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ASYMPTOTIC BEHAVIOUR FOR A PHASE-FIELD MODEL WITH HYSTERESIS IN ONE-DIMENSIONAL THERMO-VISCO-PLASTICITY*

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Abstract. The asymptotic behaviour for $t \to \infty$ of the solutions to a one-dimensional model for thermo-visco-plastic behaviour is investigated in this paper. The model consists of a coupled system of nonlinear partial differential equations, representing the equation of motion, the balance of the internal energy, and a phase evolution equation, determining the evolution of a phase variable. The phase evolution equation can be used to deal with relaxation processes. Rate-independent hysteresis effects in the strain-stress law and also in the phase evolution equation are described by using the mathematical theory of hysteresis operators.

Keywords: phase-field system, phase transition, hysteresis operator, thermo-visco-plasticity, asymptotic behaviour

MSC 2000: 74N30, 35B40, 47J40, 34C55, 35K60, 74K05

1. Introduction

In this paper, an initial-boundary value problem for a system of partial differential equations involving hysteresis operators is considered, and the asymptotic behaviour of the solutions to this system is investigated. The system has been derived in [25] to model one-dimensional thermo-visco-plastic developments connected with solid-solid phase transitions taking also into account the hysteresis effects appearing on the macroscopic scale as a consequence of effects on the micro- and/or mesoscale.

To model such developments, one is considering the evolution of several quantities: the displacement u, the absolute temperature θ, and a phase variable w, which is

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usually a so-called *generalized freezing index*, see [21]. For a wire of unit length, the evolution of these fields is determined by the following system:

$$\varrho u_{tt} - \mu u_{xxt} = \sigma_x + f(x,t) \quad \text{a.e. in } \Omega_\infty, \tag{1.1}$$

$$\sigma = \mathcal{H}_1[u_x, w] + \theta \mathcal{H}_2[u_x, w] \quad \text{a.e. in } \Omega_\infty, \tag{1.2}$$

$$(C_V\theta + \mathcal{F}_1[u_x, w])_t - \kappa\theta_{xx} = \mu u_{xt}^2 + \sigma u_{xt} + g(x,t,\theta) \quad \text{a.e. in } \Omega_\infty, \tag{1.3}$$

$$\nu w_t = -\psi \quad \text{a.e. in } \Omega_\infty, \tag{1.4}$$

$$\psi = \mathcal{H}_3[u_x, w] + \theta \mathcal{H}_4[u_x, w] \quad \text{a.e. in } \Omega_\infty, \tag{1.5}$$

$$u(0,t) = 0, \ \ \mu u_{xt}(1,t) + \sigma(1,t) = 0, \ \ \theta_x(0,t) = \theta_x(1,t) = 0 \quad \text{a.e. in } (0,\infty), \tag{1.6}$$

$$u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad \theta(\cdot,0) = \theta_0, \quad w(\cdot,0) = w_0 \quad \text{a.e. in } \Omega, \tag{1.7}$$

with $\Omega_\infty := \Omega \times (0,\infty)$ and $\Omega := [0,1]$.

The equation (1.1) is the equation of motion, (1.3) is the balance of internal energy, and (1.4) is the phase evolution equation. By the constitutive law (1.2), the elastoplastic stress σ is determined, and the constitutive law (1.4) defines the thermodynamic force ψ. The boundary condition (1.6) means that the wire is fixed at $x = 0$, stress-free at $x = 1$, and thermally insulated at both ends. Here, x denotes the space variable, t denotes the time, and the indices x and t denote the differentiation with respect to space and time, respectively.

The mass density ϱ, the viscosity μ, the specific heat C_V, the heat conductivity κ, and the kinetic relaxation coefficient ν are supposed to be positive constants. The initial data for the displacement, the velocity, the temperature, and the phase variable considered in (1.7) are denoted by u_0, u_1, θ_0, and w_0, respectively. Finally, the nonlinearities $\mathcal{H}_i$, $1 \leqslant i \leqslant 4$, and $\mathcal{F}_1$ are hysteresis operators (see below), where one needs to take into account $u_x(x,\cdot)|_{[0,t]}$ and $w(x,\cdot)|_{[0,t]}$ to compute $\mathcal{H}_i[u_x, w](x,t)$ and $\mathcal{F}_1[u_x, w](x,t)$.

These operators are supposed to reflect some *memory* in the material on the macroscale, resulting from effects in the micro/mesoscale. Such effects can lead to *hysteresis loops*, as they are for example observed in the macroscopic strain-stress relation (ε-σ, where $\varepsilon = u_x$ is the linearized strain) determined from measurements in uniaxial load-deformation of materials like *shape memory alloys*, see, e.g., [2], [4], [6], [7], [8], [9], [10], [30], [31], [38]. The curves show a strong dependence on the temperature, but many of them are *rate-independent*, i.e., they are independent of the speed with which they are traversed.

There are other approaches to model hysteretic behaviour by considering systems similar to parts of (1.1)–(1.5), where the operators $\mathcal{F}_1$ and $\mathcal{H}_i$, for $1 \leqslant i \leqslant 4$, are superposition operators. These models are derived by considering a free energy, which is a superposition operator, involving a potential which has (one or more) concave parts. The concave parts of the potential correspond to unstable physical states, and these instabilities are supposed to produce the observed hysteresis effects. Such approaches have successfully been used and investigated in a number of papers, see, e.g., [3], [5], [7], [9], [33], [37], [39] and the references therein, but the modelling by non-convex free energies has its limits, since a non-convex part of the potential alone does not ensure that hysteresis loops are present, see, e.g., [29]. Moreover, the simple superposition operator cannot represent all the complicated hysteresis curves that are observed in experiments.

