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Applications of Mathematics, Vol. 48 (2003), No. 6, 547–558

Persistent URL: `http://dml.cz/dmlcz/134550`

THE BOUNDARY REGULARITY OF A WEAK SOLUTION OF THE NAVIER-STOKES EQUATION AND ITS CONNECTION TO THE INTERIOR REGULARITY OF PRESSURE*

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Abstract. We assume that $\boldsymbol{v}$ is a weak solution to the non-steady Navier-Stokes initial-boundary value problem that satisfies the strong energy inequality in its domain and the Prodi-Serrin integrability condition in the neighborhood of the boundary. We show the consequences for the regularity of $\boldsymbol{v}$ near the boundary and the connection with the interior regularity of an associated pressure and the time derivative of $\boldsymbol{v}$.

Keywords: Navier-Stokes equations, regularity

MSC 2000: 35Q30, 76D05

1. Introduction

Suppose that Ω is a bounded domain in $\mathbb{R}^3$ with a C^∞ boundary $\partial\Omega$ such that Ω is locally on one side of $\partial\Omega$. Let $T>0$ and $Q_T=\Omega\times(0,T)$. We deal with the Navier-Stokes initial-boundary value problem

$$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=-\nabla p+\nu\Delta\boldsymbol{v} \quad \text{in } Q_T, \tag{1}$$

$$\nabla\cdot\boldsymbol{v}=0 \quad \text{in } Q_T, \tag{2}$$

$$\boldsymbol{v}=\boldsymbol{0} \quad \text{on } \partial\Omega\times(0,T), \tag{3}$$

$$\boldsymbol{v}\big|_{t=0}=\boldsymbol{v}_0 \tag{4}$$

where $\boldsymbol{v}=(v_1,v_2,v_3)$ and p denote the velocity and the pressure and $\nu>0$ is the viscosity coefficient. We will assume for simplicity that $\nu=1$.

* This work was supported by the Northern Illinois University, DeKalb, by the Grant Agency of the Czech Republic (grant No. 201/02/0684) and by the research plan of the Ministry of Education of the Czech Republic No. MSM 98/210000010.

We deal with a weak solution $\boldsymbol{v}$ of the problem (1)–(4) that satisfies a strong energy inequality. (Such a solution can be constructed.) The notion of a weak solution of the problem (1)–(4) is well known. The readers can find the definition and a survey of important properties e.g. in [3]. Let us only recall that $\boldsymbol{v} \in L^2(0,T;W_0^{1,2}(\Omega)^3) \cap L^\infty(0,T;L^2(\Omega)^3)$. The associated pressure is a scalar function p such that $\boldsymbol{v}$ and p satisfy equation (1) in Q_T in the sense of distributions. p is defined a.e. in Q_T, it is determined modulo an additive function of time and can be chosen so that it belongs to $L^{5/3}((\varepsilon,T)\times\Omega)$ for each $\varepsilon \in (0,T)$ (see [13]).

A point $(\boldsymbol{x},t) \in \overline{\Omega}\times(0,T)$ is called a *regular point* of the weak solution $\boldsymbol{v}$ if there exists a neighborhood U of $(\boldsymbol{x},t)$ such that $\boldsymbol{v}$ is essentially bounded in $U\cap Q_T$. The points of $\overline{\Omega}\times(0,T)$ which are not regular are called *singular.*

The following lemma gives more information on interior regularity of the weak solution $\boldsymbol{v}$ of the problem (1)–(4). t_1 and t_2 will always denote instants of time such that $0 \leqslant t_1 < t_2 \leqslant T$.

Lemma 1. *Let Ω_1 be a subdomain of Ω and let at least one of the conditions*

(i) $\boldsymbol{v} \in L^a(t_1,t_2;L^b(\Omega_1)^3)$ *for some* $a \in [2,+\infty)$, $b \in (3,+\infty)$ *such that* $2/a+3/b = 1$,

(i)$'$ $\boldsymbol{v} \in L^\infty(t_1,t_2;L^3(\Omega_1)^3)$ *and the norm of* $\boldsymbol{v}$ *in* $L^\infty(t_1,t_2;L^3(\Omega_1)^3)$ *is sufficiently small*

be satisfied. Let Ω_2 be a sub-domain of Ω_1 such that $\overline{\Omega_2}\subset\Omega_1$ and let ζ be a positive number such that $t_1+\zeta < t_2-\zeta$. Then

a) $\boldsymbol{v}$ *and its space derivatives of arbitrary orders belong to* $L^\infty(\Omega_2\times(t_1+\zeta,t_2-\zeta))^3$ *and*

b) ∇p *and* $\partial\boldsymbol{v}/\partial t$ *and their space derivatives of arbitrary orders belong to* $L^\alpha(t_1+\zeta,t_2-\zeta;L^\infty(\Omega_2)^3)$ *for each* $\alpha\in[1,2)$.

Statement a) follows from [11], while b) is proved e.g. in [10].

Regularity up to the boundary of a weak solution $\boldsymbol{v}$ of the problem (1)–(4) was studied by S. Takahashi [14]. S. Takahashi worked with a domain Ω_1 of the form $\Omega_1 = U_\delta(\mathbf{x}_0)\cap\Omega$ for some $\boldsymbol{x}_0\in\partial\Omega$ under the assumption that $\partial\Omega_1\cap\partial\Omega$ is part of a plane. He has shown that if $\boldsymbol{v}$ satisfies condition (i) or condition (i)$'$ then it has no singular points in $U_{\delta'}(\boldsymbol{x}_0)\cap\overline{\Omega}$ in the time interval $(t_1+\zeta,t_2-\zeta)$ for all $\zeta\in(0,(t_2-t_1)/2)$ and $\delta'<\delta$.

We shall use the following notation:

- $\boldsymbol{n}$ is the outer normal vector on $\partial\Omega$.
- $L^2_\sigma(\Omega)^3$ is the closure of $\{\Phi\in C_0^\infty(\Omega)^3;\ \nabla\cdot\Phi=0\ \text{in}\ \Omega\}$ in $L^2(\Omega)^3$. Functions from $L^2_\sigma(\Omega)^3$ have the normal component on $\partial\Omega$ equal to zero in the sense of traces and $[L^2_\sigma(\Omega)^3]^\perp = \{\nabla\varphi\in L^2(\Omega)^3;\ \varphi\in W^{1,2}_{\mathrm{loc}}(\Omega)\}$ (see e.g. [3], Chap. III).

