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NONLINEAR MODELS OF SUSPENSION BRIDGES: DISCUSSION OF THE RESULTS*

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Abstract. In this paper we present several nonlinear models of suspension bridges; most of them have been introduced by Lazer and McKenna. We discuss some results which were obtained by the authors and other mathematicians for the boundary value problems and initial boundary value problems. Our intention is to point out the character of these results and to show which mathematical methods were used to prove them instead of giving precise proofs and statements.

Keywords: beam equation, system of beam wave equation, initial boundary value problem, bifurcation, Fučík spectrum

MSC 2000: 35B10, 35B40, 34B15, 74K10

1. Introduction and historical review

Although it is probably not possible to find when the very first bridge was built, it was surely deep in the human history. These structures have fascinated people since that time. Bridges present technology and knowledge of the era in which they were created. They became kind of "engineering art".

The typical bridge is supported by pilots. It depends on the material and the type of bridge what distance could be between two pilots but, in general, the following statement must be satisfied: "The longer bridge we would like to build the more pilots we will need."

Suspension bridges were the answer to the question how to cross long distances between two points while putting the pilots just close to the ends of the road-bed. They offer an elegant solution to going over the bays of sea or wild deep rivers, etc.

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If we compare the lengths of recent suspension bridges with other types of bridges we find out that their main spans[1] are more than twice longer (see [14]).

It is really impressive that already in 1826 Menai Suspension Bridge, the first major suspension bridge, was completed. It was designed by Thomas Telford, self-taught Scottish engineer. The bridge has center span 176 meters long and it is maybe interesting to mention that it was built twenty years before the steam engine was fully developed. From that time this type of bridges holds leading in the length of the main span.

The revolution in technology and science at the break of the nineteenth and twentieth centuries let arise bridges with lengths that none could have imagined before (for example: Brooklyn Bridge, 1883, 486 meters, or Golden Gate, 1937, 1,280 meters). However, all kinds of bridges were constructed more or less just from the engineering point of view. The only way how to model the behaviour of the designed bridge were the simulations on a miniature model and application of the linear mechanical theory.

Several collapses of suspension bridges in the past—let us mention the collapse of the Tacoma Narrows Bridge as the most famous one—brought the problem to the interest of theoretical scientists. It turned out that the methods of the designing and constructing should be revisited. It was pointed out that the knowledge of the global behaviour and of the dynamics of such monumental structures is necessary.

Let us briefly recall what happened with the first Tacoma Narrows Bridge. July 1, 1940, a suspended road-bed 854 meters long crossed the river Narrows. The elegant deck was slim and flexible. Moreover, the deck was made from plate girders instead of strengthening trusses. They were catching the wind from the side, unlike an open truss construction which would let it blow through. On November 7, 1940, the bridge went to devastating vertical, torsional and side-way oscillatory motions. After some time, the center span was torn and felt down to the river (see Fig. 1).

This accident warned of one important aspect which arises in constructing bridges (and other structures as well). By a model, we can catch only the main features of the system, but we do not know much about the "inner" dynamics. This became fatal to Tacoma Narrows Bridge.

A new bridge was built across the river Narrows. It is heavy, rigid and resists wind forces much better than the old Tacoma Narrows Bridge. On the other hand, the construction of the old bridge cost $6 million. The new one cost $15 million. It is obvious that the research could spare much money and material. Moreover, this could be more important for small bridges which are build in towns. They have to be often light and flexible because the price is crucial.

[1] Main span is the distance between two principal supports of the bridge.

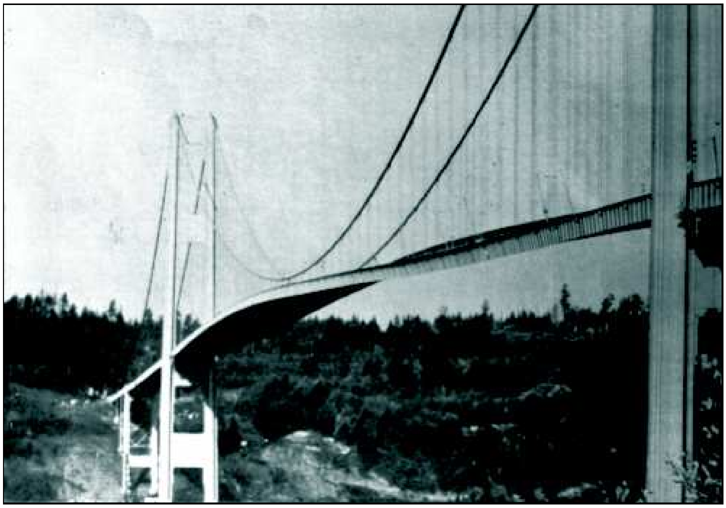

Figure 1. Vertical and torsional oscillations in Tacoma Narrows Bridge.

