligned}$$

where we have made use of (3.8), (3.9) and (3.10). Passing to the limit in $\big(\tilde{\mathscr{M}}(\alpha_h)\big)_h^H$ and using the previous limit processes we see that

$$\begin{cases} \big(\Lambda\varepsilon(u), \varepsilon(\hat{v} - u)\big)_{0,\Omega(\alpha)} \geqslant L_\alpha(\hat{v} - u) + (g\lambda_1, \hat{v}_1 - u_1)_{0,\alpha},\\ \big(g(\overline{\mu}_1 - \lambda_1), u_1\big)_{0,\alpha} \geqslant 0 \end{cases} \tag{3.11}$$

holds for any $\hat{v} \in \mathbb{H}^1(\hat{\Omega})$ such that $\hat{v}\big|_{\Omega(\alpha)} \in K(\alpha)$ and any $\overline{\mu}_1 \in \Lambda_1$. It follows from (3.11) that

$$\big(\Lambda\varepsilon(u), \varepsilon(v - u)\big)_{0,\Omega(\alpha)} + j_\alpha(v) - j_\alpha(u) \geqslant L_\alpha(v - u)$$

holds for any $v \in K(\alpha)$, i.e. u solves $\big(\mathscr{P}(\alpha)\big)$.

(*iv*) It remains to verify that the sequence $\{\lambda_H\}$ tends weakly to $\lambda(\alpha)$ in the sense of Definition 3.1. Let $\{\psi_h\}$ be a sequence with the properties required by Definition 3.1. These functions can be inserted into the first equation in $\big(\mathscr{M}(\alpha_h)\big)_h^H$. Passing there to the limit with $h, H \to 0+$, we obtain

$$\begin{aligned}
&\lim_{h,H\to 0+} \big\{(g\lambda_1^H, \psi_1^h)_{0,\alpha_h} + (\lambda_2^H, \psi_2^h)_{0,\alpha_h}\big\}\\
&\quad = \lim_{h\to 0+} \big\{\big(\Lambda\varepsilon(u_h), \varepsilon(\psi_h)\big)_{0,\Omega(\alpha_h)} - L_{\alpha_h}(\psi_h)\big\}\\
&\quad = \big(\Lambda\varepsilon(u), \varepsilon(\psi)\big)_{0,\Omega(\alpha)} - L_\alpha(\psi).
\end{aligned} \tag{3.12}$$

Since u solves $\big(\mathscr{P}(\alpha)\big)$, the right hand side of the last equality in (3.12) is equal to $(g\lambda_1, \psi_1)_{0,\alpha} + \langle\lambda_2, \psi_2\rangle_\alpha$ as follows from $\big(\mathscr{M}(\alpha)\big)$. Let us also mention that for any function $\psi \in \mathbb{H}^1(\hat{\Omega})$ vanishing in a neighbourhood of $\Gamma_u(\alpha)$ one can construct a sequence $\{\psi_h\}$ with the properties mentioned in Definition 3.1. $\square$

Let $E_j^h: \mathscr{U}_{ad}^h \times \mathbf{\Lambda}_H \to \mathbb{R}^1$ be *an approximation* of the functional E_j, $j = 0, \ldots, s$. *The approximation of the optimal shape design problem* $(\mathbb{P})$ introduced in Section 2 reads as follows:

$$(\mathbb{P})_h^H \qquad \begin{cases} \qquad \text{Minimize } E_0^h\big(\alpha_h, \lambda_H(\alpha_h)\big) \\ \text{subject to } E_j^h\big(\alpha_h, \lambda_H(\alpha_h)\big) \leqslant 0, \quad j = 1, \ldots, s,\ \alpha_h \in \mathscr{U}_{ad}^h, \end{cases}$$

with $\lambda_H(\alpha_h) \in \mathbf{\Lambda}_H$ being the solution of $\big(\mathscr{R}(\alpha_h)\big)_H$.

In what follows we will analyze:

(k) the existence of at least one solution of $(\mathbb{P})_h^H$;

(kk) the relation between $(\mathbb{P})_h^H$ and $(\mathbb{P})$ when $h, H \to 0+$.

In order to prove (k) one has to suppose that $\{E_j^h\}_{j=0}^s$ are *lower semicontinuous* in the following sense:

$$(\mathbb{A})_h^H \qquad \begin{cases} \left.\begin{array}{l} \alpha_h^{(n)} \to \alpha_h \quad \text{as} \quad n \to \infty; \quad \alpha_h^{(n)}, \alpha_h \in \mathscr{U}_{ad}^h \\ \lambda_H^{(n)} \to \lambda_H \quad \text{as} \quad n \to \infty; \quad \lambda_H^{(n)}, \lambda_H \in \mathbf{\Lambda}_H \end{array}\right\} \Longrightarrow \\ \Longrightarrow \liminf\limits_{n\to\infty} E_j^h\big(\alpha_h^{(n)}, \lambda_H^{(n)}\big) \geqslant E_j^h(\alpha_h, \lambda_H) \\ \qquad \forall j = 0, \ldots, s; \quad \forall h, H > 0. \end{cases}$$

Using the same approach as in Lemma 3.1 and making use of the assumptions concerning the family $\{\mathscr{T}(h, \alpha_h)\}$, $\alpha_h \in \mathscr{U}_{ad}^h$ one can prove that the mapping $\alpha_h \mapsto \lambda_H(\alpha_h)$ with $\alpha_h \in \mathscr{U}_{ad}^h$ and $\lambda_H(\alpha_h)$ being the solution of $\big(\mathscr{R}(\alpha_h)\big)_H$ is continuous. From this we directly obtain

Theorem 3.1. *Let the assumption* $(\mathbb{A})_h^H$ *be satisfied. Then* $(\mathbb{P})_h^H$ *has a solution for any* $h, H > 0$.

To establish a relation between $(\mathbb{P})$ and $(\mathbb{P})_h^H$ when $h, H \to 0+$ we restrict ourselves to the case when there are *no state* constraints, i.e. $s = 0$. We need the following assumption relating E_0^h to E_0:

$$(\mathbb{B}) \qquad \begin{cases} \left.\begin{array}{l} \alpha_h \rightrightarrows \alpha \quad \text{in } [a,b];\ \alpha_h \in \mathscr{U}_{ad}^h,\ \alpha \in \mathscr{U}_{ad} \\ \lambda_H(\alpha_h) \rightharpoonup \lambda(\alpha) \quad \text{in } \mathbb{H}^{-1/2}(\hat{\Gamma}); \quad h, H \to 0+ \\ \text{(in the sense of Definition 3.1)} \end{array}\right\} \Longrightarrow \\ \Longrightarrow \lim\limits_{h,H\to 0+} E_0^h\big(\alpha_h, \lambda_H(\alpha_h)\big) = E_0\big(\alpha, \lambda(\alpha)\big). \end{cases}$$