- $\|\cdot\|_q$ and $\|\cdot\|_{s,q}$, will denote the norm in $L^q(\Omega)$ and in $W^{s,q}(\Omega)$, respectively. The norms of vector-valued or tensor-valued functions will be denoted in the same way as the norms of scalar-valued functions.
- P_σ is the orthogonal projector of $L^2(\Omega)^3$ onto $L^2_\sigma(\Omega)^3$. Put $Q_\sigma = I - P_\sigma$. If $\boldsymbol{w}$ is smooth enough, i.e. if $\nabla\cdot\boldsymbol{w} \in L^2(\Omega)^3$, then $Q_\sigma\boldsymbol{w}$ has the form $\nabla\varphi$ where φ satisfies the Neumann problem

$$\Delta\varphi = \nabla\cdot\boldsymbol{w} \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\boldsymbol{n}}\bigg|_{\partial\Omega} = (\boldsymbol{w}\cdot\boldsymbol{n})\big|_{\partial\Omega}.$$

  Using the assumption about the smoothness of $\partial\Omega$, one can deduce from the results on the regularity of solutions of this problem (see e.g. [5], p. 15) that P_σ and Q_σ are continuous linear operators in $W^{s,q}(\Omega)^3$ for all $s \geqslant 0$ and $q \geqslant 2$.
- $A = -P_\sigma \circ \Delta$ with $D(A) = W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3 \cap L^2_\sigma(\Omega)^3$. A is a selfadjoint positive operator in $L^2_\sigma(\Omega)^3$. It was proved in [1] and [4] that the domain of the fractional power A^s $(0 \leqslant s \leqslant 1)$ is $D(A^s) = D((-\Delta)^s) \cap L^2_\sigma(\Omega)^3$ where $-\Delta$ is considered to be the operator in $L^2(\Omega)^3$ with the domain $D(-\Delta) = W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3$. Since $D((-\Delta)^{1/2}) = W^{1,2}_0(\Omega)^3$ and consequently $D((-\Delta)^s)$ is the interpolation space $[L^2(\Omega)^3, W^{1,2}_0(\Omega)^3]_{2s} = W^{2s,2}(\Omega)^3$ $(0 \leqslant s < \frac{1}{4})$, we have $D(A^s) = W^{2s,2}(\Omega)^3 \cap L^2_\sigma(\Omega)^3$ $(0 \leqslant s < \frac{1}{4})$. It can be also deduced from [4] that A^s is a continuous operator from $W^{2s,q}(\Omega)^3$ into $L^q(\Omega)^3$ $(0 \leqslant s \leqslant 1,\ q \geqslant 2)$.
- $U^*_r = U_r(\partial\Omega) \cap \Omega$ (for $r > 0$).

We shall further use the conditions

(ii) $\boldsymbol{v} \in L^a(t_1, t_2; L^b(U^*_r)^3)$ for some $r > 0$ and $a \in [2, +\infty)$, $b \in (3, +\infty)$ satisfying $2/a + 3/b = 1$,

(ii)′ $\boldsymbol{v} \in L^\infty(t_1, t_2; L^3(U^*_r)^3)$ and the norm of $\boldsymbol{v}$ in $L^\infty(t_1, t_2; L^3(U^*_r)^3)$ is sufficiently small.

Both the conditions (ii) and (ii)′ are obviously fulfilled if $\boldsymbol{v}$ has no singular points on $\partial\Omega$ in the time interval $[t_1, t_2]$. The main results of this paper are given by the next two theorems.

Theorem 1. *Let condition* (ii) *or condition* (ii)′ *be fulfilled and let $\zeta > 0$ be such a number that $t_1 + \zeta < t_2 - \zeta$. Then $\boldsymbol{v} \in L^\infty(t_1 + \zeta, t_2 - \zeta; W^{2+\delta,2}(U^*_\varrho)^3)$ and both $\partial\boldsymbol{v}/\partial t$ and ∇p belong to $L^\infty(t_1 + \zeta, t_2 - \zeta; W^{\delta,2}(U^*_\varrho)^3)$ for each $\delta \in ([0, \frac{1}{2})$ and $\varrho \in (0, r)$.*

Let us note that statement b) of Lemma 1 holds with $\alpha = +\infty$ in the case when $\Omega = \mathbb{R}^3$. (This will easily follow from Lemma 2 and the identity $p^{II} = 0$. It was also independently proved by P. Kučera and Z. Skalák—see [6] and [12], where this question and other related topics are also discussed.) Thus, a challenging question arises about the influence of the boundary of Ω on the interior regularity of pressure and the time derivative of velocity, even if $\partial\Omega$ is arbitrarily far from the considered domains Ω_1 and Ω_2. Theorem 2 shows that conditions (ii) or (ii)$'$ enable us to obtain the same result as in the case when $\Omega = \mathbb{R}^3$.

Theorem 2. *Let Ω_1 and Ω_2 be subdomains of Ω such that $\overline{\Omega_2} \subset \Omega_1$ and let ζ be a positive number such that $t_1 + \zeta < t_2 - \zeta$. Suppose that at least one of the conditions* (i) *and* (i)$'$ *and at least one of the conditions* (ii) *and* (ii)$'$ *are satisfied. Then ∇p, $\partial \boldsymbol{v}/\partial t$ and their space derivatives of arbitrary orders belong to $L^\infty(\Omega_2 \times (t_1 + \zeta, t_2 - \zeta))^3$.*

2. Proofs of Theorem 1 and Theorem 2

The problem (1)–(4) can be localized to U^*_r in a standard way: Let $\varrho \in (0, r)$ and let η be a C^∞ cut-off function such that $\eta(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in U^*_\varrho$, $0 \leqslant \eta(\boldsymbol{x}) \leqslant 1$ for $\boldsymbol{x} \in U^*_{(r+2\varrho)/3} - U^*_\varrho$ and $\eta(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \in \Omega - U^*_{(r+2\varrho)/3}$. Put $\boldsymbol{u} = \eta\boldsymbol{v} - \boldsymbol{V}$ where $\nabla \cdot \boldsymbol{V} = \nabla\eta \cdot \boldsymbol{v}$. Function $\boldsymbol{V}$ can be constructed so that it has a compact support in $[U^*_{(2r+\varrho)/3} - \overline{U^*_{\varrho/2}}] \times [t_1, t_2]$ and

$$\|\nabla^{m+1}\boldsymbol{V}\|_2 \leqslant c(m)\|\nabla^m \boldsymbol{v}\|_2 \tag{5}$$

for all $m \in \mathbb{N}$. (See e.g. [2], Theorem 3.2, Chap. III.3.) $\boldsymbol{u}$ satisfies the equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla[\eta(p - \overline{p})] + \Delta\boldsymbol{u} + \boldsymbol{h} \quad \text{in } \Omega \times (t_1, t_2), \tag{6}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega \times (t_1, t_2) \tag{7}$$

where

$$\begin{aligned}
\overline{p}(t) &= \int_{U^*_{(2r+\varrho)/3} - U^*_{\varrho/2}} p(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}, \\
\boldsymbol{h} &= -\frac{\partial \boldsymbol{V}}{\partial t} - (\boldsymbol{V} \cdot \nabla)(\eta\boldsymbol{v}) - ((\eta\boldsymbol{v}) \cdot \nabla)\boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} + (\eta\boldsymbol{v} \cdot \nabla\eta)\boldsymbol{v} \\
&\quad - \eta(1 - \eta)(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - 2\nabla\eta \cdot \nabla\boldsymbol{v} - \boldsymbol{v}\Delta\eta + \Delta\boldsymbol{V} + (p - \overline{p})\nabla\eta.
\end{aligned}$$