The aspect which distinguishes suspension bridges from other ones is their fundamental nonlinearity. It is caused by the presence of supporting cable stays which restrain the movement of the center span in the downward direction, but have no influence on its behaviour in the opposite direction.

This type of nonlinearity, often called *jumping* or *asymmetric*, has given rise to the following principle:

> Systems with asymmetry and large uni-directional loading tend to have multiple oscillatory solutions; the greater the asymmetry, the larger the number of oscillatory solutions, the greater the loading, the larger the amplitude of the oscillations.

The aim of our paper is to summarize several models describing the behaviour of suspension bridges and to pick up the main results as well as open problems that can be found in this field.

Nowadays, there exists a large amount of papers concerning these systems. Thus we do not set as a goal to mention here all of them. Our choice is motivated by our own research in this direction and by the limitations of our resources. See, e.g., [1]–[34]. Joint effort of these papers is to determine under what conditions the existence of a unique stable solution is guaranteed or, on the other hand, when some other (possibly dangerous from the practical point of view) solutions could appear.

Going through literature, we can find two main trends in modelling suspension bridges: the first is represented by the effort of creating the most realistic models, whereas the other trend is to deal with models which are as simple as possible. Both approaches have their advantages and disadvantages. If we work with simpler models, we can use finer analytical methods and obtain—in some sense—more interesting results. However, the correspondence of these results with the real situation is contentious. On the other hand, if even the simple (e.g. ordinary differential) model exhibits the behaviour that can be considered dangerous for the bridge structure, we can expect that the more precise model will behave in the similar—or even worse—way. Of course, all models considered keep the fundamental asymmetric feature which is reflected by the jumping nonlinearity terms in the particular equations—as will be seen later.

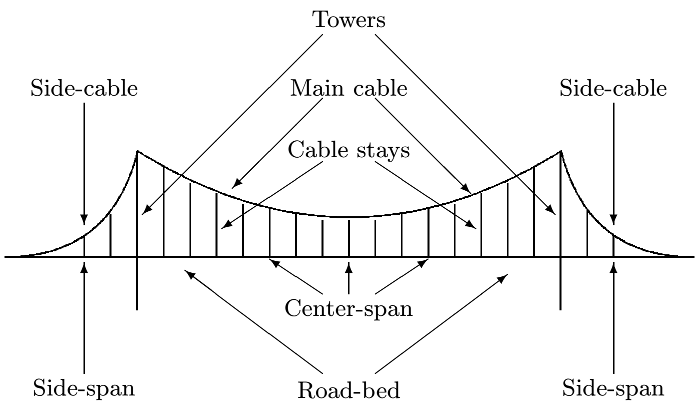

Figure 2. Sideview of the suspension bridge.

2.1. Single beam model.

We can start our list with the basic model introduced by Lazer and McKenna [24]. It describes the vertical oscillations of a one-dimensional beam (which represents the center span of a road-bed) with simply supported ends, hanging on nonlinear cable-stays (see Fig. 3). It neglects the influence of the towers and side parts and ignores the coupling of the main cable and the road-bed. The construction holding the cable stays is taken as a solid and immovable object.

If we consider that oscillations are measured as positive in the downward direction, the mathematical formulation is the following (see [24]):

$$(\mathrm{SB}_1) \qquad \begin{cases} mu_{tt} + ku_{xxxx} + \delta u_t + bu^+ = W(x) + f(x,t), \\ u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0. \end{cases}$$

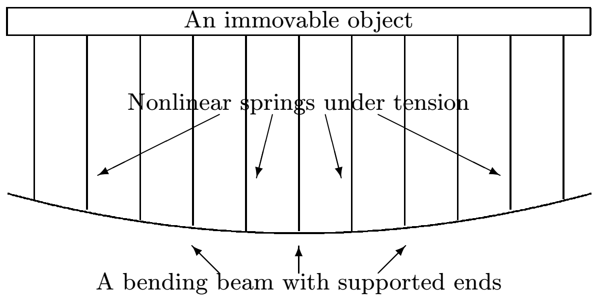

Figure 3. Center span hanged by fixed frame.

Here u is the displacement of the beam of length L. The first term in the equation (SB_1) represents an inertial force (with constant mass m), the second term is an elastic force (with material constant k) and the third term on the left hand side describes a viscous damping (with the coefficient δ). The cable stays are taken as one-sided springs, obeying Hooke's law, with a restoring force proportional to the displacement if they are stretched, and with no restoring force if they are compressed. This fact is described by the fourth term bu^+, where $u^+ = \max\{0, u\}$ and b is a coefficient which characterizes the stiffness of the cable stays. On the right-hand side, we have the gravitation force $W(x)$ and the external force $f(x,t)$, e.g. due to the wind.