Hence, to describe such structures, the more general *hysteresis operators* have been introduced and used in a number of papers, see, e.g., the monographs [3], [14], [15], [36] on this subject and the references therein. For a final time $T > 0$, an operator $\mathcal{H}\colon C[0,T] \to \mathrm{Map}[0,T] := \{v\colon [0,T] \to \mathbb{R}\}$ is a *hysteresis operator* if it is rate-independent and causal according to the following definitions. The operator $\mathcal{H}$ is called *rate-independent*, if for every $v \in C[0,T]$ and every continuous increasing (not necessarily strictly increasing) function $\alpha\colon [0,T] \to [0,T]$ with $\alpha(0) = 0$ and $\alpha(T) = T$ it holds that $\mathcal{H}[v \circ \alpha](t) = \mathcal{H}[v](\alpha(t))$ for all $t \in [0,T]$.

An operator $\mathcal{H}\colon D(\mathcal{H})$ ($\subseteq \mathrm{Map}[0,T]$) $\to \mathrm{Map}[0,T]$ is said to be *causal*, if for every $v_1, v_2 \in D(\mathcal{H})$ and every $t \in [0,T]$ we have the implication

$$v_1(\tau) = v_2(\tau) \quad \forall\, \tau \in [0,t] \Rightarrow \mathcal{H}[v_1](t) = \mathcal{H}[v_2](t). \tag{1.8}$$

An example of a hysteresis operator is the *stop operator*, which is also called *Prandtl's normalized elastic-perfectly plastic element.* To define the stop operator, we consider some yield limit $r > 0$, an initial stress $\sigma_r^0 \in [-r,r]$, and a final time $T > 0$. For each input function $\varepsilon \in W^{1,1}(0,T)$, we have (see, e.g., [3], [14], [15], [36]) a unique solution $\sigma_r \in W^{1,1}(0,T)$ to the variational inequality

$$\sigma_r(t) \in [-r,r] \quad \forall\, t \in [0,T], \quad \sigma_r(0) = \sigma_r^0, \tag{1.9}$$

$$(\varepsilon_t(t) - \sigma_{r,t}(t))(\sigma_r(t) - \eta) \geqslant 0 \quad \forall\, \eta \in [-r,r], \quad \text{a.e. in } (0,T). \tag{1.10}$$

This defines the stop operator $\mathcal{S}_r\colon [-r,r] \times W^{1,1}(0,T) \to W^{1,1}(0,T)\colon (\sigma_r^0, \varepsilon) \mapsto \sigma_r$. An example for the evolution of the input and the output for the stop operator is presented in Fig. 1, showing the input-output relation of $\mathcal{S}_2[0,\varepsilon]$ for an input function ε which initially increases from 0 to 5, then decreases to -6, then increases to 0, then decreases to -3, and finally increases to 6.

Connected with the stop operator $\mathcal{S}_r$ is another important hysteresis operator, the so-called *play operator* $\mathcal{P}_r$ defined by

$$\mathcal{P}_r\colon [-r,r] \times W^{1,1}(0,T) \to W^{1,1}(0,T)\colon (\sigma_r^0, \varepsilon) \mapsto \varepsilon - \mathcal{S}_r[\sigma_r^0, \varepsilon]. \tag{1.11}$$

It is well-known, see, e.g., [3], [14], [15], that the stop and the play operator can be extended to Lipschitz continuous operators on $[-r,r] \times C[0,T]$. Moreover, using the

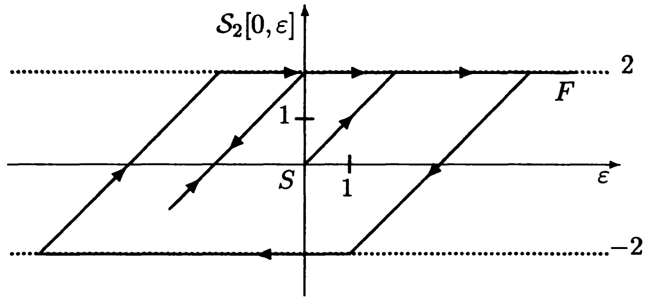

Figure 1. An example for the evolution of $(\varepsilon(t), \mathcal{S}_2[0,\varepsilon](t))$, starting in $S = (0,0)$ and finishing in $F = (6,2)$.

notation of [3, Chapter 2.5], one has for all $\sigma_r^0 \in [-r, r]$ that $\frac{1}{2}\mathcal{S}_r^2[\sigma_r^0, \cdot]$ is the *clockwise admissible potential* and $r\mathcal{P}_r[\sigma_r^0, \cdot]$ is the corresponding *dissipation operator* for the operator $\mathcal{S}_r[\sigma_r^0, \cdot]$, i.e., for all $\varepsilon \in W^{1,1}(0,T)$ it holds that

$$\left(\frac{1}{2}\mathcal{S}_r^2[\sigma_r^0, \varepsilon]\right)_t + |(r\mathcal{P}_r[\sigma_r^0, \varepsilon])_t| = \mathcal{S}_r[\sigma_r^0, \varepsilon]\varepsilon_t \quad \text{a.e. in } (0,T). \tag{1.12}$$

Let $\mathrm{Map}[0,\infty) := \{v\colon [0,\infty) \to \mathbb{R}\}$. An operator $\mathcal{H}\colon D(\mathcal{H})$ $(\subset \mathrm{Map}[0,\infty) \times \mathrm{Map}[0,\infty) \to \mathrm{Map}[0,\infty)$ is said to be *causal*, if for every $(\varepsilon_1, w_1), (\varepsilon_2, w_2) \in D(\mathcal{H})$ and every $t \geqslant 0$ we have the implication

$$\varepsilon_1(\tau) = \varepsilon_2(\tau),\ \ w_1(\tau) = w_2(\tau) \quad \forall\, \tau \in [0,t] \Rightarrow \mathcal{H}[\varepsilon_1, w_1](t) = \mathcal{H}[\varepsilon_2, w_2](t).$$