Note that $\operatorname{supp} \boldsymbol{h} \subset \big(U^*_{(2r+\varrho)/3} - \overline{U^*_{\varrho/2}}\big) \times [t_1, t_2]$. $\boldsymbol{u}$ satisfies the boundary condition

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega \times (t_1, t_2). \tag{8}$$

An analysis of the system (6)–(8) requires some information about regularity of the function $\boldsymbol{h}$, which is closely connected with the interior regularity of functions p and the time derivative of $\boldsymbol{v}$. p can be written as a sum $p^I + p^{II}$ where $\nabla p^I = -Q_\sigma(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ and $\nabla p^{II} = Q_\sigma \Delta\boldsymbol{v}$. Then for a.a. $t \in (t_1, t_2)$ one has

$$\Delta p^I = -v_{i,\,j}\, v_{j,\,i} \quad \text{in } \Omega, \qquad \frac{\partial p^I}{\partial \boldsymbol{n}}(\boldsymbol{x}, t)\bigg|_{x\in\partial\Omega} = 0, \tag{9}$$

$$\Delta p^{II} = 0 \quad \text{in } \Omega, \qquad \frac{\partial p^{II}}{\partial \boldsymbol{n}}(\boldsymbol{x}, t)\bigg|_{x\in\partial\Omega} = (\Delta\boldsymbol{v}(\boldsymbol{x}, t)\cdot\boldsymbol{n})\big|_{x\in\partial\Omega}. \tag{10}$$

The harmonic part p^{II} of pressure is connected with velocity only through the behavior of $\Delta\boldsymbol{v}$ on the boundary. This is also observed and discussed in [9], pp. 83–85.

Lemma 2. *Let Ω_1 be a subdomain of Ω and let at least one of the conditions* (i) *and* (i)$'$ *be satisfied. Let Ω_2 be a subdomain of Ω_1 such that $\overline{\Omega_2} \subset \Omega_1$ and let ζ be a positive number such that $t_1 + \zeta < t_2 - \zeta$. Then ∇p^I and its space derivatives of arbitrary orders belong to $L^\infty(\Omega_2 \times (t_1+\zeta, t_2-\zeta))^3$.*

P r o o f. A solution $\boldsymbol{v}$ can have singularities only at time instants $t \in \Gamma$ where the set Γ is closed in $(0, T)$ and its measure is zero. Moreover, $\boldsymbol{v}$ is of class C^∞ on $\overline{\Omega} \times ((0,T) - \Gamma)$. (See e.g. [3].) Suppose that $t \in (t_1+\zeta, t_2-\zeta) - \Gamma$ and $\boldsymbol{a}$ is a unit vector. Let μ be a C^∞ cut-off function such that $\mu(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in \Omega_2$, $0 \leqslant \mu(\boldsymbol{x}) \leqslant 1$ for $\boldsymbol{x} \in \Omega_1 - \Omega_2$ and $\mu(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \notin \Omega_1$. Let $\boldsymbol{x} \in \Omega_2$. Then

$$\begin{aligned}
\boldsymbol{a}\cdot\mu(\boldsymbol{x})\nabla p^I(\boldsymbol{x}, t) &= -\frac{1}{4\pi}\int_{\mathbb{R}^3} \boldsymbol{a}\cdot\frac{\Delta_y[\mu(\boldsymbol{y})\nabla_y p^I(\boldsymbol{y}, t)]}{|\boldsymbol{y}-\boldsymbol{x}|}\,\mathrm{d}\boldsymbol{y} \\
&= \frac{1}{4\pi}\int_\Omega \Delta_y\Big(\frac{\boldsymbol{a}\mu(\boldsymbol{y})}{|\boldsymbol{y}-\boldsymbol{x}|}\Big)\cdot\nabla_y p^I(\boldsymbol{y}, t)\,\mathrm{d}\boldsymbol{y} \\
&\quad + \frac{\boldsymbol{a}}{4\pi}\cdot\int_\Omega \frac{\mu(\boldsymbol{y})}{|\boldsymbol{y}-\boldsymbol{x}|}\nabla_y[v_{i,\,j}(\boldsymbol{y}, t)v_{j,\,i}(\boldsymbol{y}, t)]\,\mathrm{d}\boldsymbol{y} \\
&= \frac{1}{4\pi}\int_\Omega \nabla_y\varphi^{x,a}(\boldsymbol{y})\cdot\nabla_y p^I(\boldsymbol{y}, t)\,\mathrm{d}\boldsymbol{y} + \frac{\boldsymbol{a}}{4\pi}\cdot\boldsymbol{I}(\boldsymbol{x}, t)
\end{aligned}$$

where the integral $\boldsymbol{I}$ belongs to $L^\infty(\Omega_1 \times (t_1+\zeta, t_2-\zeta))^3$ (due to Lemma 1) and $\nabla_y\varphi^{x,a}(y) = Q_\sigma\Delta_y(\boldsymbol{a}\mu(\boldsymbol{y})/|\boldsymbol{y}-\boldsymbol{x}|)$. One can derive that

$$\varphi^{x,a}(\boldsymbol{y}) = \boldsymbol{a}\cdot\Big[\nabla_y\frac{\mu(\boldsymbol{y})-\mu(\boldsymbol{x})}{|\boldsymbol{y}-\boldsymbol{x}|} + \boldsymbol{w}^x(\boldsymbol{y})\Big]$$

where

$$\begin{aligned}
&\Delta_y\boldsymbol{w}^x(\boldsymbol{y}|) = 0 \ \text{ in } \Omega, \\
&\frac{\partial\boldsymbol{w}^x(\boldsymbol{y})}{\partial_y\boldsymbol{n}}\bigg|_{y\in\partial\Omega} = \Big(-\frac{\boldsymbol{n}}{|\boldsymbol{y}-\boldsymbol{x}|^3} + 3\frac{(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}}{|\boldsymbol{y}-\boldsymbol{x}|^5}(\boldsymbol{y}-\boldsymbol{x})\Big)\bigg|_{y\in\partial\Omega}.
\end{aligned}$$