Now we have

Theorem 3.2. *Let the assumption* $(\mathbb{B})$ *be satisfied. Let* $\alpha_h^* \in \mathscr{U}_{ad}^h$ *be a solution of* $(\mathbb{P})_h^H$ *and* $\lambda_H(\alpha_h^*) \in \mathbf{\Lambda}_H$ *the solution of* $\big(\mathscr{R}(\alpha_h^*)\big)_H$. *Then there exist subsequences* *of* $\{\alpha_h^*\}$ *and* $\{\lambda_H(\alpha_h^*)\}$ *(still denoted by the same symbols, respectively) and elements* $\alpha^* \in \mathscr{U}_{ad}$, $\lambda \in \mathbf{\Lambda}$ *such that*

$$\begin{aligned} \alpha_h^* &\rightrightarrows \alpha^* \quad \textit{in } [a,b], \quad h \to 0+;\\ \lambda_H(\alpha_h^*) &\rightharpoonup \lambda \quad \textit{in } \mathbb{H}^{-1/2}(\hat\Gamma), \quad h, H \to 0+, \end{aligned}$$

(in the sense of Definition 3.1*). Moreover,* α^* *is a solution of* $(\mathbb{P})$ *and* $\lambda \equiv \lambda(\alpha^*)$ *solves* $(\mathscr{R}(\alpha^*))$.

P r o o f. Since $\mathscr{U}_{ad}$ is compact in the C-norm, one can find a subsequence of $\{\alpha_h^*\}$ (still denoted by the same symbol) and an element $\alpha^* \in \mathscr{U}_{ad}$ such that

$$\alpha_h^* \rightrightarrows \alpha^* \quad \text{in } [a,b], \quad h \to 0+\,.$$

At the same time we may assume that

$$\lambda_H(\alpha_h^*) \rightharpoonup \lambda(\alpha^*) \quad \text{in } \mathbb{H}^{-1/2}(\hat\Gamma), \quad h, H \to 0+,$$

as follows from Lemma 3.1.

Let $\overline{\alpha} \in \mathscr{U}_{ad}$ be given. One can find a sequence $\{\overline{\alpha}_h\}$, $\overline{\alpha}_h \in \mathscr{U}_{ad}^h$ such that

$$\overline{\alpha}_h \rightrightarrows \overline{\alpha} \quad \text{in } [a,b], \quad h \to 0+$$

and at the same time

$$\lambda_H(\overline{\alpha}_h) \rightharpoonup \lambda(\overline{\alpha}) \quad \text{in } \mathbb{H}^{-1/2}(\hat\Gamma), \quad h, H \to 0+\,.$$

It follows from the definition of $(\mathbb{P})_h^H$ that

$$E_0^h\big(\alpha_h^*, \lambda_H(\alpha_h^*)\big) \leqslant E_0^h\big(\overline{\alpha}_h, \lambda_H(\overline{\alpha}_h)\big). \tag{3.13}$$

Passing to the limit with $h, H \to 0+$ in (3.13) and using $(\mathbb{B})$ we arrive at

$$E_0\big(\alpha^*, \lambda(\alpha^*)\big) \leqslant E_0\big(\overline{\alpha}, \lambda(\overline{\alpha})\big),$$

i.e. $\alpha^* \in \mathscr{U}_{ad}$ solves $(\mathbb{P})$. $\square$

R e m a r k 3.3. If state constraints were presented, the problem would be more involved. Besides $(\mathbb{B})$, an additional assumption concerning the constraint functionals would have to be satisfied:

$$\begin{cases} \text{for any } \{\alpha_h\},\ \{\lambda_H(\alpha_h)\} \text{ such as in } (\mathbb{B}) \text{ one has}\\ \lim\limits_{h,H\to 0+} \inf E_j^h\big(\alpha_h, \lambda_H(\alpha_h)\big) \geqslant E_j\big(\alpha, \lambda(\alpha)\big), \quad j = 1, \dots, s. \end{cases} \tag{3.14}$$

Moreover, the following density result has to be satisfied:

$$\tilde{\mathscr{U}}_{ad} = \overline{\bigcup \tilde{\mathscr{U}}^h_{ad}}, \tag{3.15}$$

where

$$\begin{aligned}\tilde{\mathscr{U}}_{ad} &= \big\{\alpha \in \mathscr{U}_{ad} \mid E_j\big(\alpha, \lambda(\alpha)\big) \leqslant 0, \quad j = 1, \dots, s\big\},\\ \tilde{\mathscr{U}}^h_{ad} &= \big\{\alpha_h \in \mathscr{U}^h_{ad} \mid E^h_j\big(\alpha_h, \lambda_H(\alpha_h)\big) \leqslant 0, \quad j = 1, \dots, s\big\}\end{aligned}$$

and the closure in (3.15) is taken with respect to the C-norm. If $(\mathbb{B})$, (3.14) and (3.15) were satisfied, then Theorem 3.2 would hold again. Nevertheless, the verification of (3.15) in concrete examples is difficult.

Example 3.1. Define

$$E^h_0\big(\alpha_h, \lambda_H(\alpha_h)\big) = \frac{1}{2}\|\lambda_H(\alpha_h)\|^2_{-1/2,\alpha_h,h},$$

where

$$\|\lambda_H(\alpha_h)\|_{-1/2,\alpha_h,h} \equiv \|v_h\|_{1,\Omega_h}, \tag{3.16}$$

with $v_h \equiv v_h(u_h) \in \mathbb{V}_h(\alpha_h)$ being the solution of

$$\big(\Lambda\varepsilon(v_h), \varepsilon(\psi_h)\big)_{0,\Omega_h} = \big(\Lambda\varepsilon(u_h), \varepsilon(\psi_h)\big)_{0,\Omega_h} - L_{\alpha_h}(\psi_h) \qquad \forall \psi_h \in \mathbb{V}_h(\alpha_h). \tag{3.17}$$

The pair $\big(u_h, \lambda_H(\alpha_h)\big)$ is the solution of $\big(\mathscr{M}(\alpha_h)\big)^H_h$. We shall show that the system $\{E^h_0\}$, $h \to 0+$ satisfies $(\mathbb{B})$. Indeed, from (3.16) and (3.17) it follows that

$$\begin{aligned}2E^h_0\big(\alpha_h, \lambda_H(\alpha_h)\big) &= \big(\Lambda\varepsilon(u_h), \varepsilon(v_h)\big)_{0,\Omega_h} - L_{\alpha_h}(v_h)\\ &= \big(\Lambda\varepsilon(v_h), \varepsilon(u_h)\big)_{0,\Omega_h} - L_{\alpha_h}(v_h)\\ &= \big(\Lambda\varepsilon(u_h), \varepsilon(u_h)\big)_{0,\Omega_h} - L_{\alpha_h}(u_h) - L_{\alpha_h}(v_h).\end{aligned}$$