If the force $f(x,t)$ is time periodic with a period $\tau > 0$ then it is reasonable to look for time periodic oscillations with the same period, i.e. we look for u satisfying (SB_1) and periodic conditions

$$u(x, t+\tau) = u(x,t), \quad (x,t) \in (0,L) \times \mathbb{R}. \tag{1}$$

Then (SB_1) together with periodic condition (1) forms the *periodic boundary value problem.*

We can also look for another type of solution u to (SB_1). Namely, if we substitute (1) by

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in (0,L), \tag{2}$$

then (SB_1) together with initial conditions (2) forms the *initial boundary value problem.* We then look for a solution on a finite interval $[0,T]$ with a fixed $T \in \mathbb{R}$ or study the global behaviour of the solution $u = u(x,t)$ for $t \to +\infty$.

As we mentioned above, this model was introduced by Lazer and McKenna and is used as a starting point for studies of suspension bridges in most of the cited works. It does not describe exactly the behaviour of a suspension bridge but on the other hand it reflects the influence of the cable stays and is reasonably simple and applicable.

2.2. String beam model.

In the second iteration, we can take into account not only the motion of the road-bed (represented by a vibrating beam), but also the oscillations of the main cable which can be replaced by a vibrating string. The string is coupled with the beam by one-sided springs (see Fig. 4). Thus we end up with the following model (see [24]):

$$
(\mathrm{SB}_2) \qquad \begin{cases} m_1 v_{tt} - k_1 v_{xx} + \delta_1 v_t - b(u-v)^+ = W_1(x) + f_1(x,t), \\ m_2 u_{tt} + k_2 u_{xxxx} + \delta_2 u_t + b(u-v)^+ = W_2(x) + f_2(x,t), \\ u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \\ v(0,t) = v(L,t) = 0. \end{cases}
$$

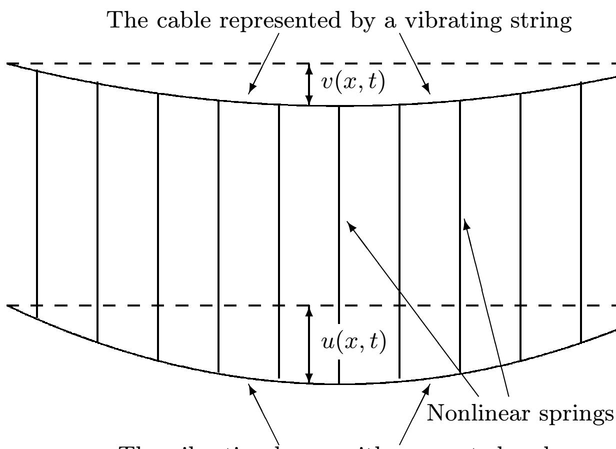

Figure 4. Center span hanged by main cable.

Here $v(x,t)$ measures the displacement of the vibrating string representing the main cable and $u(x,t)$ means—as in the previous case—the displacement of the bending beam standing for the road-bed of the bridge. The nonlinear stays connecting the beam and the string pull the cable down, hence we have the minus sign in front of $b(u-v)^+$ in the first equation, and hold the road-bed up, therefore we consider the plus sign in front of the same term in the second equation. The meaning of all other data in system (SB_2) is similar to the scalar model (SB_1).

Similarly as before, if $f_1(x,t)$ and $f_2(x,t)$ are periodic with period $\tau > 0$ then we look for u and v which satisfy (SB_2) and periodic conditions

$$
u(x,t+\tau) = u(x,t), \quad v(x,t+\tau) = v(x,t), \quad x \in (0,L), t \in \mathbb{R}.
$$

We thus get a periodic boundary value problem. If (3) are substituted by initial conditions

$$\begin{aligned} u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x), \\ v(x,0) &= v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in (0,L), \end{aligned} \tag{4}$$

then (SB_2) together with (4) forms an initial boundary value problem for unknown functions u and v.

2.3. Torsional vertical model.

The two previous models considered only vertical oscillations of suspension bridges. Now, we can choose another approach. Namely, to include also torsional oscillations, which are certainly not quite negligible. (They are said to be one of the direct causes of the destruction of the above mentioned Tacoma Narrows Bridge—see Fig. 1.) Thus we obtain the following model (see [24]):

$$(\mathrm{SB}_3) \qquad \begin{cases} m_1 y_{tt} + k_1 y_{xxxx} + \delta_1 y_t + b[(y - l\sin\varphi)^+ + (y + l\sin\varphi)^+] \\ \quad = W_1(x) + f_1(x,t), \\ m_2 \varphi_{tt} - k_2 \varphi_{xx} + \delta_2 \varphi_t - bl\cos\varphi[(y - l\sin\varphi)^+ - (y + l\sin\varphi)^+] \\ \quad = W_2(x) + f_2(x,t), \\ y(0,t) = y(L,t) = y_{xx}(0,t) = y_{xx}(L,t) = 0, \\ \varphi(0,t) = \varphi(L,t) = 0. \end{cases}$$

Here $y(x,t)$ is the displacement measured at the centre of gravity, $\varphi(x,t)$ measures the deflection of the cross section from the horizontal, l is the half length of the cross section of the road-bed—see Fig. 5. The meaning of the other parameters is analogous to those of previous models.