Then we have

$$\begin{aligned}\boldsymbol{a}\cdot\mu(\boldsymbol{x})\nabla p^{I}(\boldsymbol{x},t) &= -\frac{1}{4\pi}\int_{\Omega}\varphi^{x,a}(\boldsymbol{y})v_{i,\,j}(\boldsymbol{y},t)v_{j,\,i}(\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y}+\boldsymbol{a}\cdot\boldsymbol{I}(\boldsymbol{x},t)\\ &= -\frac{1}{4\pi}\int_{\Omega}\varphi^{x,a}_{i,\,j}(\boldsymbol{y})v_i(\boldsymbol{y},\,t)v_j(\boldsymbol{y},\,t)\,\mathrm{d}\boldsymbol{y}+\boldsymbol{a}\cdot\boldsymbol{I}(\boldsymbol{x},t).\end{aligned}$$

This shows that ∇p^I belongs to $L^\infty(\Omega_2\times(t_1+\zeta,t_2-\zeta))^3$. The same statement about the space derivatives of ∇p^I can be obtained analogously, provided we deal with $D_x^{|k|}\nabla p^I$ (where $D_x^{|k|}=\partial^{|k|}/\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}$, $k=(k_1,k_2,k_3)$ is a multiindex) instead of ∇p^I.

Lemma 3. *Let Ω_2 be a subdomain of Ω such that $\overline{\Omega_2}\subset\Omega$. Let $\partial\boldsymbol{v}/\partial\boldsymbol{n}\in L^\beta(t_1,t_2;L^1(\partial\Omega)^3)$ (where $\beta\geqslant 1$) and let ζ be a positive number such that $t_1+\zeta<t_2-\zeta$. Then ∇p^{II} and its space derivatives of arbitrary orders belong to $L^\beta(t_1+\zeta,t_2-\zeta;L^\infty(\Omega_2)^3)$.*

Proof. Let Ω_1 be a domain in Ω such that $\overline{\Omega_2}\subset\Omega_1\subset\Omega$. Suppose that t, $\boldsymbol{x}$, $\boldsymbol{a}$, $\varphi^{x,a}$ and μ have the same meaning as in the proof of Lemma 2. Then

$$\begin{aligned}\boldsymbol{a}\cdot\mu(\boldsymbol{x})\nabla p^{II}(\boldsymbol{x},t) &= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\boldsymbol{a}\cdot\frac{\Delta_y[\mu(\boldsymbol{y})\nabla_y p^{II}(\boldsymbol{y},t)]}{|\boldsymbol{y}-\boldsymbol{x}|}\,\mathrm{d}\boldsymbol{y}\\ &= \frac{1}{4\pi}\int_{\Omega}\Delta_y\Big(\frac{\boldsymbol{a}\mu(\boldsymbol{y})}{|\boldsymbol{y}-\boldsymbol{x}|}\Big)\cdot\nabla_y p^{II}(\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y}\\ &= \frac{1}{4\pi}\int_{\Omega}\Delta_y\Big(\frac{\boldsymbol{a}\mu(\boldsymbol{y})}{|\boldsymbol{y}-\boldsymbol{x}|}\Big)\cdot Q_\sigma\Delta_y\boldsymbol{v}(\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y}\\ &= \frac{1}{4\pi}\int_{\Omega}Q_\sigma\Delta_y\Big(\frac{\boldsymbol{a}\mu(\boldsymbol{y})}{|\boldsymbol{y}-\boldsymbol{x}|}\Big)\cdot\Delta_y\boldsymbol{v}(\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y}\\ &= \frac{1}{4\pi}\int_{\Omega}\nabla_y\varphi^{x,a}(\boldsymbol{y})\cdot\Delta_y\boldsymbol{v}(\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y}=\frac{1}{4\pi}\int_{\partial\Omega}\nabla_y\varphi^{x,a}(\boldsymbol{y})\cdot\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}(\boldsymbol{y},t)\,\mathrm{d}_{\boldsymbol{y}}S\\ &\quad-\frac{1}{4\pi}\int_{\Omega}\varphi^{x,a}_{i,j}(\boldsymbol{y})v_{i,j}(\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y}=\frac{1}{4\pi}\int_{\partial\Omega}\nabla_y\varphi^{x,a}(\boldsymbol{y})\cdot\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}(\boldsymbol{y},t)\,\mathrm{d}_{\boldsymbol{y}}S\\ &\quad+\frac{1}{4\pi}\int_{\Omega}\nabla_y\Delta_y\varphi^{x,a}(\boldsymbol{y})\cdot\boldsymbol{v}(\boldsymbol{y},t)\,\mathrm{d}\boldsymbol{y}.\end{aligned}$$

This proves the statement about ∇p^{II}. The same statement about the space derivatives of ∇p^{II} can be obtained analogously. □

The conclusions of Lemma 2 and Lemma 3 imply that if at least one of the conditions (i), (i)′ is fulfilled and $\partial\boldsymbol{v}/\partial\boldsymbol{n}\in L^\beta(t_1,t_2;L^1(\partial\Omega)^3)$ for some $\beta\geqslant 2$ then ∇p has all space derivatives in $L^\beta(t_1+\zeta,t_2-\zeta;L^\infty(\Omega_2)^3)$. Using also Lemma 1 and equation (1), one can obtain the same statement about $\partial\boldsymbol{v}/\partial t$.

Thus, conditions (ii) or (ii)$'$, Lemma 1 (used with $\Omega_1 = U_r^*$ and $\Omega_2 = U^*_{(2r+\varrho)/3} - \overline{U^*_{\varrho/2}}$), the assumption that $\partial \boldsymbol{v}/\partial \boldsymbol{n} \in L^\beta(t_1, t_2; L^1(\partial\Omega)^3)$ for some $\beta \geqslant 2$ and inequality (5) imply that the function $\boldsymbol{h}$ has all space derivatives in $L^\beta(t_1+\zeta, t_2-\zeta; L^\infty(\Omega)^3)$.

We shall further assume that (ii) or (ii)$'$ holds. At the beginning, we do not have sufficient information on the integrability of $\partial \boldsymbol{v}/\partial \boldsymbol{n}$ on $\partial\Omega \times (t_1, t_2)$ and we can only derive by means of Lemma 1 that $\boldsymbol{h}$ has all space derivatives in $L^\alpha(t_1 + \zeta, t_2 - \zeta; L^\infty(\Omega)^3)$ for each $\alpha \in [1, 2)$. However, this enables us to prove a higher smoothness of $\boldsymbol{u}$ in $\Omega \times (t_1 + \zeta, t_2 - \zeta)$ (Lemma 4). It implies certain integrability of $\partial \boldsymbol{v}/\partial \boldsymbol{n}$ on $\partial\Omega \times (t_1 + \zeta, t_2 - \zeta)$ (see estimate (13) which further makes it possible (by means of Lemmas 1, 2 and 3) to improve the information on function $\boldsymbol{h}$, etc. This procedure will be repeated several times.