Moreover, the operator $\mathcal{H}$ generates an operator $\overline{\mathcal{H}}$ mapping (ε, w) with $\varepsilon, w\colon \Omega \times [0,\infty) \to \mathbb{R}$ such that $(\varepsilon(x,\cdot), w(x,\cdot)) \in D(\mathcal{H})$ for a.e. $x \in \Omega$ to the function on $\Omega \times [0,\infty)$ defined by $\overline{\mathcal{H}}[\varepsilon, w](x,t) = \mathcal{H}[\varepsilon(x,\cdot), w(x,\cdot)](t)$ for all $t \geqslant 0$ and for a.e. $x \in \Omega$. In the sequel, we will no longer distinguish between $\mathcal{H}$ and the generated operator $\overline{\mathcal{H}}$.

The hysteresis phenomena described by hysteresis operators are often related to changes between different configurations within the wire. In the system above, these configurations are described by the phase parameter w, and the evolution of these configurations is described by the phase evolution equation (1.4). By considering such an equation, one can take into account relaxation processes that appear in addition to the rate independent hysteresis loops, which are modelled by the hysteresis operators.

Let us recall some results for systems with hysteresis operators similar to the one above. In [11], [17], [20], [21], [23], [26], [27], a multi-dimensional phase transition is considered without taking mechanical effects into account. This corresponds to investigating (1.3)–(1.5) without a dependence on u or σ. The one-dimensional thermoelastoplastic hysteresis without considering relaxation processes in the phase transition, i.e., (1.1)–(1.3) with no dependence on w, has been studied in [16], [18].

For the complete system (1.1)–(1.7) above with an additional Ginzburg term u_{xxxx} on the left-hand side of (1.1) and boundary condition $u = u_{xx} = 0$ on $\partial\Omega$ for u, the global existence and uniqueness of a solution has been shown in [24].

The system (1.1)–(1.7) has been derived and investigated in [25]. Therein, the existence, uniqueness, and regularity of a strong solution has been proved (see Theorem 3 in Section 2.3), and it has also been shown that the Clausius-Duhem inequality and therefore the second principle of thermodynamics is satisfied for the solution.

In the present work, we are dealing with the asymptotic behaviour for $t \to \infty$ of the system under consideration. After discussing the assumptions in Section 2.1, the results are presented in Theorem 1 and Theorem 2 in Section 2.2. The a priori estimates derived in Section 3 are used in Section 4 to prove these theorems.

2. Asymptotic behaviour of solutions

2.1. Assumptions

The assumptions used in the investigation of the asymptotic behaviour of the solution to (1.1)–(1.7) are now presented and discussed. Let $C[0,\infty)$ denote the set of all continuous functions from $[0,\infty)$ to $\mathbb{R}$, including also the unbounded ones. For $t \geqslant 0$, the seminorm $|\cdot|_{[0,t]}$ on $C[0,\infty)$ and on $C[0,T]$ for $T \geqslant t$ is defined by

$$|f|_{[0,t]} = \max_{0 \leqslant s \leqslant t} |f(s)|. \tag{2.1}$$

We will use the following assumptions:

(H1) We have $u_0 \in H^2(\Omega)$, $u_1 \in W^{1,\infty}(\Omega)$, $\theta_0 \in H^1(\Omega)$, $w_0 \in H^1(\Omega)$, and there is some $\delta > 0$ such that $\theta_0(x) \geqslant \delta$ for all $x \in \overline{\Omega}$. Moreover, the compatibility condition $u_0(0) = u_1(0) = 0$ is satisfied.

(H2) We assume that $g\colon \Omega \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that there are functions $g_1, g_2\colon \Omega_\infty \to [0,\infty)$, with

$$\begin{gathered} g_1 \in L^1(\Omega_\infty) \cap L^2(\Omega_\infty), \quad g_2 \in L^1(0,\infty; L^\infty(\Omega)) \cap L^2(0,\infty; L^\infty(\Omega)), \\ |g(x,t,s) - g_1(x,t)| \leqslant g_2(x,t)s, \quad g(x,t,-s) = g_1(x,t) \quad \forall\,(x,t) \in \Omega_\infty,\ s \geqslant 0. \end{gathered}$$

(H3) The operators $\mathcal{H}_1, \dots, \mathcal{H}_4$, $\mathcal{F}_1\colon C[0,\infty) \times C[0,\infty) \to C[0,\infty)$ are causal and map $W^{1,1}_{\mathrm{loc}}(0,\infty) \times W^{1,1}_{\mathrm{loc}}(0,\infty)$ into $W^{1,1}_{\mathrm{loc}}(0,\infty)$. The operators map $C[0,T] \times C[0,T]$ continuously into $C[0,T]$ for all $T > 0$, and for all $\varepsilon, w \in C[0,\infty)$

$$\mathcal{F}_1[\varepsilon, w](t) \geqslant 0 \quad \forall\, t \geqslant 0.$$

(H4) There exist causal operators $\mathcal{F}_2\colon W^{1,1}_{\mathrm{loc}}(0,\infty)\times W^{1,1}_{\mathrm{loc}}(0,\infty)\to W^{1,1}_{\mathrm{loc}}(0,\infty)$, $\mathcal{D}_1,\mathcal{D}_2\colon W^{1,1}_{\mathrm{loc}}(0,\infty)\times W^{1,1}_{\mathrm{loc}}(0,\infty)\to L^1_{\mathrm{loc}}(0,\infty)$, $\mathcal{G}\colon W^{1,1}_{\mathrm{loc}}(0,\infty)\to W^{1,1}_{\mathrm{loc}}(0,\infty)$, and a non-decreasing function k_1 such that for all $\varepsilon,w\in W^{1,1}_{\mathrm{loc}}(0,\infty)$

i) $|\mathcal{D}_1[\varepsilon,w]| = \varepsilon_t\mathcal{H}_1[\varepsilon,w]+(\mathcal{G}[w])_t\mathcal{H}_3[\varepsilon,w]-(\mathcal{F}_1[\varepsilon,w])_t$ a.e. in $(0,\infty)$,