Similarly as in the previous two cases, we can formulate the periodic boundary value problem adding periodic conditions on y and φ under the assumption that f_1 and f_2 are periodic in t, or the initial boundary value problem adding the initial conditions on y and φ.

In literature, we can find another variant of model (SB_3): functions $\sin\varphi$ and $\cos\varphi$ are linearized and replaced by φ and 1, respectively. However, in many cases, this partial linearization brings no significant advantage and, conversely, the boundedness of functions sine and cosine can be helpful.

2.4. Complex model.

The fourth model combines all the previous aspects together. It considers vertical oscillations $y(x,t)$ of the road-bed, its torsional motion $\varphi(x,t)$, and coupling with

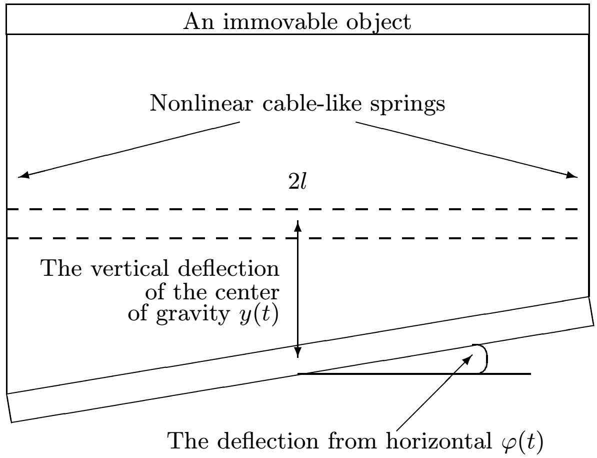

Figure 5. Cross section of the suspension bridge.

two main cables—the displacement of the right cable is described by a function $u(x,t)$ and the displacement of the left one is described by $v(x,t)$. Moreover, we can generalize the description of the nonlinear cable stays and, instead of $b(\cdot)^+$, assume that the interaction between the main cables and the road-bed is given by more general nonlinear functions $E_1(\cdot)$ and $E_2(\cdot)$. Thus we end up with the complex torsional-vertical model in the following form (see [31]):

$$
(\mathrm{SB}_4)\qquad \begin{cases} m_1 y_{tt} + k_1 y_{xxxx} + \delta_1 y_t + E_1(y - u - l\sin\varphi) \\ \quad + E_2(y - v + l\sin\varphi) = W_1(x) + f_1(x,t), \\ m_2\varphi_{tt} - k_2\varphi_{xx} + \delta_2\varphi_t - l\cos\varphi[E_1(y - u - l\sin\varphi) \\ -E_2(y - v + l\sin\varphi)] = W_2(x) + f_2(x,t), \\ m_3 u_{tt} - k_3 u_{xx} + \delta_3 u_t - E_1(y - u - l\sin\varphi) \\ \quad = W_3(x) + f_3(x,t), \\ m_4 v_{tt} - k_4 v_{xx} + \delta_4 v_t - E_2(y - v + l\sin\varphi) \\ \quad = W_4(x) + f_4(x,t), \\ y(0,t) = y(L,t) = y_{xx}(0,t) = y_{xx}(L,t) = 0, \\ \varphi(0,t) = \varphi(L,t) = 0, \\ u(0,t) = u(L,t) = v(0,t) = v(L,t) = 0. \end{cases}
$$

The models ($\mathrm{SB}_{1\text{–}4}$) have many similar features and all of them are in fact hidden in the matrix scheme

$$
(5)\qquad \mathbf{M}\mathbf{w}_{tt} + \mathbf{A}\mathbf{w} + \mathbf{D}\mathbf{w}_t + \mathbf{N}(\mathbf{w}) = \mathbf{W}(x) + \mathbf{F}(x,t).
$$

Namely, in the case of model (SB_4), we have $\mathbf{w} = [y, \varphi, u, v]^t$ the vector of unknown functions, $\mathbf{M}$, $\mathbf{D}$ are constant diagonal matrices:

$$\mathbf{M} = \begin{bmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & m_3 & \\ 0 & & & m_4 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} \delta_1 & & & 0 \\ & \delta_2 & & \\ & & \delta_3 & \\ 0 & & & \delta_4 \end{bmatrix}.$$

The symbol $\mathbf{A}$ represents an operator of the second and fourth space derivatives and $\mathbf{W}(x) + \mathbf{F}(x,t)$ is the vector of the right-hand sides, i.e.

$$\mathbf{A}\mathbf{w} = [k_1 y_{xxxx}, -k_2\varphi_{xx}, -k_3 u_{xx}, -k_4 v_{xx}]^t,$$
$$\mathbf{W}(x) = [W_i(x)]^t, \quad \mathbf{F}(x,t) = [f_i(x,t)]^t, \quad i = 1, \ldots, 4.$$

The nonlinear terms are given by

$$\begin{aligned} \mathbf{N}(\mathbf{w}) &= E_1(g_1(\mathbf{w}))\nabla g_1 + E_2(g_2(\mathbf{w}))\nabla g_2, \\ g_1(\mathbf{w}) &= y - u - l\sin\varphi, \\ g_2(\mathbf{w}) &= y - v + l\sin\varphi, \end{aligned}$$

where ∇g_1, ∇g_2 are gradients with respect to the vector $\mathbf{w} = [y, \varphi, u, v]^t$.