In the sequel, c will denote a generic constant, i.e. a constant whose value may change from line to line. It will depend on the function $\boldsymbol{u}$, but it will be always independent of time.

Lemma 4. *Let condition* (ii) *or condition* (ii)$'$ *be satisfied and let $\zeta > 0$ be such a number that $t_1 + \zeta < t_2 - \zeta$. Then $A^{1/2}\boldsymbol{u} \in L^\infty(t_1 + \zeta, t_2 - \zeta; L^2(\Omega)^3)$ and $A\boldsymbol{u} \in L^2(t_1 + \zeta, t_2 - \zeta; L^2(\Omega)^3)$.*

P r o o f. Assume that e.g. condition (ii) holds. (The case of (ii)$'$ could be treated analogously.) Suppose that $t \in (t_1 + \zeta, t_2 - \zeta) - \Gamma$. ($\Gamma$ is the set from the proof of Lemma 2.) If we multiply equation (6) by $A\boldsymbol{u}$ and integrate over Ω, we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}} \frac{1}{2} \int_\Omega |A^{1/2}\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} + \int_\Omega (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \cdot A\boldsymbol{u} \,\mathrm{d}\boldsymbol{x} + \int_\Omega |A\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{h} \cdot A\boldsymbol{u} \,\mathrm{d}\boldsymbol{x} \tag{11}$$

where

$$\begin{aligned}
\left| \int_\Omega (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \cdot A\boldsymbol{u} \,\mathrm{d}\boldsymbol{x} \right| &\leqslant \frac{1}{8} \int_\Omega |A\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} + c \int_\Omega |\boldsymbol{u}|^2 |\nabla \boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} \\
&\leqslant \frac{1}{8} \int_\Omega |A\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} + c \Big(\int_\Omega |\boldsymbol{u}|^b \,\mathrm{d}\boldsymbol{x} \Big)^{2/b} \Big(\int_\Omega |\nabla \boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} \Big)^{\frac{b-3}{b}} \Big(\int_\Omega |\nabla \boldsymbol{u}|^6 \,\mathrm{d}\boldsymbol{x} \Big)^{1/b} \\
&\leqslant \frac{1}{8} \|A\boldsymbol{u}\|_2^2 + \delta \Big(\int_\Omega |\nabla \boldsymbol{u}|^6 \,\mathrm{d}\boldsymbol{x} \Big)^{1/3} + c(\delta) \Big(\int_\Omega |\boldsymbol{u}|^b \,\mathrm{d}\boldsymbol{x} \Big)^{\frac{2}{b-3}} \Big(\int_\Omega |\nabla \boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} \Big) \\
&\leqslant \frac{1}{4} \|A\boldsymbol{u}\|_2^2 + c \Big(\int_\Omega |\boldsymbol{u}|^b \,\mathrm{d}\boldsymbol{x} \Big)^{a/b} \|A^{1/2}\boldsymbol{u}\|_2^2.
\end{aligned}$$

(δ is an appropriate positive number.) Let $0 \leqslant s < 1/4$. Then $D(A^s) = W^{2s,2}(\Omega)^3 \cap L^2_\sigma(\Omega)^3$ (see Sec. 1). Thus, $P_\sigma \boldsymbol{h}(\cdot, t) \in D(A^s)$. Let us further choose $\gamma \in (0, 1)$ and

$q \geqslant 2$ so that $2-\gamma \leqslant q$ and $3\gamma/4q \leqslant s$. Then $2q(1-\gamma)/(q-\gamma) \leqslant q$ and

$$\begin{aligned}
\left|\int_\Omega \boldsymbol{h}\cdot A\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\right| &= \left|\int_\Omega A^s P_\sigma \boldsymbol{h}\cdot A^{1-s}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\right| \leqslant \int_\Omega |A^s P_\sigma \boldsymbol{h}|^\gamma\,|A^s P_\sigma \boldsymbol{h}|^{1-\gamma}\,|A^{1-s}\boldsymbol{u}|\,\mathrm{d}\boldsymbol{x}\\
&\leqslant \|A^s P_\sigma \boldsymbol{h}\|_q^\gamma \left(\int_\Omega |A^s P_\sigma \boldsymbol{h}|^{\frac{2q(1-\gamma)}{(q-\gamma)}}\,\mathrm{d}\boldsymbol{x} + \int_\Omega |A^{1-s}\boldsymbol{u}|^{\frac{2q}{(q-\gamma)}}\,\mathrm{d}\boldsymbol{x}\right)^{\frac{(q-\gamma)}{q}}\\
&\leqslant c\|\boldsymbol{h}\|_{2s,\,q}^\gamma \|A^s P_\sigma \boldsymbol{h}\|_{2q(1-\gamma)/(q-\gamma)}^{2(1-\gamma)} + c\|\boldsymbol{h}\|_{2s,\,q}^\gamma \|A^{1-s}\boldsymbol{u}\|_{2q/(q-\gamma)}^2\\
&\leqslant c\|\boldsymbol{h}\|_{2s,\,q}^{2-\gamma} + c\|\boldsymbol{h}\|_{2s,\,q}^\gamma \|A^{1-s}\boldsymbol{u}\|_{3\gamma/2q,\,2}\\
&\leqslant c\|\boldsymbol{h}\|_{2s,\,q}^{2-\gamma} + c\|\boldsymbol{h}\|_{2s,\,q}^\gamma \|A^{1-s+3\gamma/4q}\boldsymbol{u}\|_2^2\\
&\leqslant c\|\boldsymbol{h}\|_{2s,\,q}^{2-\gamma} + c\|\boldsymbol{h}\|_{2s,\,q}^\gamma \|A^{1/2}\boldsymbol{u}\|_2^{4s-3\gamma/q}\|A\boldsymbol{u}\|_2^{2-4s+3\gamma/q}\\
&\leqslant c\|\boldsymbol{h}\|_{2s,\,q}^{2-\gamma} + \frac{1}{4}\|A\boldsymbol{u}\|_2^2 + c\|\boldsymbol{h}\|_{2s,\,q}^{2\gamma q/(4sq-3\gamma)}\|A^{1/2}\boldsymbol{u}\|_2^2.
\end{aligned}$$

Substituting this to (11), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A^{1/2}\boldsymbol{u}\|_2^2 + \|A\boldsymbol{u}\|_2^2 \leqslant c(\|\boldsymbol{u}\|_b^a + \|\boldsymbol{h}\|_{2s,\,q}^{2\gamma q/(4sq-3\gamma)})\|A^{1/2}\boldsymbol{u}\|_2^2 + c\|\boldsymbol{h}\|_{2s,\,q}^{2-\gamma}. \tag{12}$$

$\|\boldsymbol{u}\|_b^a$ is, due to condition (ii), an integrable function of t on (t_1, t_2). We can choose $\gamma \in (0,1)$ so small and $q > 2$ so large that $(1+3/q)\gamma < 4s$. Then $2\gamma q/(4sq-3\gamma) < 2$ and therefore $\|\boldsymbol{h}\|_{2s,\,q}^{2\gamma q/(4sq-3\gamma)}$ and $\|\boldsymbol{h}\|_{2s,\,q}^{2-\gamma}$ are integrable functions of t on $[t_1+\zeta, t_2-\zeta]$.