Arguing in the same way as in Lemma 3.1 one can prove that

$$\begin{cases} L_{\alpha_h}(u_h) \to L_\alpha(u), & h \to 0+;\\ L_{\alpha_h}(v_h) \to L_\alpha(v), & h \to 0+,\end{cases} \tag{3.18}$$

where $u \in K(\alpha)$ solves $\big(\mathscr{P}(\alpha)\big)$ and $v \in \mathbb{V}(\alpha)$ satisfies

$$\big(\Lambda\varepsilon(v), \varepsilon(\psi)\big)_{0,\Omega(\alpha)} = \big(\Lambda\varepsilon(u), \varepsilon(\psi)\big)_{0,\Omega(\alpha)} - L_\alpha(\psi) \quad \forall \psi \in \mathbb{V}(\alpha).$$

Assume for the moment that we have already proved that

$$(3.19)\qquad \big(\Lambda\varepsilon(u_h),\varepsilon(u_h)\big)_{0,\Omega_h} \longrightarrow \big(\Lambda\varepsilon(u),\varepsilon(u)\big)_{0,\Omega(\alpha)},\quad h\to 0+ .$$

From this and (3.18) we have

$$\begin{aligned}\lim_{h,H\to 0+} 2E_0^h\big(\alpha_h,\lambda_H(\alpha_h)\big) &= \big(\Lambda\varepsilon(u),\varepsilon(u)\big)_{0,\Omega(\alpha)} - L_\alpha(u) - L_\alpha(v)\\ &= \big(\Lambda\varepsilon(v),\varepsilon(u)\big)_{0,\Omega(\alpha)} - L_\alpha(v)\\ &= \big(\Lambda\varepsilon(u),\varepsilon(v)\big)_{0,\Omega(\alpha)} - L_\alpha(v)\\ &= \big(\Lambda\varepsilon(v),\varepsilon(v)\big)_{0,\Omega(\alpha)} = |\!|\!| v|\!|\!|^2_{1,\Omega(\alpha)}\\ &= \|\lambda(\alpha)\|^2_{-1/2,\alpha} \equiv 2E_0\big(\alpha,\lambda(\alpha)\big).\end{aligned}$$

In order to verify (3.19) we first prove an auxiliary result. Recall that

$$\begin{aligned} j_\alpha(v) &= \int_a^b g|v_1\circ F_\alpha|\,\mathrm{d}x_1 = \sup_{\mu_1\in\Lambda_1}\big\{-(g\mu_1,v_1)_{0,\alpha}\big\};\\ j^H_{\alpha_h}(v_h) &= \sup_{\mu_1^H\in\Lambda_1^H}\big\{-(g\mu_1^H,v_1^h)_{0,\alpha_h}\big\},\end{aligned}$$

where $v=(v_1,v_2)\in\mathbb{V}(\alpha)$, $v_h=(v_1^h,v_1^h)\in\mathbb{V}_h(\alpha_h)$.

Lemma 3.2. *Let $\alpha_h\rightrightarrows\alpha$ in $[a,b]$, $\alpha_h\in\mathscr{U}^h_{ad}$, $\alpha\in\mathscr{U}_{ad}$ and*

$$\tilde v_h \rightharpoonup v \quad \text{in } H^1(\hat\Omega),\quad h\to 0+,$$

where $\tilde v_h$ stands for the uniform extension of $v_h\in V_h(\alpha_h)$ from $\Omega(\alpha_h)$ onto $\hat\Omega$. Then[1]

$$(3.20)\qquad j^H_{\alpha_h}(v_h)\to j_\alpha(v)\quad \text{as}\quad h,H\to 0+ .$$

Proof. We have

$$j^H_{\alpha_h}(v_h) \leqslant \sup_{\mu_1\in\Lambda_1}\big\{-(g\mu_1,v_h)_{0,\alpha_h}\big\} = \int_a^b g\big|v_h\big(x_1,\alpha_h(x_1)\big)\big|\,\mathrm{d}x_1 \equiv j_{\alpha_h}(v_h).$$

It is an easy exercise to show that $j_{\alpha_h}(v_h)\to j_\alpha(v)$ as $h\to 0+$ (see Lemma 7.10 in [7]) so that

$$(3.21)\qquad \lim_{h,H\to 0+}\sup j^H_{\alpha_h}(v_h)\leqslant j_\alpha(v).$$

[1] Scalar functions v_h, v are used as the arguments of $j^H_{\alpha_h}$, j_α, respectively.

Let $\overline{\mu}_1 \in \Lambda_1$ be such that

$$j_\alpha(v) = -(g\overline{\mu}_1, v)_{0,\alpha}.$$

Then one can find a sequence $\{\overline{\mu}_1^H\}$, $\overline{\mu}_1^H \in \Lambda_1^H$ such that $\overline{\mu}_1^H \to \overline{\mu}_1$ in $L^2(\hat{\Gamma})$, $H \to 0+$. However,

$$j_{\alpha_h}^H(v_h) \geqslant -(g\overline{\mu}_1^H, v_h)_{0,\alpha_h} \longrightarrow -(g\overline{\mu}_1, v)_{0,\alpha} = j_\alpha(v).$$

Hence

$$\lim_{h,H\to 0+} \inf j_{\alpha_h}^H(v_h) \geqslant j_\alpha(v),$$

which together with (3.21) yields (3.20). $\square$

Corollary 3.1. *It follows from Lemma* 3.2 *that*

$$j_{\alpha_h}^H(u_h) \to j_\alpha(u), \quad h, H \to 0+,$$

where u_h *is the same as in Lemma* 3.1.

P r o o f o f (3.19). Let $(u_h, \lambda_H) \in \mathbb{V}_h(\alpha_h) \times \mathbf{\Lambda}_H$ be the solution of $\big(\mathscr{M}(\alpha_h)\big)_h^H$. Then the first component $u_h \in K_h^H(\alpha_h)$ solves the problem $\big(\mathscr{P}(\alpha_h)\big)_h^H$ which can be equivalently expressed as follows:

$$\text{(3.22)} \qquad \begin{cases} \text{Find } u_h \in K_h^H(\alpha_h) \text{ such that} \\ \mathscr{J}_h^H(u_h) \leqslant \mathscr{J}_h^H(v_h) \quad \forall v_h \in K_h^H(\alpha_h), \end{cases}$$

where

$$\mathscr{J}_h^H(v_h) \equiv \mathscr{J}_{\alpha_h}(v_h) + j_{\alpha_h}^H(v_h).$$

Since

$$\big(\Lambda\varepsilon(u_h), \varepsilon(u_h)\big)_{0,\Omega_h} = 2\mathscr{J}_h^H(u_h) + 2L_{\alpha_h}(u_h) - 2j_{\alpha_h}^H(u_h),$$

then taking into account (3.18) and Corollary 3.1 we conclude that (3.19) holds if and only if

$$\text{(3.23)} \qquad \mathscr{J}_h^H(u_h) \to \mathscr{J}_\alpha(u), \quad h, H \to 0+.$$