$|\mathcal{D}_2[\varepsilon,w]| = \varepsilon_t\mathcal{H}_2[\varepsilon,w]+(\mathcal{G}[w])_t\mathcal{H}_4[\varepsilon,w]-(\mathcal{F}_2[\varepsilon,w])_t$ a.e. in $(0,\infty)$

ii) $|(\mathcal{G}[w])_t(t)|^2\leqslant k_1(|w|_{[0,t]})w_t(t)(\mathcal{G}[w])_t(t)$ for a.e. $t\in(0,\infty)$.

(H5) We have $\mathcal{F}_{1,0},\mathcal{F}_{2,0}\in L^1(\Omega)$ such that for all $\varepsilon,w\in W^{1,1}_{\mathrm{loc}}(0,\infty;L^2(\Omega))$ with $\varepsilon(\cdot,0)=u_{0,x}$ and $w(\cdot,0)=w_0$ a.e. on Ω it holds that

$$\mathcal{F}_1[\varepsilon,w](\cdot,0)=\mathcal{F}_{1,0},\quad \mathcal{F}_2[\varepsilon,w](\cdot,0)=\mathcal{F}_{2,0}\quad \text{a.e. in } \Omega.$$

(H6) There are non-decreasing functions $k_2,k_3,k_4\colon [0,\infty)\to[0,\infty)$ such that for all $\varepsilon,w\in C[0,\infty)$

i) $\max\limits_{1\leqslant i\leqslant 4}|\mathcal{H}_i[\varepsilon,w](t)|\leqslant k_2(|\varepsilon|_{[0,t]}+|w|_{[0,t]})\quad \forall\, t\geqslant 0.$

ii) $-\mathcal{F}_2[\varepsilon,w](t)\leqslant k_3(|\varepsilon|_{[0,t]}+|w|_{[0,t]})(1+\mathcal{F}_1[\varepsilon,w](t))\quad \forall\, t\geqslant 0.$

iii) If $\varepsilon,w\in W^{1,1}_{\mathrm{loc}}(0,\infty)$ then

$$\begin{aligned}&\max_{1\leqslant i\leqslant 4}|(\mathcal{H}_i[\varepsilon,w])_t(t)|+|(\mathcal{F}_1[\varepsilon,w])_t(t)|\\ &\qquad\leqslant k_4(|\varepsilon|_{[0,t]}+|w|_{[0,t]})\big(|\varepsilon_t(t)|+\sqrt{w_t(t)(\mathcal{G}[w])_t(t)}\big)\quad \text{for a.e. } t\in(0,\infty).\end{aligned}$$

(H7) We have $f\in L^\infty(0,\infty;L^2(\Omega))$ and there exist functions $f_\infty\in L^2(\Omega)$, $F\in L^2(0,\infty;H^1(\Omega)\cap H^1(0,\infty;L^2(\Omega)\cap L^\infty(\Omega_\infty)$, and positive constants K_0, K_1 such that

$$f-f_\infty\in L^1(0,\infty;L^2(\Omega)),\quad F(x,t)=\int_1^x f(\xi,t)\,\mathrm{d}\xi\quad \text{for a.e. } (x,t)\in\Omega_\infty,$$

$$\|f_\infty\|_{L^1(\Omega)}|\varepsilon(t)|\leqslant(1-K_0)|\mathcal{F}_1[\varepsilon,w](t)|+K_1\quad \forall\,\varepsilon,w\in C[0,\infty),\ t\geqslant 0. \tag{2.2}$$

For the formulation of the remaining assumptions, we use the following notations, which are well defined by (H1):

$$\varepsilon_{0,\min}:=\min\{u_{0,x}(x)\colon x\in\overline{\Omega}\},\quad \varepsilon_{0,\max}:=\max\{u_{0,x}(x)\colon x\in\overline{\Omega}\}, \tag{2.3}$$

$$w_{0,\min}:=\min\{w_0(x)\colon x\in\overline{\Omega}\},\quad w_{0,\max}:=\max\{w_0(x)\colon x\in\overline{\Omega}\}. \tag{2.4}$$

(H8) For each $\varepsilon_\Delta>0$, there exists $\varepsilon_-\leqslant\varepsilon_{0,\min}$, $\varepsilon_+\geqslant\varepsilon_{0,\max}$, $w_\Delta>0$, $w_-\leqslant w_{0,\min}$, and $w_+\geqslant w_{0,\max}$ such that for all $\varepsilon,w\in C[0,\infty)$ and all $t\geqslant 0$,

i) If $\varepsilon(t) \geqslant \varepsilon_+$,

$$\varepsilon_{0,\min} \leqslant \varepsilon(0) \leqslant \varepsilon_{0,\max}, \qquad \varepsilon_- - \varepsilon_\Delta \leqslant \varepsilon(\tau) \leqslant \varepsilon_+ + \varepsilon_\Delta \quad \forall \tau \in [0,t], \tag{2.5}$$

$$w_{0,\min} \leqslant w(0) \leqslant w_{0,\max}, \quad w_- - w_\Delta \leqslant w(\tau) \leqslant w_+ + w_\Delta \quad \forall \tau \in [0,t], \tag{2.6}$$

hold then we have

$$\mathcal{H}_1[\varepsilon, w](t) \geqslant \|F\|_{L^\infty(\Omega_\infty)}, \quad \mathcal{H}_2[\varepsilon, w](t) \geqslant 0. \tag{2.7}$$