The periodic boundary value problem and the initial boundary value problem for (SB_4) are defined as in the previous cases. Let us note that the boundary conditions, the periodic conditions and the initial conditions influence the choice of the function spaces where the solution is looked for.

Due to this unified formulation, we can find a group of results which are valid—with some modifications—for all previous models ($\mathrm{SB}_{1\text{–}4}$).

2.5. Results concerning periodic boundary value problems for ($\mathrm{SB}_{1\text{–}4}$).

Existence results. If we consider boundary value problems formulated in Subsections 2.1–2.4, the existence of at least one time periodic solution is guaranteed for any periodic right-hand side and all positive values of the bridge parameters.

The proof for model (SB_1) can be found in [10], for model (SB_2) in [12] (here the proofs are based on the degree theory). The existence result for model (SB_2) is also proved in [27] (but there the author uses Galerkin method of approximative solutions).

Similar results can be obtained also for model (SB_3) and for the complex model (SB_4), but here under the additional assumption in the form of some growth conditions on the general nonlinearities E_1, E_2.

Uniqueness results. More interesting results concern the question of uniqueness of periodic solutions. Here we can find two main approaches. The first is based on the direct application of the Banach contraction principle. Thus we obtain the following assertion:

The boundary value problems (SB_{1-2}) have unique periodic solutions for any periodic right-hand sides if the stiffness parameter b is "sufficiently" small.

In the case of model (SB_1) this means

$$b < m \min\{\alpha^2, \beta\}, \tag{6}$$

where

$$\alpha^2 = \frac{\pi^2}{4}\frac{k\tau^2}{mL^4} > 0, \quad \beta = \frac{1}{2\pi}\frac{\delta\tau}{m} > 0.$$

In the case of model (SB_2) the corresponding condition has the form

$$b < \frac{m_1 m_2}{m_1 + m_2} \min\{\alpha_1^2, \beta_1, \alpha_2^2, \beta_2\}, \tag{7}$$

where

$$\alpha_1^2 = \frac{\pi^2 k_1 \tau^2}{4m_1 L^2}, \quad \alpha_2^2 = \frac{\pi^2 k_2 \tau^2}{4m_2 L^4}, \qquad \beta_i = \frac{\delta_i \tau}{2\pi m_i}, \quad i = 1, 2.$$

For the proofs see [34]. In the same way we can proceed also in the other two cases and obtain similar conditions (under suitable assumptions about E_1 and E_2).

The second approach to the uniqueness problem is a little bit different. It works with the right-hand side in the form of a constant weight and a sufficiently small periodic perturbation. On the other hand, the bridge parameters can be arbitrary positive real numbers.

In the case of the single beam model (SB_1), the constant weight leads to the strictly positive equilibrium. Sufficiently "small" periodic perturbations of the right-hand side result in a periodic solution which corresponds to small periodic oscillations around this equilibrium and still stays positive. Thus the problem is in fact linear and uniqueness is guaranteed. For details see [3].

The open problem is to give a precise meaning to the formulation "sufficiently small" and to determine the allowed magnitude of the perturbation.

In the case of the string beam model (SB_2), the situation is more complicated. If we want to proceed in the same way as in the previous case, we have to ensure the existence of a *strictly positive* equilibrium which here means that the difference $(u - v)$ is strictly positive. This is not true for any constant right-hand side and thus another assumption must be added. In the simplest way, it can be written as $W_1 \ll W_2$. Then we can continue with the same arguments and again obtain the uniqueness result. The precise proof can be found in [12].

As for the models ($SB_{3–4}$), we can expect similar results under similar assumptions (and with some restrictive conditions on general nonlinearities E_1, E_2).

2.6. Results concerning initial boundary value problems for ($SB_{1–4}$).
Methods for initial boundary value problems for partial differential equations (especially in one space dimension) make it possible to study models of suspension bridges in their whole complexity. The Faedo-Galerkin method is the main tool used to obtain the existence, regularity and uniqueness results.

One of the most important questions concerning the qualitative behaviour of a solution is its boundedness. Quite a standard tool for obtaining a priori estimates of solutions is the Gronwall lemma. This technique enables us to find a solution bounded on any finite interval (which may be possibly unbounded on an infinite time interval). Wide spectrum of systems of partial differential equations have been studied in this way e.g. in [1], [28], [29], [30].