The number ζ can be chosen not only arbitrarily small, but also such that $t_1+\zeta \notin \Gamma$, i.e. $\|A^{1/2}\boldsymbol{u}(\cdot, t_1+\zeta)\|_2 < +\infty$.

Recall that inequality (12) holds for $t \in (t_1+\zeta, t_2-\zeta) - \Gamma$. It implies that $A^{1/2}\boldsymbol{u}$ and $A\boldsymbol{u}$ satisfy the statement of the lemma if $\|A^{1/2}\boldsymbol{u}\|_2$ is a left-lower and right-upper semi-continuous function of t at instants of time $t \in \Gamma$. (Or in other words, unless $\|A^{1/2}\boldsymbol{u}\|_2$ has jumps up at the time instants $t \in \Gamma$.) This would be an easy consequence of classical results about the Navier-Stokes equations (see e.g. [3] or [7]) if $\boldsymbol{h}$, in addition to its space regularity, were at least square integrable in time. However, we actually know that the function $\boldsymbol{h}$ is only integrable in time with an arbitrary exponent $\alpha \in [1,2)$. Nevertheless, we can exclude the jumps up by means of the following argument: Let $t' \in (t_1+\zeta, t_2-\zeta) \cap \Gamma$. We can choose $t_0' < t'$ arbitrarily close to t' and construct a local in time strong solution $\boldsymbol{u}'$ to the problem (6)–(8) on a time interval $(t_0', t_0'+T')$ overlapping $(t_0', t']$, such that $\boldsymbol{u}'(t_0') = \boldsymbol{u}(t_0')$. The existence of a local in time strong solution is well known—see e.g. [3] or [7] for details. In fact, we only need $\boldsymbol{u}'$ to satisfy the energy inequality and the norm $\|\nabla\boldsymbol{u}'\|_2$ to have no jumps up and such a solution can be constructed even if $\boldsymbol{h}$ is integrable in time only with an exponent strictly less than two, but arbitrarily close to two. Since $\boldsymbol{u}$ satisfies the Prodi-Serrin integrability condition, $\boldsymbol{u}$ coincides with $\boldsymbol{u}'$ on the interval $(t_0', t_0'+T')$ and therefore its norm $\|A^{1/2}\boldsymbol{u}\|_2$ has no jump up at the time instant t'. $\square$

The theorem on traces now implies that

$$(13)\qquad \left(\int_{\partial\Omega} |\nabla \boldsymbol{u}|\,\mathrm{d}S\right)^4 \leqslant c\|\boldsymbol{u}\|_{3/2,\,2}^4 \leqslant c\|A^{3/4}\boldsymbol{u}\|_2^4 + c \leqslant c\|A^{1/2}\boldsymbol{u}\|_2^2\,\|A\boldsymbol{u}\|_2^2 + c \leqslant c\|A\boldsymbol{u}\|_2^2 + c.$$

Since the right hand side is an integrable function of time on $(t_1+\zeta, t_2-\zeta)$ and $\boldsymbol{v}$ coincides with $\boldsymbol{u}$ on $\partial\Omega\times(t_1,t_2)$, we also have $\partial\boldsymbol{v}/\partial\boldsymbol{n} \in L^4(t_1+\zeta, t_2-\zeta; L^1(\partial\Omega)^3)$. Due to Lemma 2 and Lemma 3, ∇p and $\partial\boldsymbol{v}/\partial t$ have all space derivatives in $L^4(t_1+\zeta, t_2-\zeta; L^\infty(\Omega_2)^3)$ (where $\Omega_2 = U^*_{(2r+\varrho)/3} - \overline{U^*}_{\varrho/2}$). Hence $\boldsymbol{h}$ and all its space derivatives belong to $L^4(t_1+\zeta, t_2-\zeta; L^\infty(\Omega)^3)$.

Lemma 5. *Let condition* (ii) *or condition* (ii)$'$ *be fulfilled,* $0<\varepsilon\leqslant 1$ *and* $t_1+\zeta < t_2-\zeta$*. Then* $A^{1-\varepsilon}\boldsymbol{u} \in L^\infty(t_1+\zeta, t_2-\zeta; L^2(\Omega)^3)$.

Proof. We can assume without loss of generality that ζ is chosen such that $t_1+\zeta\notin\Gamma$, i.e. $\|A\boldsymbol{u}(\cdot, t_1+\zeta)\|_2 < +\infty$. Let $t\in(t_1+\zeta, t_2-\zeta)$. We will denote $t_0 = t_1+\zeta$ for simplicity. We can obviously deal only with $\varepsilon\in(0,\frac12)$. Using the integral representation of $\boldsymbol{u}(\cdot,t)$ by means of the semigroup e^{At}, we have

$$(14)\qquad A^{1-\varepsilon}\boldsymbol{u}(\cdot,t) = A^{1-\varepsilon}\mathrm{e}^{A(t-t_0)}\boldsymbol{u}(\cdot,t_0) + \int_{t_0}^t A^{1-\varepsilon}\mathrm{e}^{A(t-\tau)}P_\sigma\boldsymbol{h}(\cdot,\tau)\,\mathrm{d}\tau - \int_{t_0}^t A^{1-\varepsilon}\mathrm{e}^{A(t-\tau)}P_\sigma(\boldsymbol{u}(\cdot,\tau)\cdot\nabla)\boldsymbol{u}(\cdot,\tau)\,\mathrm{d}\tau.$$