Let us prove (3.23). The inequality

$$\text{(3.24)} \qquad \lim_{h,H\to 0+} \inf \mathscr{J}_h^H(u_h) \geqslant \mathscr{J}_\alpha(u)$$

is obvious by virtue of Lemma 3.1, (3.18) and Corollary 3.1. Arguing in the same way as in Step (*iii*) in the proof of Lemma 3.1 one can find a sequence $\{\overline{v}_h\}$, $\overline{v}_h \in K_h(\alpha_h)$ such that

$$\|\overline{v}_h - \tilde{u}\|_{1,\Omega(\alpha_h)} \to 0, \quad h \to 0+, \tag{3.25}$$

where $\tilde{u}$ is the uniform extension of u from $\Omega(\alpha)$ onto $\hat{\Omega}$. Since at the same time $\overline{v}_h \in K_h^H(\alpha_h)$, it follows from (3.22) that

$$\mathscr{I}_h^H(u_h) \leqslant \mathscr{I}_h^H(\overline{v}_h). \tag{3.26}$$

It is again very easy to verify that

$$\mathscr{I}_h^H(\overline{v}_h) \to \mathscr{I}_\alpha(u),$$

making use of (3.25) and the fact that $\alpha_h \rightrightarrows \alpha$ in $[a,b]$, $h \to 0+$ (see [7]). From this and (3.26) we obtain that

$$\lim_{h,H\to 0+} \sup \mathscr{I}_h^H(u_h) \leqslant \mathscr{I}_\alpha(u),$$

which together with (3.24) proves (3.19). □

R e m a r k 3.4. It follows from (3.16) that the equivalent expression of E_0^h from the previous example reads as follows:

$$E_0^h\bigl(\alpha_h, \lambda_H(\alpha_h)\bigr) = \frac{1}{2}(g\lambda_1^H, v_1^h)_{0,\alpha_h} + \frac{1}{2}(\lambda_2^H, v_2^h)_{0,\alpha_h},$$

i.e. only the boundary data are necessary when evaluating E_0^h. This fact can be used in the practical realization of $(\mathbb{P})_h^H$ (see also Remark 2.2).

R e m a r k 3.5. Until now functionals defined on $\mathscr{U}_{ad} \times \mathbf{\Lambda}$, i.e. depending on the design variable α and the contact stress λ, were considered. It is readily seen that the previous analysis can be extended to the more general case, namely when the functionals in addition to α, λ depend also on the solution $u(\alpha)$ itself. In this case the mixed variational formulation $\bigl(\mathscr{M}(\alpha)\bigr)$ and its discretization $\bigl(\mathscr{M}(\alpha_h)\bigr)_h^H$ will be used for the numerical realization of the Signorini problem. After an appropriate modification of assumptions, all our results remain valid.

In this section we will study the differentiability of the mapping $\alpha_h \mapsto u_h(\alpha_h)$, $\lambda_H(\alpha_h)$, where the couple $\big(u_h(\alpha_h), \lambda_H(\alpha_h)\big)$ is the solution of $\big(\mathscr{M}(\alpha_h)\big)_h^H$. We shall show that this mapping is *directionally differentiable* and the corresponding derivatives are given by another quadratic programming problem. For the sake of simplicity of the presentation we restrict ourselves to the frictionless case, i.e. $g \equiv 0$ on $\hat{\Gamma}$ and consequently, only one set defining the Lagrange multipliers is present, namely Λ_2^H. The sensitivity analysis will be done for the matrix formulation of $\big(\mathscr{M}(\alpha_h)\big)_h^H$.

Let $\mathbb{V}_h(\alpha_h)$, Λ_2^H and $\mathscr{U}_{ad}^h$ be isometrically isomorphic to $\mathbb{R}^n$, $\mathbb{R}_+^d$ and $\mathscr{U}$, respectively. The elements $\vec{\alpha} \in \mathscr{U} \subseteq \mathbb{R}^{D+1}$ will be called *discrete design variables.* Then using the standard approach, problem $\big(\mathscr{M}(\alpha_h)\big)_h^H$ takes the following algebraic form:

$$
\big(\vec{\mathscr{M}}(\vec{\alpha})\big) \qquad \begin{cases} \qquad \text{Find } \big(\vec{x}(\vec{\alpha}), \vec{\lambda}(\vec{\alpha})\big) \in \mathbb{R}^n \times \mathbb{R}_+^d \quad \text{such that} \\ \mathbb{A}(\vec{\alpha})\vec{x}(\vec{\alpha}) = \vec{F}(\vec{\alpha}) + \mathbb{B}^T(\vec{\alpha})\vec{\lambda}(\vec{\alpha}) \\ \big(\mathbb{B}(\vec{\alpha})\big(\vec{x}(\vec{\alpha}) + \underset{\sim}{\alpha}\big), \vec{\mu} - \vec{\lambda}(\vec{\alpha})\big) \geqslant 0 \quad \forall \vec{\mu} \in \mathbb{R}_+^d. \end{cases}
$$

Here $\mathbb{A}(\vec{\alpha})$ is the stiffness matrix of the problem, $\vec{F}(\vec{\alpha})$ is the right hand side arising from the discretization of the applied forces and $\mathbb{B}(\vec{\alpha})$ is the so-called kinematic transformation matrix characterizing the unilateral contact condition along $\hat{\Gamma}$. All data depend on the discrete design variable $\vec{\alpha}$. Taking into account our special geometry of $\Omega(\alpha_h)$, the components of the vector $\vec{\alpha} \in \mathscr{U}$ are given by the values of $\alpha_h \in \mathscr{U}_{ad}^h$ at the nodal points, i.e.

$$
\vec{\alpha} = (\alpha_0, \ldots, \alpha_D), \quad \text{where} \quad \alpha_i = \alpha_h(x_1^i), \quad i = 0, \ldots, D,
$$

while the components of $\underset{\sim}{\alpha} \in \mathbb{R}^n$ used in $\big(\vec{\mathscr{M}}(\vec{\alpha})\big)_3$ are copies of the components of $\vec{\alpha} \in \mathbb{R}^{D+1}$ corresponding to the contact nodes and completed by zeros at the remaining positions. The nodal displacement field $\vec{x} \in \mathbb{R}^n$ can be arranged in the following way: $\vec{x} = (\vec{x}_0, \vec{x}_c)$, where $\vec{x}_c$ is the subvector the components of which are the x_2-coordinates of the displacement field $\vec{x}$ at the contact nodes. Since the unilateral conditions concern the subvector $\vec{x}_c$ only, we will suppose that the kinematic matrix $\mathbb{B}(\vec{\alpha}) \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^d)$ is such that

$$
\mathbb{B}(\vec{\alpha})\vec{x} = \vec{0} \quad \text{for any} \quad \vec{x} = (\vec{x}_0, \vec{x}_c) \quad \text{with} \quad \vec{x}_c = \vec{0}.
$$

Moreover, we will suppose that the stability condition

$$
\mathbb{B}^T(\vec{\alpha})\vec{\mu} = \vec{0} \Longrightarrow \vec{\mu} = \vec{0} \tag{4.1}
$$

is satisfied for any $\vec{\alpha} \in \mathscr{U}$.