The authors of [17] study the initial boundary value problem for (SB_3) where $b(\cdot)^+$ is replaced by more general nonlinear functions $E_1(\cdot)$ and $E_2(\cdot)$ describing the influence of the coupling of the road-bed and the fixed frame. More restrictive assumptions on the right-hand side allow them to avoid the use of the Gronwall lemma and to get a globally bounded solution. On the other hand, the nonlinear terms E_1 and E_2 can be rather general.

The results of [17] could be formulated in the following way:

> We assume that nonlinear functions E_1, E_2 are Lipschitz continuous and their primitive functions are bounded from below by a linear function. Moreover, let E_1, E_2 have a polynomial growth at infinity (with an arbitrarily large power) and let the right-hand side be uniformly bounded and time independent. Then for any initial conditions there exists a unique (weak) solution which is bounded on every finite time interval $(0, T)$ by a constant independent of T (i.e. the solution is bounded on the whole positive real line).

In contrast to periodic boundary value problems, here the open questions concern the global properties and asymptotic behaviour of solutions. However, one problem is common. It arises when we want to consider arbitrary time dependent right-hand sides. Most of the results for both types of boundary value problems deal with the right-hand sides in the form of a constant weight and a small time dependent perturbation. The relation between the magnitude of this perturbation and the qualitative properties of the solution is a very interesting but difficult open problem.

3. Simplified models

3.1. Periodic symmetric model.

In order to obtain other—perhaps more detailed—results, we come back to the model (SB_1). If we neglect the damping term, i.e. we put $\delta = 0$, add symmetry conditions with respect to both the variables x and t, and "normalize" the problem in some sense, we obtain the boundary value problem

$$(\mathrm{SB}_5) \qquad \begin{cases} u_{tt} + u_{xxxx} + bu^+ = W(x) + f(x,t) \ \text{ in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\ u\left(\pm\frac{\pi}{2}, t\right) = u_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\ u(x,t) = u(-x,t) = u(x,-t) = u(x,t+\pi). \end{cases}$$

Many variants of this model assume $W(x) \equiv 1$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This description of a suspension bridge was used e.g. in [33]. The authors showed that if the parameter b crosses a certain eigenvalue of a related eigenvalue problem, an additional solution appears. In particular, they proved that for $-1 < b < 3$, the problem (SB_5) has a unique solution, however for $3 < b < 15$ and f small enough in some sense another solution exists.

This result was extended first in [8] where the existence of at least three solutions for $3 < b < 15$ by a variational reduction method is proved, and then in [19] where the authors showed that for $15 < b < 15 + \varepsilon$, $\varepsilon > 0$, at least four solutions exist. Moreover, the additional solutions tend to have large amplitudes.

These results hint that the number of solutions could increase with respect to the number of crossed eigenvalues. That is why the authors in [11] decided to formulate the problem (SB_5) "in the language of the bifurcation theory" and to explain this phenomenon from this point of view. In their considerations they assumed $f(x,t) \equiv 0$, proved that the problem (SB_5) with $f \equiv 0$ possesses multiple solutions for $b \geqslant 3$ and provided also some qualitative information about the solution set of (SB_5) (see [11]). Moreover, they in fact showed that the multiple solutions of (SB_5) exist not because of the perturbation term $f(x,t)$, but because of the absence of the damping term δu_t.

3.2. Vertical ordinary differential model.

Actually, the bifurcation results for the problem (SB_5) were inspired by a similar multiplicity result for another, more simplified model.

In fact, the authors of [24] suggested to consider the right-hand side in (SB_5) of a special form $W(x) + f(x,t) = \cos x + f(t) \cos x$ and to expect the solution to have a similar character $u(x,t) = y(t) \cos x$. If we put these relations into the problem (SB_5), we obtain the ordinary differential model

$$(\mathrm{SB}_6) \qquad \begin{cases} y''(t) + y(t) + by^+(t) = 1 + f(t) \quad \text{in } \mathbb{R}, \\ y(t) = y(-t) = y(t+\pi). \end{cases}$$

In [24] we can find a theorem which in fact says that the number of solutions increases as b crosses the eigenvalues corresponding to the linear part of the equation (SB_6). The authors of [11] specify this phenomenon and show that there is a sequence $b_m = 4m^2 - 1$, $m \in \mathbb{N} \cup \{0\}$, such that ($SB_6$) with $f(t) \equiv 0$ has exactly $2m+1$ solutions if $b \in (b_m, b_{m+1})$. Moreover, the set of all solutions is described in a rather explicit form (for illustration see Fig. 6).