Let us choose a number $\xi\in[0,\frac14)$ such that $\varepsilon+\xi>\frac14$. Then $4(1-\varepsilon-\xi)/3<1$ and $P_\sigma\boldsymbol{h}(\cdot,\tau)\in D(A^\xi)$ for a.a. $\tau\in(t_0,t)$. Thus, we obtain

$$(15)\qquad \left\|\int_{t_0}^t A^{1-\varepsilon}\mathrm{e}^{A(t-\tau)}P_\sigma\boldsymbol{h}(\cdot,\tau)\,\mathrm{d}\tau\right\|_2 = \left\|\int_{t_0}^t A^{1-\varepsilon-\xi}\mathrm{e}^{A(t-\tau)}A^\xi P_\sigma\boldsymbol{h}(\cdot,\tau)\,\mathrm{d}\tau\right\|_2 \leqslant c\int_{t_0}^t \frac{\|A^\xi P_\sigma\boldsymbol{h}(\cdot,\tau)\|_2}{(t-\tau)^{1-\varepsilon-\xi}}\,\mathrm{d}\tau \leqslant c\left(\int_{t_0}^t \frac{\mathrm{d}\tau}{(t-\tau)^{4(1-\varepsilon-\xi)/3}}\right)^{3/4}\left(\int_{t_0}^t \|\boldsymbol{h}(\cdot,\tau)\|_{\xi,2}^4\,\mathrm{d}\tau\right)^{1/4} \leqslant c.$$

Suppose that $\varepsilon=\frac14+\kappa$ where $\kappa\in(0,\frac14]$ for a while. (Hence $4(1-\varepsilon)/3<1$.) Using the results of Lemma 4, we can derive that

$$(16)\qquad \left\|\int_{t_0}^t A^{1-\varepsilon}\mathrm{e}^{A(t-\tau)}P_\sigma\big(\boldsymbol{u}(\cdot,\tau)\cdot\nabla\big)\boldsymbol{u}(\cdot,\tau)\,\mathrm{d}\tau\right\|_2 \leqslant \int_{t_0}^t \frac{c}{(t-\tau)^{1-\varepsilon}}\|A\boldsymbol{u}(\cdot,\tau)\|_2^{1/2}\,\mathrm{d}\tau \leqslant c\left(\int_{t_0}^t \frac{\mathrm{d}\tau}{(t-\tau)^{4(1-\varepsilon)/3}}\right)^{3/4}\left(\int_{t_0}^t \|A\boldsymbol{u}(\cdot,\tau)\|_2^2\,\mathrm{d}\tau\right)^{1/4} \leqslant c.$$

Inequalities (15) and (16), together with Lemma 4 and identity (14), imply that $A^{1-\varepsilon}\boldsymbol{u} = A^{3/4-\kappa}\boldsymbol{u} \in L^\infty(t_1+\zeta, t_2-\zeta; L^2(\Omega)^3)$.

Let $\varepsilon \in (0, \frac{1}{2})$ now. Let us choose $\kappa > 0$ so small that $1-\varepsilon < (1+2\kappa)/(1+4\kappa)$. Using the above information on $A^{3/4-\kappa}\boldsymbol{u}$, we can replace estimates (16) by

$$
\begin{aligned}
(17)\qquad &\left\| \int_{t_0}^{t} A^{1-\varepsilon} \mathrm{e}^{A(t-\tau)} P_\sigma(\boldsymbol{u}(\cdot,\tau)\cdot\nabla)\boldsymbol{u}(\cdot,\tau)\,\mathrm{d}\tau \right\|_2 \\
&\leqslant c\int_{t_0}^{t} \frac{1}{(t-\tau)^{1-\varepsilon}} \|A^{3/4}\boldsymbol{u}(\cdot,\tau)\|_2\,\mathrm{d}\tau \\
&\leqslant c\int_{t_0}^{t} \frac{1}{(t-\tau)^{1-\varepsilon}} \|A^{3/4-\kappa}\boldsymbol{u}(\cdot,\tau)\|_2^{1/(1+4\kappa)} \|A\boldsymbol{u}(\cdot,\tau)\|^{4\kappa/(1+4\kappa)}\,\mathrm{d}\tau \\
&\leqslant c\left(\int_{t_0}^{t} \frac{\mathrm{d}\tau}{(t-\tau)^{\frac{(1-\varepsilon)(1+4\kappa)}{1+2\kappa}}}\right)^{\frac{1+2\kappa}{1+4\kappa}} \left(\int_{t_0}^{t} \|A\boldsymbol{u}(\cdot,\tau)\|_2^2\,\mathrm{d}\tau\right)^{\frac{2\kappa}{1+4\kappa}} \leqslant c.
\end{aligned}
$$

The statement of the lemma follows from Lemma 4, (14), (15) and (17). □

We can now proceed similarly as after the proof of Lemma 4: We have

$$
(18)\qquad \int_{\partial\Omega} |\nabla\boldsymbol{u}|\,\mathrm{d}S \leqslant c\|\boldsymbol{u}\|_{3/2,\,2} \leqslant c\|A^{3/4}\boldsymbol{u}\|_2 + c.
$$

This estimate and Lemma 5 imply that $\partial\boldsymbol{v}/\partial\boldsymbol{n} \in L^\infty(t_1+\zeta, t_2-\zeta; L^1(\partial\Omega)^3)$. Thus, ∇p and $\partial\boldsymbol{v}/\partial t$ have all space derivatives in $L^\infty((U^*_{(2r+\varrho)/3} - \overline{U^*_{\varrho/2}}) \times (t_1+\zeta, t_2-\zeta))^3$ and consequently, $\boldsymbol{h}$ and all its space derivatives belong to $L^\infty(\Omega\times(t_1+\zeta, t_2-\zeta))^3$.

Lemma 6. *Let $\boldsymbol{g} \in L^\infty(t_1+\zeta, t_2-\zeta; W^{2-\xi,2}(\Omega)^3)$ for some $\xi \in [0, \frac{1}{2})$. Then the operator $B_t\boldsymbol{w} = (\boldsymbol{g}(\cdot,t)\cdot\nabla)\boldsymbol{w}$ is for a.a. $t \in (t_1+\zeta, t_2-\zeta)$ and for $0 \leqslant s \leqslant 1$ a continuous linear operator from $W^{s+1,2}(\Omega)^3$ into $W^{s,2}(\Omega)^3$ and the estimate*

$$
(19)\qquad \|B_t\boldsymbol{w}\|_{s,\,2} \leqslant c\|\boldsymbol{w}\|_{s+1,\,2}
$$

holds uniformly for a.a. $t \in (t_1+\zeta, t_2-\zeta]$.