ii) If $\varepsilon(t) \leqslant \varepsilon_-$, (2.5), and (2.6) hold then we have

$$\mathcal{H}_1[\varepsilon, w](t) \leqslant -\|F\|_{L^\infty(\Omega_\infty)}, \quad \mathcal{H}_2[\varepsilon, w](t) \leqslant 0. \tag{2.8}$$

iii) If $w(t) \geqslant w_+$, (2.5), and (2.6) hold then we have

$$\mathcal{H}_3[\varepsilon, w](t) \geqslant 0, \quad \mathcal{H}_4[\varepsilon, w](t) \geqslant 0. \tag{2.9}$$

iv) If $w(t) \leqslant w_-$, (2.5), and (2.6) hold then we have

$$\mathcal{H}_3[\varepsilon, w](t) \leqslant 0, \quad \mathcal{H}_4[\varepsilon, w](t) \leqslant 0. \tag{2.10}$$

(H9) For every $\varepsilon, w \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ with ε and w bounded and

$$\int_0^\infty (|\mathcal{D}_1[\varepsilon, w](t)| + |\mathcal{D}_2[\varepsilon, w](t)|)\,\mathrm{d}t < \infty,$$

there exists $\varepsilon_\infty \in \mathbb{R}$ such that $\lim\limits_{t\to\infty} \varepsilon(t) = \varepsilon_\infty$.

(H10) For every ε, w as in (H9), there exists $w_\infty \in \mathbb{R}$ such that $\lim\limits_{t\to\infty} w(t) = w_\infty$.

Before the asymptotic results will be presented in Section 2.2, the above assumptions are discussed, starting with considerations concerning relations to the physical background.

Remark 2.1. Thanks to (H1), there is a positive lower bound for the initial temperature and the lower bound for g in (H2) ensures that this function does not model any further cooling at absolute zero. Considering the free energy $\mathcal{F}$, the entropy $\mathcal{S}$, and the internal energy $\mathcal{U}$ as in [25], i.e.

$$\begin{aligned}
\mathcal{F}[\varepsilon, w, \theta] &:= C_V\theta(1 - \ln(\theta)) + \mathcal{F}_1[\varepsilon, w] + \theta\mathcal{F}_2[\varepsilon, w],\\
\mathcal{S}[\varepsilon, w, \theta] &:= C_V\theta - \mathcal{F}_2[\varepsilon, w],\\
\mathcal{U}[\varepsilon, w, \theta] &:= C_V\theta + \mathcal{F}_1[\varepsilon, w],
\end{aligned}$$

the lower bound for $\mathcal{F}_1$ in (H3) yields that the internal energy is nonnegative. Moreover, the nonnegativity of the expressions on the right-hand sides of the equations in (H5) i) is combined with (H5) ii) to prove that the system (1.1)–(1.7) is thermodynamically consistent, see [25, Remark 3]. The functions $\mathcal{D}_1$ and $\mathcal{D}_2$ arising in (H4) are related to the energy dissipation during a hysteresis loop.

Remark 2.2. There are cases where the operators $\mathcal{H}_i$ are decoupled. For example, the model for phase transition without mechanical effects as studied in [11], [17], [20], [21], [23], [26] can be combined with the model considered in [16], [18], that is the thermoelastoplastic hysteresis model without relaxation processes. In that case, if one does not take into account any direct coupling between phase transitions and mechanical effects, but only a coupling via the energy balance, one ends up with the system (1.1)–(1.7) with $\mathcal{H}_1$ and $\mathcal{H}_2$ depending only on u_x, and $\mathcal{H}_3$ and $\mathcal{H}_4$ depending only on w. Moreover, one is sometimes dealing with hysteresis operators arising as the sum of a superposition operator and some well-known hysteresis operator. Hence, we will investigate decoupled $\mathcal{H}_i$ of this form. Considering causal operators $\tilde{\mathcal{H}}_1,\dots,\tilde{\mathcal{H}}_4\colon C[0,\infty)\to C[0,\infty)$ and nonnegative functions $h_1,\dots,h_4\in C^2(\mathbb{R})$, we can define the operators $\mathcal{H}_1,\dots,\mathcal{H}_4$ by setting, for all $\varepsilon, w\in C[0,\infty)$ and all $t\geqslant 0$,

$$\mathcal{H}_i[\varepsilon,w](t) := \begin{cases} h_i'(\varepsilon(t)) + \tilde{\mathcal{H}}_i[\varepsilon](t) & \text{for } i=1,2,\\ h_i'(w(t)) + \tilde{\mathcal{H}}_i[w](t) & \text{for } i=3,4.\end{cases} \tag{2.11}$$

For $1\leqslant i\leqslant 4$, we assume that we have a *clockwise admissible potential* and the corresponding *dissipation operator* for $\tilde{\mathcal{H}}_i$, i.e. (see [3, Chapter 2.5]), we assume that we have a causal operator $\tilde{\mathcal{F}}_i\colon C[0,\infty)\to C[0,\infty)$ which is mapping $W^{1,1}_{\rm loc}(0,\infty)$ in $W^{1,1}_{\rm loc}(0,\infty)$ and a causal operator $\tilde{\mathcal{D}}_i\colon W^{1,1}_{\rm loc}(0,\infty)\to L^1_{\rm loc}(0,\infty)$ with

$$|\tilde{\mathcal{D}}_i[v]| = v_t\tilde{\mathcal{H}}_i[v] - (\tilde{\mathcal{F}}_i[v])_t \quad \text{a.e. in } (0,\infty), \quad \forall\, v\in W^{1,1}_{\rm loc}[0,\infty). \tag{2.12}$$

Then (H4) holds with $\mathcal{G}$ being the identity and $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{D}_1$, $\mathcal{D}_2$ defined by

$$\mathcal{F}_j[\varepsilon,w](t) := h_j(\varepsilon(t)) + \tilde{\mathcal{F}}_j[\varepsilon](t) + h_{j+2}(w(t)) + \tilde{\mathcal{F}}_{j+2}[w](t), \tag{2.13}$$

$$\mathcal{D}_j[\varepsilon,w](t) := |\tilde{\mathcal{D}}_j[\varepsilon](t)| + |\tilde{\mathcal{D}}_{j+2}[w](t)|, \tag{2.14}$$

for all $\varepsilon, w\in C[0,\infty)$, $t\geqslant 0$, and $j\in\{1,2\}$.