Finally, we will suppose that the mappings

$$\vec{\alpha} \longmapsto \mathbb{A}(\vec{\alpha}),\ \vec{F}(\vec{\alpha}),\ \mathbb{B}(\vec{\alpha}) \tag{4.2}$$

are *once continuously differentiable* in an open set $\tilde{\mathscr{U}} \supset \mathscr{U}$ and the matrix $\mathbb{A}(\vec{\alpha})$ is *uniformly positive definite with respect to* $\vec{\alpha} \in \mathscr{U}$.

We start with

Lemma 4.1. *The mappings*

$$\begin{aligned} &\vec{\alpha} \mapsto \vec{x}(\vec{\alpha}) \\ &\vec{\alpha} \mapsto \vec{\lambda}(\vec{\alpha}), \quad \vec{\alpha} \in \mathscr{U}, \end{aligned}$$

where $\big(\vec{x}(\vec{\alpha}), \vec{\lambda}(\vec{\alpha})\big)$ *solves* $\big(\vec{\mathscr{M}}(\vec{\alpha})\big)$ *are Lipschitz continuous.*

Proof. Due to (4.1), problem $\big(\vec{\mathscr{M}}(\vec{\alpha})\big)$ has a unique solution $\big(\vec{x}(\vec{\alpha}), \vec{\lambda}(\vec{\alpha})\big)$ for any $\vec{\alpha} \in \mathscr{U}$. First we prove the Lipschitz continuity of the mapping $\vec{\alpha} \mapsto \vec{\lambda}(\vec{\alpha})$. From $\big(\vec{\mathscr{M}}(\vec{\alpha})\big)_2$ we have

$$\vec{x}(\vec{\alpha}) = \mathbb{A}^{-1}(\vec{\alpha})\big(\vec{F}(\vec{\alpha}) + \mathbb{B}^T(\vec{\alpha})\vec{\lambda}(\vec{\alpha})\big). \tag{4.3}$$

Substituting (4.3) into $\big(\vec{\mathscr{M}}(\vec{\alpha})\big)_3$ we see that $\vec{\lambda}(\vec{\alpha}) \in \mathbb{R}^d_+$ solves the variational inequality

$$\begin{aligned} \big(\mathbb{C}(\vec{\alpha})\vec{\lambda}(\vec{\alpha}), \vec{\mu} - \vec{\lambda}(\vec{\alpha})\big) \geqslant &- \big(\mathbb{B}(\vec{\alpha})\mathbb{A}^{-1}(\vec{\alpha})\vec{F}(\vec{\alpha}), \vec{\mu} - \vec{\lambda}(\vec{\alpha})\big) \\ &- \big(\mathbb{B}(\vec{\alpha})\underset{\sim}{\alpha}, \vec{\mu} - \vec{\lambda}(\vec{\alpha})\big) \quad \forall \vec{\mu} \in \mathbb{R}^d_+, \end{aligned} \tag{4.4}$$

where $\mathbb{C}(\vec{\alpha}) \equiv \mathbb{B}(\vec{\alpha})\mathbb{A}^{-1}(\vec{\alpha})\mathbb{B}^T(\vec{\alpha})$. The inequality (4.4) is nothing else than the algebraic representation of the reciprocal variational formulation $\big(\mathscr{R}(\alpha_h)\big)_H$. The fact that the mapping $\vec{\alpha} \mapsto \vec{\lambda}(\vec{\alpha})$ is Lipschitz continuous now easily follows from (4.4) by virtue of (4.2) and the positive definiteness of $\mathbb{C}(\vec{\alpha})$. The Lipschitz continuity of $\vec{\alpha} \mapsto \vec{x}(\vec{\alpha})$ now follows from (4.3). $\square$

Let $\vec{\beta} \in \mathbb{R}^{D+1}$ be a fixed direction and let $\big(\vec{x}(\vec{\alpha} + t\vec{\beta}), \vec{\lambda}(\vec{\alpha} + t\vec{\beta})\big)$, $t \to 0+$ be a solution to $\big(\vec{\mathscr{M}}(\vec{\alpha} + t\vec{\beta})\big)$. It follows from Lemma 4.1 that the finite differences

$$\left\{ \frac{\vec{x}(\vec{\alpha} + t\vec{\beta}) - \vec{x}(\vec{\alpha})}{t} \right\}, \quad \left\{ \frac{\vec{\lambda}(\vec{\alpha} + t\vec{\beta}) - \vec{\lambda}(\vec{\alpha})}{t} \right\}$$

are bounded for $t \to 0+$. Thus there exist a sequence $\{t_n\}$ and elements $\dot{\vec{x}} \in \mathbb{R}^n$, $\dot{\vec{\lambda}} \in \mathbb{R}^d_+$ such that

$$\text{(4.5)} \qquad \begin{cases} \dfrac{\vec{x}(\vec{\alpha}+t_n\vec{\beta})-\vec{x}(\vec{\alpha})}{t_n} \longrightarrow \dot{\vec{x}}, \\ \dfrac{\vec{\lambda}(\vec{\alpha}+t_n\vec{\beta})-\vec{\lambda}(\vec{\alpha})}{t_n} \longrightarrow \dot{\vec{\lambda}}, \quad \text{as} \quad n \to \infty. \end{cases}$$

Next we shall show that the elements $\dot{\vec{x}}$, $\dot{\vec{\lambda}}$ are uniquely determined and do not depend on the specific choice of $\{t_n\}$. They will be called the *directional derivatives* of $\vec{x}$, $\vec{\lambda}$, respectively at the point $\vec{\alpha}$ and the direction $\vec{\beta}$.

Writing down problems $\big(\vec{\mathscr{M}}(\vec{\alpha})\big)$ and $\big(\vec{\mathscr{M}}(\vec{\alpha}+t_n\vec{\beta})\big)$, subtracting them and dividing by $t_n \to 0+$ we arrive at the relation

$$\text{(4.6)} \qquad \dot{\mathbb{A}}(\vec{\alpha})\vec{x}(\vec{\alpha}) + \mathbb{A}(\vec{\alpha})\dot{\vec{x}}(\vec{\alpha}) = \dot{\vec{F}}(\vec{\alpha}) + \dot{\mathbb{B}}^T(\vec{\alpha})\vec{\lambda}(\vec{\alpha}) + \mathbb{B}^T(\vec{\alpha})\dot{\vec{\lambda}}.$$

It follows from (4.2) that

$$\dot{\mathbb{A}}(\vec{\alpha}) = \lim_{t_n \to 0+} \frac{\mathbb{A}(\vec{\alpha}+t_n\vec{\beta}) - \mathbb{A}(\vec{\alpha})}{t_n} = \mathbb{A}'(\vec{\alpha},\vec{\beta}) = \big(\nabla_\alpha \mathbb{A}(\vec{\alpha}), \vec{\beta}\big)$$

is the directional derivative of $\mathbb{A}$ at the point $\vec{\alpha}$ and the direction $\vec{\beta}$. Here the symbol ∇_α stands for the gradient of $\mathbb{A}$ with respect to $\vec{\alpha}$. The symbols $\dot{\vec{F}}(\vec{\alpha})$, $\dot{\mathbb{B}}(\vec{\alpha})$ have a similar meaning and can be computed in a similar way. Below we prove that $\dot{\vec{x}}$ belongs to a certain convex set.