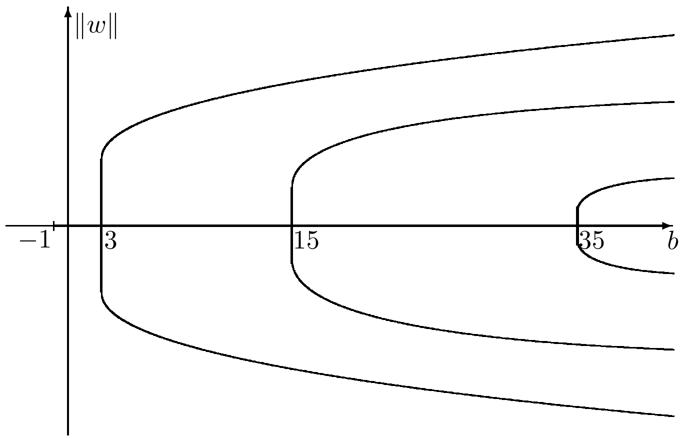

Figure 6. Bifurcation diagram of model (SB_6). Here the equilibrium solution $\frac{1}{b+1}$ is shifted to the zero.

New and in some sense surprising results were obtained in [13]. Here the authors realized that the model (SB_6) is in fact only a very specific case and that the time period plays an important role.

They consider the problem (SB_6) with $f(t) \equiv 0$ and solutions that are even and T-periodic in time. Then the model (SB_6) reads

$$\begin{cases} y''(t) + y(t) + by^+(t) = 1 & \text{in } \mathbb{R}, \\ y(t) = y(-t) = y(t+T). \end{cases} \tag{8}$$

The solutions are all uniformly bounded if $T \in (0, \pi)$, there are solutions with arbitrarily large amplitude if $T \geqslant \pi$ and there are “blow up points” if $T > \pi$ in the sense that for bounded values of b there exist nonstationary solutions of (8) with the amplitude approaching infinity. In Fig. 7, the horizontal axis represents the parameter $b \in \mathbb{R}$ and the vertical axis the space of nonstationary solutions of (8) that form two branches. The pattern of the bifurcation diagram is richer in the case $T > \pi$ and more blow up points appear when T is crossing the values $k\pi$, $k = 1, 2, \ldots$.

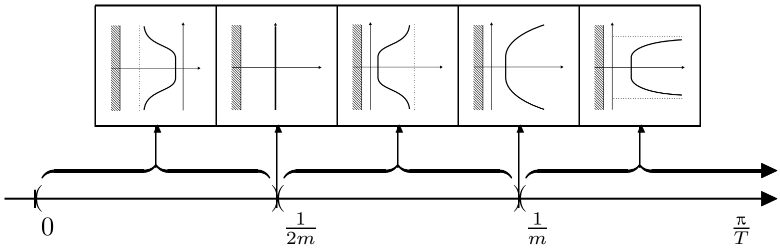

Figure 7. Qualitative behaviour of bifurcating branches with respect to π/T.

It is worth mentioning that the problem (8) is closely connected to the Fučík spectrum of

$$\begin{aligned} &y''(t) + \alpha y^+(t) - \beta y^-(t) = 0 \quad \text{in } \mathbb{R},\\ &y(t) = y(-t) = y(t+T) \end{aligned}$$

(see Fig. 8). For a fixed period T, the point b will be the "blow up point" if and only if $(b+1, 1)$ belongs to the Fučík spectrum.

The structure of the set of the blow up points depends on the value of π/T and can be summarized as follows:

$$\begin{array}{lll} \pi/T \geqslant 1, & \text{no blow up point,} & \\ \pi/T \in \left(\frac{1}{2}, 1\right), & \text{exactly one blow up point} & b_1 > 0,\\ \pi/T \in \left(\frac{1}{3}, \frac{1}{2}\right), & \text{exactly two blow up points} & b_1 < 0,\ b_2 > 0,\\ \pi/T \in \left(\frac{1}{4}, \frac{1}{3}\right), & \text{exactly three blow up points} & b_1 < 0,\ b_2 > 0,\ b_3 > 0,\\ \pi/T \in \left(\frac{1}{5}, \frac{1}{4}\right), & \text{exactly four blow up points} & b_1 < 0,\ b_2 < 0,\ b_3 > 0,\ b_4 > 0,\\ \vdots & \vdots & \vdots \end{array}$$

Roughly speaking, as T increases to $+\infty$ (i.e. $\pi/T \searrow 0+$) the number of blow up points increases. They are "travelling" to the left and crossing zero. For a given T, all blow up points (if there are any) are greater than or equal to $(\pi/T)^2 - 1$. Fig. 8 depicts a special case for $T = 16\pi/5$ with two blow up points b_1 and b_2.

For the right-hand side in (SB_6) in the form $1 + \varepsilon f(t)$ with f even and T-periodic, the authors in [13] prove that the structure of the solution set of this perturbed problem is determined for ε small enough just by the structure of the solution set of the unperturbed problem (8). This can be done due to the precise description of the structure of the set of all solutions of problem (8), the main tools being the homotopy invariance property of the Leray-Schauder degree and the Leray-Schauder index formula.