If $h_1(r) = h_1^* r^2$ with some positive constant h_1^* then the corresponding operator $\mathcal{H}_1$ models a linear elasticity with a hysteretic modification.

Remark 2.3. A sufficient condition for (H8) to be satisfied is that the two following assumptions (H11) and (H12) hold. These assumptions are especially useful, if the operators $\mathcal{H}_1, \dots, \mathcal{H}_4$ are decoupled as in the Remarks 2.2, 2.5–2.6. The notation of an *outward pointing* operator used in these assumptions is introduced and discussed in [13].

The more general formulation in (H8) is helpful, if the operators are coupled, e.g., if they are derived from multi-dimensional stop or Prandtl-Ishlinskii operators (see, e.g., [15], [21], [22], [23]).

(H11) For each $\varepsilon_\Delta > 0$, there exists $\varepsilon_- \leqslant \varepsilon_{0,\min}$ and $\varepsilon_+ \geqslant \varepsilon_{0,\max}$ such that for all $w \in C[0,\infty)$ with $w_{0,\min} \leqslant w(0) \leqslant w_{0,\max}$ the operator mapping $\varepsilon \in C[0,\infty)$ to $\mathcal{H}_1[\varepsilon, w] \in C[0,\infty)$ is *pointing outwards with bound* $\|F\|_{L^\infty(\Omega_\infty)}$ *in the* ε_Δ*-neighbourhood of* $[\varepsilon_-, \varepsilon_+]$ *for initial values in* $[\varepsilon_{0,\min}, \varepsilon_{0,\max}]$ and that the same holds for $\mathcal{H}_2$ just with bound 0, that is to say for all $\varepsilon \in C[0,\infty)$ and all $t \geqslant 0$ holds:
 i) If $\varepsilon(t) \geqslant \varepsilon_+$ and (2.5) hold then we have (2.7).
 ii) If $\varepsilon(t) \leqslant \varepsilon_-$ and (2.5) hold then we have (2.8).

(H12) There are $w_\Delta > 0$, $w_- \leqslant w_{0,\min}$, and $w_+ \geqslant w_{0,\max}$ such that for all $\varepsilon \in C[0,\infty)$ with $\varepsilon_{0,\min} \leqslant \varepsilon(0) \leqslant \varepsilon_{0,\max}$ the operators $C[0,\infty) \ni w \mapsto \mathcal{H}_3[\varepsilon, w]$ and $C[0,\infty) \ni w \mapsto \mathcal{H}_4[\varepsilon, w]$ are pointing outwards with bound 0 in the w_Δ-neighbourhood of $[w_-, w_+]$ for initial values in $[w_{0,\min}, w_{0,\max}]$, that is to say for all $w \in C[0,\infty)$ and $t \geqslant 0$ it holds that:
 i) If $w(t) \geqslant w_+$ and (2.6) hold then we have (2.9).
 ii) If $w(t) \leqslant w_-$ and (2.6) hold then we have (2.10).

Remark 2.4. If we use $\tilde{\mathcal{H}}_3 = \tilde{\mathcal{H}}_4 \equiv 0$ in Remark 2.2 then $\mathcal{H}_3$ and $\mathcal{H}_4$ are superposition operators and the assumption (H12) holds if and only if there are $w_\Delta > 0$, $w_- \leqslant w_{0,\min}$, and $w_+ \geqslant w_{0,\max}$ such that
- For all $s \in [w_+, w_+ + w_\Delta]$ holds $h_3'(s) \geqslant 0$, $h_4'(s) \geqslant 0$.
- For all $s \leqslant [w_- - w_\Delta, w_-]$ holds $h_3'(s) \leqslant 0$, $h_4'(s) \leqslant 0$.

A similar condition has been used in [1], [32], [33]. If this condition is directly adapted to hysteresis operators, one ends up with an assumption similar to (H12), but with the condition (2.6) replaced by $w_- - w_\Delta \leqslant w(t) \leqslant w_+ + w_\Delta$ only. This assumption is stronger than (H12) and will be denoted by (H12+). There are important hysteresis operators satisfying (H12), but not (H12+).

In a similar way, one can consider a stronger version (H11+) of (H11), where $\varepsilon_- - \varepsilon_\Delta \leqslant \varepsilon(t) \leqslant \varepsilon_+ + \varepsilon_\Delta$ is used instead of (2.5).

Remark 2.5. If for the functions and operators in Remark 2.2 there are positive constants $K_{2,1}, \dots, K_{2,4}$ such that

$$|\tilde{\mathcal{H}}_i[v](t)| \leqslant K_{2,i} \quad \forall t \geqslant 0,\ v \in C[0,\infty),\ 1 \leqslant i \leqslant 4, \tag{2.15}$$

$$\pm \lim_{r \to \pm\infty} h_1'(r) > K_{2,1} + \|F\|_{L^\infty(\Omega_\infty)}, \tag{2.16}$$

$$\pm \lim_{r \to \pm\infty} h_j'(r) > K_{2,j} \quad \forall 2 \leqslant j \leqslant 4, \tag{2.17}$$

then the assumptions (H11+) and (H12+) are satisfied. Hence, (H11), (H12), and (H8) hold. Moreover, the condition (2.2) in (H7) is satisfied if the other assumptions in (H7) hold.