The solution $\vec{x}(\vec{\alpha})$ satisfies d linear inequality constraints

$$f_i(\vec{\alpha}) \geqslant 0 \quad \forall i = 1, \dots, d,$$

where

$$f_i(\vec{\alpha}) \equiv b_{ij}(\vec{\alpha})\big(x_j(\vec{\alpha}) + \underset{\sim}{\alpha}_j\big)$$

with $b_{ij}(\vec{\alpha})$, $i = 1, \dots, d$; $j = 1, \dots, n$ being the elements of $\mathbb{B}(\vec{\alpha})$ and $\underset{\sim}{\alpha}_j$ the j-th component of $\vec{\underset{\sim}{\alpha}}$.

The index set $I = \{1, \dots, d\}$ will be splitted into 3 disjoint subsets as follows:

$$\begin{aligned} I_+(\vec{\alpha}) &= \big\{i \in I \mid f_i(\vec{\alpha}) > 0\big\}, \\ I_{0,+}(\vec{\alpha}) &= \big\{i \in I \mid f_i(\vec{\alpha}) = 0 \quad \& \quad \lambda_i(\vec{\alpha}) > 0\big\}, \\ I_{0,0}(\vec{\alpha}) &= \big\{i \in I \mid f_i(\vec{\alpha}) = 0 \quad \& \quad \lambda_i(\vec{\alpha}) = 0\big\}. \end{aligned}$$

Let the i-th constraint be *non-active*, i.e. $i \in I_+(\vec{\alpha})$. Then it remains non-active for small changes of t due to the continuity of the mapping $\vec{\alpha} \mapsto \vec{x}(\vec{\alpha})$. Then $\lambda_i(\vec{\alpha}+t\vec{\beta}) = 0$ for any $t \geqslant 0$ sufficiently small and consequently $\dot{\lambda}_i(\vec{\alpha}) = 0$ for any $i \in I_+(\vec{\alpha})$.

Let the i-th constraint be *strongly active*, i.e. $i \in I_{0,+}(\vec{\alpha})$. Then it remains strongly active for small perturbations of $t > 0$ due to the continuity of the mapping $\vec{\alpha} \mapsto \vec{\lambda}(\vec{\alpha})$. Thus

$$f_i(\vec{\alpha} + t\vec{\beta}) = 0$$

for $t > 0$ sufficiently small so that

$$\dot{f}_i(\vec{\alpha}) = 0 \quad \forall i \in I_{0,+}(\vec{\alpha}) \tag{4.7}$$

or

$$b_{ij}(\vec{\alpha})\dot{x}_j(\vec{\alpha}) = -\dot{b}_{ij}(\vec{\alpha})\big(x_j(\vec{\alpha}) + \underset{\sim}{\alpha}_j\big) - b_{ij}(\vec{\alpha})\underset{\sim}{\beta}_j \equiv d_i \tag{4.8}$$

holds for any $i \in I_{0,+}(\vec{\alpha})$. Here $\underset{\sim}{\vec{\beta}} = (\underset{\sim}{\beta}_1, \ldots, \underset{\sim}{\beta}_n) \in \mathbb{R}^n$ is the extension of $\vec{\beta} \in \mathbb{R}^{D+1}$ constructed in the same way as $\underset{\sim}{\alpha}$ from $\vec{\alpha}$. In other words, at the points where the constraint is strongly active, the equality constraint (4.8) is satisfied.

Finally, let the i-th constraint be *semi-active*, i.e. $i \in I_{0,0}(\vec{\alpha})$. Since at the same time $f_i(\vec{\alpha} + t\vec{\beta}) \geqslant 0$ for any $t \geqslant 0$, one has

$$\dot{f}_i(\vec{\alpha}) \geqslant 0 \Longleftrightarrow b_{ij}(\vec{\alpha})\dot{x}_j(\vec{\alpha}) \geqslant d_i \quad \forall i \in I_{0,0}(\vec{\alpha}), \tag{4.9}$$

i.e. the linear inequality constraint is satisfied at any point where the semi-active constraint is realized.

From (4.8) and (4.9) one has

Lemma 4.2. *The element $\dot{\vec{x}}$ belongs to the convex set $\mathscr{K}(\vec{\alpha}, \vec{\beta})$, where*

$$\mathscr{K}(\vec{\alpha}, \vec{\beta}) = \big\{\vec{z} \in \mathbb{R}^n \mid b_{ij}(\vec{\alpha})z_j = d_i \ \forall i \in I_{0,+}(\vec{\alpha}),\ b_{ij}(\vec{\alpha})z_j \geqslant d_i \ \forall i \in I_{0,0}(\vec{\alpha})\big\}.$$

Moreover, we have

$$\dot{\lambda}_i(\vec{\alpha}) \geqslant 0 \quad \forall i \in I_{0,0}(\vec{\alpha}), \tag{4.10}$$

$$\big(b_{ij}(\vec{\alpha})\dot{x}_j(\vec{\alpha}) - d_i\big)\dot{\lambda}_i(\alpha) = 0 \quad \forall i \in I \ (\textit{no sum in } i). \tag{4.11}$$

Proof. It remains to verify (4.10) and (4.11). For any $i \in I_{0,0}(\vec{\alpha})$ we have

$$\dot{\lambda}_i(\vec{\alpha}) = \lim_{t_n \to 0+} \frac{\lambda_i(\vec{\alpha} + t_n\vec{\beta}) - \lambda_i(\vec{\alpha})}{t_n} = \lim_{t_n \to 0+} \frac{\lambda_i(\vec{\alpha} + t_n\vec{\beta})}{t_n} \geqslant 0,$$

since $\vec{\lambda}(\vec{\alpha}) \in \mathbb{R}^d_+$ for any $\vec{\alpha} \in \mathscr{U}$. Let us prove (4.11). If $i \in I_+(\vec{\alpha})$, then $\dot{\lambda}_i(\vec{\alpha}) = 0$. If $i \in I_{0,+}(\vec{\alpha})$ then the equality constraint (4.8) holds. Finally, let $i \in I_{0,0}(\vec{\alpha})$ and $\dot{\lambda}_i(\vec{\alpha}) > 0$. Then $\lambda_i(\vec{\alpha} + t_n\vec{\beta}) > 0$ for n sufficiently large so that $f_i(\vec{\alpha} + t_n\vec{\beta}) = 0$. Thus