Proof. It can be verified that

$$
\begin{aligned}
\|B_t\boldsymbol{w}\|_2 &\leqslant \|\boldsymbol{g}(\cdot,t)\|_{2-\xi,\,2}\|\boldsymbol{w}\|_{1,\,2} \leqslant c\|\boldsymbol{w}\|_{1,\,2}, \\
\|B_t\boldsymbol{w}\|_{1,\,2} &\leqslant c(\|\boldsymbol{g}(\cdot,t)\|_{2-\xi,\,2} + \|\boldsymbol{g}(\cdot,t)\|_{1,\,2})\|\boldsymbol{w}\|_{2,\,2} \leqslant c\|\boldsymbol{w}\|_{2,\,2}
\end{aligned}
$$

uniformly for a.a. $t \in [t_1+\zeta, t_2-\zeta]$. Hence B_t is a linear continuous operator from $[W^{2,2}(\Omega)^3, W^{1,2}(\Omega)^3]_{1-s} \equiv W^{s+1,2}(\Omega)^3$ into $[W^{1,2}(\Omega)^3, L^2(\Omega)^3]_{1-s} \equiv W^{s,2}(\Omega)^3$ and the norm of this operator can be estimated by a constant which is independent of t for a.a. $t \in [t_1+\zeta, t_2-\zeta]$. (This can be deduced e.g. from [8], p. 27.) □

Lemma 5 implies that $\boldsymbol{u} \in L^\infty(t_1+\zeta, t_2-\zeta; W^{2-\xi,2}(\Omega)^3)$ for each $\xi \in (0, \frac{1}{2})$. Hence we can use Lemma 6 with $\boldsymbol{g} = \boldsymbol{u}$ and $\boldsymbol{w} = \boldsymbol{u}(\cdot, t)$ and obtaining the estimate

$$\|(\boldsymbol{u}(\cdot,t)\cdot\nabla)\boldsymbol{u}(\cdot,t)\|_{s,2} \leqslant c\|\boldsymbol{u}(\cdot,t)\|_{s+1,2} \leqslant c\|A^{(s+1)/2}\boldsymbol{u}(\cdot,t)\|_2 \tag{20}$$

for a.a. $t \in [t_1+\zeta, t_2-\zeta]$. (Of course c depends on $\boldsymbol{u}$, but it does not matter because we work only with just one function $\boldsymbol{u}$.)

P r o o f of Theorem 1. Put $\varepsilon = \delta/2$. We can assume without loss of generality that $t_1+\zeta \notin \Gamma$, i.e. $\|A^{1+\varepsilon}\boldsymbol{u}(\cdot, t_1+\zeta)\|_2 < +\infty$. Let $t \in (t_1+\zeta, t_2-\zeta)$ and $t_0 = t_1+\zeta$. Then

$$\begin{aligned} A^{1+\varepsilon}\boldsymbol{u}(\cdot,t) = A^{1+\varepsilon}\mathrm{e}^{A(t-t_0)}\boldsymbol{u}(\cdot,t_0) &+ \int_{t_0}^t A^{1+\varepsilon}\mathrm{e}^{A(t-\tau)}P_\sigma\boldsymbol{h}(\cdot,\tau)\,\mathrm{d}\tau \\ &- \int_{t_0}^t A^{1+\varepsilon}\mathrm{e}^{A(t-\tau)}P_\sigma(\boldsymbol{u}(\cdot,\tau)\cdot\nabla)\boldsymbol{u}(\cdot,\tau)\,\mathrm{d}\tau. \end{aligned} \tag{21}$$

Let us choose ξ such that $\varepsilon < \xi < \frac{1}{4}$. Then $P_\sigma\boldsymbol{h}(\cdot,\tau) \in D(A^\xi)$ and $P_\sigma(\boldsymbol{u}(\cdot,\tau)\cdot\nabla)\boldsymbol{u}(\cdot,\tau) \in D(A^\xi)$ for a.a. $\tau \in (t_0, t)$ and

$$\begin{aligned} \Big\|\int_{t_0}^t & A^{1+\varepsilon}\mathrm{e}^{A(t-\tau)}P_\sigma(\boldsymbol{u}(\cdot,\tau)\cdot\nabla)\boldsymbol{u}(\cdot,\tau)\,\mathrm{d}\tau\Big\|_2 \\ &= \Big\|\int_{t_0}^t A^{1+\varepsilon-\xi}\mathrm{e}^{A(t-\tau)}A^\xi P_\sigma(\boldsymbol{u}(\cdot,\tau)\cdot\nabla)\boldsymbol{u}(\cdot,\tau)\,\mathrm{d}\tau\Big\|_2 \\ &\leqslant \int_{t_0}^t \frac{c}{(t-\tau)^{1+\varepsilon-\xi}}\|(\boldsymbol{u}(\cdot,\tau)\cdot\nabla)\boldsymbol{u}(\cdot,\tau)\|_{2\xi,2}\,\mathrm{d}\tau \\ &\leqslant \int_{t_0}^t \frac{c}{(t-\tau)^{1+\varepsilon-\xi}}\|A^{(2\xi+1)/2}\boldsymbol{u}(\cdot,\tau)\|_2\,\mathrm{d}\tau \\ &\leqslant \int_{t_0}^t \frac{c}{(t-\tau)^{1+\varepsilon-\xi}}\,\mathrm{d}\tau \leqslant c, \end{aligned}$$

$$\begin{aligned} \Big\|\int_{t_0}^t A^{1+\varepsilon}\mathrm{e}^{A(t-\tau)}P_\sigma\boldsymbol{h}(\cdot,\tau)\,\mathrm{d}\tau\Big\|_2 &= \Big\|\int_{t_0}^t A^{1+\varepsilon-\xi}\mathrm{e}^{A(t-\tau)}A^\xi P_\sigma\boldsymbol{h}(\cdot,\tau)\,\mathrm{d}\tau\Big\|_2 \\ &\leqslant \int_{t_0}^t \frac{c}{(t-\tau)^{1+\varepsilon-\xi}}\|A^\xi P_\sigma\boldsymbol{h}(\cdot,\tau)\|_2\,\mathrm{d}\tau \\ &\leqslant \int_{t_0}^t \frac{c}{(t-\tau)^{1+\varepsilon-\xi}}\,\mathrm{d}\tau \leqslant c. \end{aligned}$$

The statement of Theorem 1 about $\boldsymbol{v}$ now follows from these estimates, (21) and the relation between the solutions $\boldsymbol{u}$ and $\boldsymbol{v}$. The statements about $\partial\boldsymbol{v}/\partial t$ and ∇p further follow from equation (6). $\square$

P r o o f of Theorem 2. Lemma 5, estimate (18) and the coincidence of $\boldsymbol{u}$ and $\boldsymbol{v}$ in the neighborhood of $\partial\Omega$ imply that $\partial\boldsymbol{v}/\partial\boldsymbol{n} \in L^\infty(t_1+\zeta, t_2-\zeta; L^1(\partial\Omega)^3)$. The statement of Theorem 2 is now an easy consequence of Lemma 2 and Lemma 3. $\square$

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