Remark 2.6. For $1 \leqslant i \leqslant 4$, we consider a nonnegative weight function $\varphi_i \in L^1(0,\infty)$ and a function $\sigma_i^0 \in W^{1,\infty}(0,\infty)$ such that $\sigma_i^0(r) \in [-r,r]$ for all $r \geqslant 0$, $|(\sigma_i^0)_r| \leqslant 1$ a.e. on $(0,\infty)$, and $\sigma_r^0(r') = 0$ for all $r' \geqslant R_i$ for some $R_i > 0$. Moreover, we consider yield limits $r_{i,j} \in \mathbb{R}$, initial values $\sigma_{i,j}^0 \in [-r_{i,j}, r_{i,j}]$, and weights $\varphi_{i,j} > 0$. Now, we define $\tilde{\mathcal{H}}_i \colon C[0,\infty) \to C[0,\infty)$ as the *Prandtl-Ishlinskii operator*

$$\tilde{\mathcal{H}}_i[v] := \int_0^\infty \varphi_i(r)\mathcal{S}_r[\sigma_i^0(r), v]\,\mathrm{d}r + \sum_j \varphi_{i,j}\mathcal{S}_{r_{i,j}}[\sigma_{i,j}^0, v] \quad \forall v \in C[0,\infty). \tag{2.18}$$

The more general definition of this operator involing a Stieljes integral, see, e.g. [15], would allow to write this sum as one integral. A clockwise admissible potential for this operator is defined by $\tilde{\mathcal{F}}_i \colon C[0,\infty) \to C[0,\infty)$ with

$$\tilde{\mathcal{F}}_i[v] := \frac{1}{2}\int_0^\infty \varphi_i(r)\mathcal{S}_r^2[\sigma_i^0(r), v]\,\mathrm{d}r + \frac{1}{2}\sum_j \varphi_{i,j}\mathcal{S}_{r_{i,j}}^2[\sigma_{i,j}^0, v] \tag{2.19}$$

for all $v \in C[0,\infty)$ since Proposition 2.5.5 in [3] and (1.12) yield that (2.12) holds for

$$\tilde{\mathcal{D}}_i[v] := \left|\frac{\partial}{\partial t}\int_0^\infty r\varphi_i(r)\mathcal{P}_r[\sigma_r^0, v]\,\mathrm{d}r\right| + \sum_j \varphi_{i,j}|(r\mathcal{P}_r[\sigma_{i,j}^0, v])_t| \tag{2.20}$$

for all $v \in W_{\mathrm{loc}}^{1,1}[0,\infty)$. Defining now $\mathcal{H}_i$ and $\mathcal{F}_i$ as in Remark 2.2, and using well-known properties of the stop operator one can show that (H3)–(H6) hold.

Since for oscillations that are smaller then the yield limit of a play operator, the operator stays constant after the first oscillation, we can apply (2.14) and (2.20) to deduce that (H9) holds, if and only if for all $s > 0$ the function $\varphi_1 + \varphi_2$ does not vanish a.e. on $[0,s]$. For (H10), we get an analogous condition, just with $\varphi_1 + \varphi_2$ replaced by $\varphi_3 + \varphi_4$. If one wants to ensure as in Remark 2.2 that (H11) and (H12) are satisfied, one has to require that 2.15 holds, which is equivalent to the condition

$$\int_0^\infty r\varphi_i(r)\,\mathrm{d}r + \sum_j \varphi_{i,j}r_{i,j} < K_{2,i} < +\infty \quad \forall\, 1 \leqslant i \leqslant 4. \tag{2.21}$$

If this condition is satisfied, we see that (H11) and (H12) hold for appropriate functions h_i, but this argumentation can not be applied if $\mathcal{H}_i = \tilde{\mathcal{H}}_i$ for some $i \in \{1,\dots,4\}$.

In [13], it is proved that (H12) holds for $\mathcal{H}_3 := \tilde{\mathcal{H}}_3$ and $\mathcal{H}_4 := \tilde{\mathcal{H}}_4$, independently of (2.21). Moreover, it is shown there that for $\mathcal{H}_1 := \tilde{\mathcal{H}}_1$ the condition in (H11) holds if and only if $\int_0^\infty r\varphi_1(r)\,\mathrm{d}r = \infty$, and that an analogous equivalence holds for $\mathcal{H}_2 := \tilde{\mathcal{H}}_2$.

2.2. The asymptotic result

The following two theorems are the main result of this paper:

Theorem 1. *Assume that* (H1)–(H8) *are satisfied. Moreover, assume that* (u, θ, w) *is a solution to* (1.1)–(1.7) *such that*

$$u \in H^2_{\mathrm{loc}}(0, \infty; L^2(\Omega)) \cap H^1_{\mathrm{loc}}(0, \infty; H^2(\Omega)), \tag{2.22}$$

$$\theta \in H^1_{\mathrm{loc}}(0, \infty; L^2(\Omega)) \cap L^2_{\mathrm{loc}}(0, \infty; H^2(\Omega)), \tag{2.23}$$

$$w \in H^2_{\mathrm{loc}}(0, \infty; L^2(\Omega)) \cap H^1_{\mathrm{loc}}(0, \infty; H^2(\Omega)), \tag{2.24}$$

$$\theta(x, t) > 0 \quad \forall x \in \overline{\Omega}, \ t \geqslant 0. \tag{2.25}$$