$$b_{ij}(\vec{\alpha})\dot{x}_j(\vec{\alpha}) = d_i$$

and (4.11) is verified. $\square$

Denote by

$$\mathscr{H}_{\vec{\alpha}}(\vec{z}) = \frac{1}{2}\big(\vec{z}, \mathbb{A}(\vec{\alpha})\vec{z}\big) - \big(\dot{\vec{F}}(\vec{\alpha}) + \dot{\mathbb{B}}^T(\vec{\alpha})\vec{\lambda}(\vec{\alpha}) - \dot{\mathbb{A}}(\vec{\alpha})\vec{x}(\vec{\alpha}), \vec{z}\big)$$

the quadratic functional. In what follows we shall prove that the element $\dot{\vec{x}}$ is a minimizer of $\mathscr{H}_{\vec{\alpha}}$ over $\mathscr{K}(\vec{\alpha}, \vec{\beta})$ and $\dot{\vec{\lambda}}$ is the corresponding Lagrange multiplier. First of all, the quadratic programming problem

$$\left\{\begin{aligned} &\text{Find } \vec{s} \in \mathscr{K}(\vec{\alpha}, \vec{\beta}) \quad \text{such that} \\ &\mathscr{H}_{\vec{\alpha}}(\vec{s}) \leqslant \mathscr{H}_{\vec{\alpha}}(\vec{z}) \quad \forall \vec{z} \in \mathscr{K}(\vec{\alpha}, \vec{\beta}) \end{aligned}\right. \tag{4.12}$$

has a unique solution $\vec{s}$ which due to (4.2) does not depend on a particular choice of $\{t_n\}$. Using the Lagrange multiplier technique, (4.12) can be formulated as a saddle-point problem:

$$\left\{\begin{aligned} &\text{Find } (\vec{s}, \vec{\kappa}) \equiv (\vec{s}, \vec{\kappa}_1, \vec{\kappa}_2) \in \mathbb{R}^n \times \mathbb{R}^{q_1} \times \mathbb{R}^{q_2}_+ \quad \text{such that} \\ &\mathscr{E}_{\vec{\alpha}}(\vec{s}, \vec{\mu}) \leqslant \mathscr{E}_{\vec{\alpha}}(\vec{s}, \vec{\kappa}) \leqslant \mathscr{E}_{\vec{\alpha}}(\vec{z}, \vec{\kappa}) \quad \forall (\vec{z}, \vec{\mu}) \in \mathbb{R}^n \times \mathbb{R}^{q_1} \times \mathbb{R}^{q_2}_+, \end{aligned}\right. \tag{4.13}$$

where

$$\mathscr{E}_{\vec{\alpha}}(\vec{z}, \vec{\mu}) = \mathscr{H}_{\vec{\alpha}}(\vec{z}) - \big(\mathbb{B}(\vec{\alpha})\vec{z} - \vec{d}, \underset{\sim}{\vec{\mu}}\big),$$

$$q_1 = \operatorname{card} I_{0,+}(\vec{\alpha}), \quad q_2 = \operatorname{card} I_{0,0}(\vec{\alpha}),$$
$$\vec{d} = (d_1, \dots, d_d)$$

and $\underset{\sim}{\vec{\mu}} \in \mathbb{R}^d$ is the extension of $\vec{\mu} \in \mathbb{R}^{q_1} \times \mathbb{R}^{q_2}_+$ by zeros at the components corresponding to $i \in I_+(\vec{\alpha})$. The equivalent form of (4.13) is

$$\left\{\begin{aligned} &\text{Find } (\vec{s}, \vec{\kappa}) \in \mathbb{R}^n \times \mathbb{R}^{q_1} \times \mathbb{R}^{q_2}_+ \quad \text{such that} \\ &\mathbb{A}(\vec{\alpha})\vec{s} = \dot{\vec{F}}(\vec{\alpha}) + \dot{\mathbb{B}}^T(\vec{\alpha})\vec{\lambda}(\vec{\alpha}) - \dot{\mathbb{A}}(\vec{\alpha})\vec{x}(\vec{\alpha}) + \mathbb{B}^T(\vec{\alpha})\underset{\sim}{\vec{\kappa}} \\ &\big(\mathbb{B}(\vec{\alpha})\vec{s} - \vec{d}; \underset{\sim}{\vec{\mu}} - \underset{\sim}{\vec{\kappa}}\big) \geqslant 0 \quad \forall \vec{\mu} \in \mathbb{R}^{q_1} \times \mathbb{R}^{q_2}_+. \end{aligned}\right. \tag{4.14}$$

Notice that (4.14) has a unique solution $(\vec{s}, \vec{\kappa})$ because of (4.1). Moreover, this solution again does not depend on a particular choise of $\{t_n\}$. From (4.11) we have

$$(4.15) \qquad \bigl(b_{ij}(\vec{\alpha})\dot{x}_j(\vec{\alpha}) - d_i\bigr)(\mu_i - \dot{\lambda}_i) \geqslant 0$$

for any $\vec{\mu} \in \mathbb{R}^d$ such that $\mu_i = 0$ if $i \in I_+(\vec{\alpha})$, $\mu_i \in \mathbb{R}^1$ if $i \in I_{0,+}(\vec{\alpha})$ and $\mu_i \geqslant 0$ if $i \in I_{0,0}(\vec{\alpha})$. Comparing (4.6), (4.15) with (4.14) and taking into account that $\dot{\lambda}_i = 0$ for any $i \in I_+(\vec{\alpha})$ we see that $\dot{\vec{x}} = \vec{s}$, $\dot{\vec{\lambda}} = \underset{\sim}{\vec{\kappa}}$.

Summarizing the previous analysis we obtain

Theorem 4.1. *The mapping*

$$(4.16) \qquad \vec{\alpha} \mapsto \bigl(\vec{x}(\vec{\alpha}), \vec{\lambda}(\vec{\alpha})\bigr),$$

where $\bigl(\vec{x}(\vec{\alpha}), \vec{\lambda}(\vec{\alpha})\bigr)$ *is the unique solution of* $\bigl(\vec{\mathscr{M}}(\vec{\alpha})\bigr)$, *is directionally differentiable at any point* $\vec{\alpha} \in \mathscr{U}$ *and any direction* $\vec{\beta} \in \mathbb{R}^{D+1}$. *The directional derivative*

$$\vec{x}' \equiv \vec{x}'(\vec{\alpha}, \vec{\beta}) = \lim_{t \to 0+} \frac{\vec{x}(\vec{\alpha} + t\vec{\beta}) - \vec{x}(\vec{\alpha})}{t}$$

is the solution of the quadratic programming problem (4.12), *while the directional derivative* $\vec{\lambda}' \equiv \vec{\lambda}'(\vec{\alpha}, \vec{\beta})$ *is the vector of the corresponding Lagrange multipliers.*

R e m a r k 4.1. Using the duality approach in (4.13) one can derive the variational formulation for the Lagrange multiplier $\vec{\kappa}$. Such formulation is useful when only the derivative $\dot{\vec{\lambda}}$ is needed.