3.3. Torsional ordinary differential model.

If we consider the periodic boundary value problem for the torsional vertical model (SB_3) and assume the terms $y \pm l \sin\varphi$ stay positive, we obtain two separate equations

$$\begin{cases} m_1 y_{tt} + k_1 y_{xxxx} + \delta_1 y_t + 2by = W_1(x) + f_1(x,t), \\ m_2 \varphi_{tt} - k_2 \varphi_{xx} + \delta_2 \varphi_t + bl^2 \sin 2\varphi = W_2(x) + f_2(x,t). \end{cases} \tag{9}$$

The first equation for the vertical displacement y is linear, however the second for the torsion φ is nonlinear.

This is, in fact, a damped and forced sine-Gordon equation and its solution set is not mapped even in the case of zero damping $\delta_2 = 0$ and zero right-hand side. Nonetheless, we can do the following simplification: if we deal with a long bridge and the behaviour in the middle of the center span is more interesting for us than the situation near the end points, we can assume the solution φ to be space independent. Thus we end up with the ordinary differential model which can be written, e.g., in the following form (here we consider zero damping and additional symmetry condition on the solution):

$$\begin{cases} \varphi''(t) + b \sin 2\varphi(t) = 0, \\ \varphi(t) = \varphi(-t) = \varphi(t + 2\pi). \end{cases} \tag{10}$$

Here, the situation is completely described:

Every point $2b = k^2$, $k \in \mathbb{N}$, is a bifurcation point and there exist just two branches emanating from these points. Each branch preserves the nodal properties of the corresponding eigenfunction, is unbounded, has no blow up points, and is described by the strictly monotone function

$$f(b) = \arcsin \frac{v}{\sqrt{2b}},$$

where v satisfies

$$\frac{\pi}{2k} = \int_0^1 \frac{\mathrm{d}y}{\sqrt{1-y^2}\sqrt{2b-(vy)^2}}.$$

Thus we can conclude that for any $b \in (k^2/2, (k+1)^2/2)$ there exist exactly $2k+1$ solutions of the problem (10) (see Fig. 9).

This result remains true even for an arbitrary time period T—only the bifurcation points will be shifted. This is different from the case of model (SB_6), where the situation was more complicated.

Quite interesting open problems arise in the case of the partial differential sine-Gordon equation, or for the systems of ordinary differential equations with nonlinear

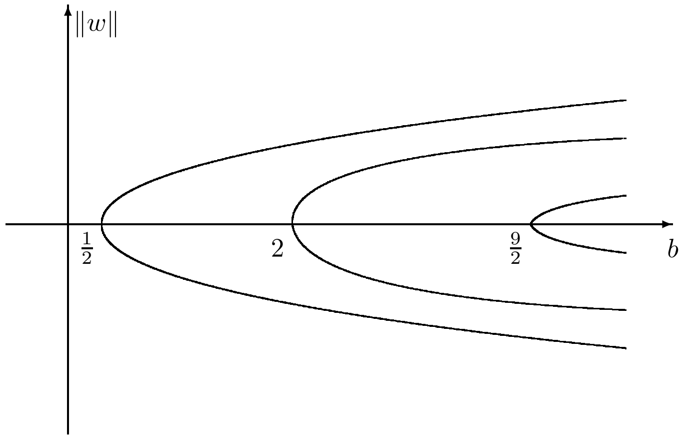

Figure 8. Bifurcation diagram of model (10).

coupling. Here we have several possibilities how to formulate the problem. We can assume as above that our solutions are space independent. Then the original torsional model (SB_3) (without damping) becomes

$$
(11)\qquad \begin{cases} m_1 y'' + b[(y - l\sin\varphi)^+ + (y + l\sin\varphi)^+] = W_1 + f_1(t), \\ m_2\varphi'' - bl\cos\varphi[(y - l\sin\varphi)^+ - (y + l\sin\varphi)^+] = W_2 + f_2(t). \end{cases}
$$

Another possibility is to replace $\sin\varphi$ by φ and $\cos\varphi$ by 1 and assume that both the right-hand sides as well as the solutions y, φ of (SB_3) have the form of $\cos x$ multiplied by time dependent functions. Then we obtain the system

$$
(12)\qquad \begin{cases} m_1 y'' + k_1 y + b[(y - l\varphi)^+ + (y + l\varphi)^+] = W_1 + f_1(t), \\ m_2\varphi'' + k_2\varphi - bl[(y - l\varphi)^+ - (y + l\varphi)^+] = W_2 + f_2(t). \end{cases}
$$

In both models (11), (12), there are still many open questions even for the simple case $W_2 \equiv f_1(t) \equiv f_2(t) \equiv 0$.

3.4. Numerical results.

To complete the survey of known results concerning our models, we should mention several numerical experiments done in this field. Let us recall at least papers [6], [9], [15], [16], [18], [20], [32], [24] and [5].

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