Since the mapping $\vec{\alpha} \mapsto \bigl(\vec{x}(\vec{\alpha}), \vec{\lambda}(\vec{\alpha})\bigr)$ is not continuously differentiable, in general, one cannot expect the differentiability of cost functionals depending on $\vec{x}(\vec{\alpha})$, $\vec{\lambda}(\vec{\alpha})$ and considered as the function of the discrete design variable $\vec{\alpha}$. In some special cases, however, the cost functional is *continuously differentiable* regardless of the fact that the mapping $\vec{\alpha} \mapsto \bigl(\vec{x}(\vec{\alpha}), \vec{\lambda}(\vec{\alpha})\bigr)$ *is not.* We will illustrate this phenomenon for the cost functional E_0^h introduced in Example 3.1. Again we restrict ourselves to the frictionless case only, i.e. $g \equiv 0$ on $\hat{\Gamma}$. In this case (since $\lambda_1^H \equiv 0$) one has

$$E_0^h\bigl(\alpha_h, \lambda_2^H(\alpha_h)\bigr) = \frac{1}{2}\|\lambda_2^H(\alpha_h)\|^2_{-1/2,\alpha_h,h} \equiv \frac{1}{2}|\!|\!|v_h|\!|\!|^2_{1,\Omega_h},$$

where $v_h \equiv v_h(\alpha_h) \in \mathbb{V}_h(\alpha_h)$ is the unique solution of

$$\bigl(\Lambda\varepsilon(v_h), \varepsilon(\psi_h)\bigr)_{0,\Omega_h} = \bigl(\lambda_2^H(\alpha_h), \psi_h\bigr)_{0,\alpha_h} \quad \forall \psi_h \in \mathbb{V}_h(\alpha_h).$$

From this and the definition of E_0^h we see that

$$E_0^h\big(\alpha_h, \lambda_2^H(\alpha_h)\big) = \frac{1}{2}\big(\lambda_2^H(\alpha_h), v_h(\alpha_h)\big)_{0,\alpha_h}. \tag{4.17}$$

The algebraic representation of (4.17) is given by

$$\mathscr{E}\big(\vec{\alpha}, \vec{\lambda}(\vec{\alpha})\big) = \frac{1}{2}\big(\vec{\lambda}(\vec{\alpha}), \mathbb{B}(\alpha)\vec{v}(\vec{\alpha})\big), \tag{4.18}$$

where $\vec{\lambda}(\vec{\alpha}) \in \mathbb{R}_+^d$ solves (4.4), $\mathbb{B}(\alpha)$ is the kinematic transformation matrix and $\vec{v}(\vec{\alpha}) \in \mathbb{R}^n$ is the solution of

$$\mathbb{A}(\vec{\alpha})\vec{v}(\vec{\alpha}) = \mathbb{B}^T(\alpha)\vec{\lambda}(\vec{\alpha}).$$

Using this in (4.18) we obtain that

$$\mathscr{E}\big(\vec{\alpha}, \vec{\lambda}(\vec{\alpha})\big) = \frac{1}{2}\big(\mathbb{C}(\vec{\alpha})\vec{\lambda}(\vec{\alpha}), \vec{\lambda}(\vec{\alpha})\big),$$

where $\mathbb{C}(\vec{\alpha}) = \mathbb{B}(\vec{\alpha})\mathbb{A}^{-1}(\vec{\alpha})\mathbb{B}^T(\vec{\alpha})$. The vector $\vec{\lambda}(\vec{\alpha})$ is the unique minimizer of the quadratic functional

$$\frac{1}{2}\big(\mathbb{C}(\vec{\alpha})\vec{\mu}, \vec{\mu}\big) - \big(\vec{Q}(\vec{\alpha}), \vec{\mu}\big)$$

over $\mathbb{R}_+^d$, where

$$\vec{Q}(\vec{\alpha}) \equiv -\mathbb{B}(\vec{\alpha})\mathbb{A}^{-1}(\vec{\alpha})\vec{F}(\vec{\alpha}) - \mathbb{B}(\vec{\alpha})\underset{\sim}{\alpha}$$

(see (4.4)). Thus

$$\begin{aligned}
\mathscr{E}\big(\vec{\alpha}, \vec{\lambda}(\vec{\alpha})\big) &= -\min_{\vec{\mu}\in\mathbb{R}_+^d}\Big\{\frac{1}{2}\big(\mathbb{C}(\vec{\alpha})\vec{\mu}, \vec{\mu}\big) - \big(\vec{Q}(\vec{\alpha}), \vec{\mu}\big)\Big\} \\
&= -\min_{\vec{\mu}\in\mathbb{R}^d}\sup_{\vec{\omega}\in\mathbb{R}_+^d}\Big\{\frac{1}{2}\big(\mathbb{C}(\vec{\alpha})\vec{\mu}, \vec{\mu}\big) - \big(\vec{Q}(\vec{\alpha}), \vec{\mu}\big) - (\vec{\mu}, \vec{\omega})\Big\}.
\end{aligned}$$

Using the classical results on the differentiability of min max functions (see [2]) we have

$$\begin{aligned}
\mathscr{E}'(\vec{\alpha}, \vec{\beta}) &= \lim_{t\to 0+}\frac{\mathscr{E}(\vec{\alpha} + t\vec{\beta}) - \mathscr{E}(\vec{\alpha})}{t} \\
&= -\Big\{\frac{1}{2}\big(\mathbb{C}'(\vec{\alpha})\vec{\lambda}(\vec{\alpha}), \vec{\lambda}(\vec{\alpha})\big) - \big(\vec{Q}'(\vec{\alpha}), \vec{\lambda}(\vec{\alpha})\big)\Big\}
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{C}'(\vec{\alpha}) &\equiv \mathbb{C}'(\vec{\alpha}, \vec{\beta}) = \big(\nabla_\alpha \mathbb{C}(\vec{\alpha}), \vec{\beta}\big), \\
\vec{Q}'(\vec{\alpha}) &= \vec{Q}'(\vec{\alpha}, \vec{\beta}) = \big(\nabla_\alpha \vec{Q}(\vec{\alpha}), \vec{\beta}\big)
\end{aligned}$$

are the directional derivatives of $\mathbb{C}$ and $\vec{Q}$. As the mappings $\vec{\alpha} \mapsto \mathbb{C}(\vec{\alpha})$, $\vec{\alpha} \mapsto \vec{Q}(\vec{\alpha})$ are continuously differentiable as follows from (4.2), the cost functional $\mathscr{E}$ is of the class C^1.

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