Then, it holds that

$$\lim_{t \to \infty} \|u_{xt}(\cdot, t)\|_{L^2(\Omega)} = 0, \quad \lim_{t \to \infty} \|u_t(\cdot, t)\|_{C(\overline{\Omega})} = 0, \tag{2.26}$$

$$\sigma(\cdot, t) \to -F_\infty \ \text{as} \ t \to \infty, \quad \text{in} \ L^2(\Omega), \tag{2.27}$$

$$\lim_{t \to \infty} \|\theta_x(\cdot, t)\|_{L^2(\Omega)} = 0, \quad \lim_{t \to \infty} \|\theta(\cdot, t) - \overline{\theta}(t)\|_{C(\overline{\Omega})} = 0, \tag{2.28}$$

with

$$F_\infty(x) := \int_1^x f_\infty(\xi)\,\mathrm{d}\xi, \quad \overline{\theta}(t) := \int_\Omega \theta(y, t)\,\mathrm{d}y \quad \forall x \in \overline{\Omega}, \ t \geqslant 0. \tag{2.29}$$

In addition, we have a constant $\theta_* > 0$ *such that*

$$\theta(x, t) \geqslant \theta_* \quad \forall x \in \overline{\Omega}, \ t \geqslant 0. \tag{2.30}$$

Remark 2.7. We see that (2.26) yields that for $t \to \infty$ the viscous part of the stress tends to zero, and by (2.27) the stress tends to $-F_\infty$, which is the potential corresponding to the limit f_∞ for $t \to \infty$ of the applied force f. Moreover, by (2.28), we see that the temperature becomes more and more uniform in space. It is an open questions whether one can show convergence for θ, u_x, or w under the general assumptions of the theorem or if oscillations can appear up to $t \to \infty$.

Also in [33], where the system (1.1)–(1.3) with $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{F}_1$ just being nonlinear superposition operators of u_x has been considered, convergence for θ and u_x could only been proved by using additional assumptions. Corresponding additional conditions are required here in part b) and c) of Theorem 2 below, and allow to show the convergence of the temperature for $t \to \infty$. If, in addition, $\mathcal{H}_2$ and $\mathcal{H}_4$ are special operators, like, e.g. stop operators, one could also show some convergence for u and w, by adapting the argument in [33, Lemma 4.5] to the more general situation considered here.

Now, convergence results are presented that can be proved using additional hypotheses.

Theorem 2. *Assume that the assumptions of Theorem 1 are satisfied.*

a) *If $\mathcal{G}$ is the identity operator, then we have*

$$\lim_{t\to\infty} \|w_t(\cdot,t)\|_{L^2(\Omega)} = 0, \quad \lim_{t\to\infty} \|\psi(\cdot,t)\|_{L^2(\Omega)} = 0, \tag{2.31}$$

$$\lim_{t\to\infty} \|(\mathcal{F}_1[u_x,w])_t(\cdot,t)\|_{L^2(\Omega)} = \sum_{i=1}^{4} \lim_{t\to\infty} \|(\mathcal{H}_i[u_x,w])_t(\cdot,t)\|_{L^2(\Omega)} = 0. \tag{2.32}$$

b) *If $\mathcal{H}_1 \equiv \mathcal{H}_3 \equiv \mathcal{F}_1 \equiv 0$, $g \equiv 0$, and $f \equiv 0$, then we have*

$$\theta(\cdot,t) \to \|\theta_0\|_{L^1(\Omega)} + \frac{\varrho}{2C_V}\|u_1\|^2_{L(\Omega)} \quad \textit{as } t\to\infty, \quad \textit{in } L^\infty(\Omega), \tag{2.33}$$

$$\lim_{t\to\infty} \|\mathcal{H}_2[u_x,w](\cdot,t)\|_{L^2(\Omega)} = 0. \tag{2.34}$$

c) *If $\mathcal{H}_1 \equiv \mathcal{H}_3 \equiv \mathcal{F}_1 \equiv 0$, $g \equiv 0$, $f \equiv 0$, and $\mathcal{G}$ is the identity operator, then we have*

$$\lim_{t\to\infty} \|\mathcal{H}_4[u_x,w](\cdot,t)\|_{L^2(\Omega)} = 0. \tag{2.35}$$

d) *If* (H9) *holds then there exists a $u_\infty \in W^{1,\infty}(\Omega)$ such that*

$$u(\cdot,t) \to u_\infty \quad \textit{as } t\to\infty, \quad \textit{weakly-star in } W^{1,\infty}(\Omega), \tag{2.36}$$

$$u_x(\cdot,t) \to u_{\infty,x} \quad \textit{as } t\to\infty, \quad \textit{a.e. in } \Omega. \tag{2.37}$$

e) *If* (H10) *holds then there exists a $w_\infty \in L^\infty(\Omega)$ such that*

$$w(\cdot,t) \to w_\infty \quad \textit{as } t\to\infty, \quad \textit{weakly-star in } L^\infty(\Omega) \quad \textit{and a.e. in } \Omega. \tag{2.38}$$

Remark 2.8. If (H8) does not hold then one can still prove the results in Theorem 1 and some of the results in Theorem 2, if some other additional assumptions are satisfied.

i) If (H4) and (H6) with $k_1, \dots, k_4$ replaced by positive constants hold then one can still show the results in Theorem 1 and the results in Theorem 2 a)–c) hold.
ii) If (H11), (H4) ii) with k_1 replaced by a positive constant, and (H6) without the $|w|_{[0,t]}$-term in the evaluation of k_2, k_3, k_4 hold then one can prove that the results in Theorem 1 and the results in Theorem 2 a)–d) hold.
iii) If (H12) and (H6) without the $|\varepsilon|_{[0,t]}$-term in the evaluation of k_2, k_3, k_4 hold then one can prove the results in Theorem 1 and the results in Theorem 2 a)–c) and e) hold.

2.3. Existence of solutions

Before proving the asymptotic result, it will be recalled that there is a solution to the problem under consideration satisfying the regularity and positivity demands presented in Theorem 1, at least if some additional assumptions are satisfied. These assumptions will be

(H13) $f \in H^1_{\text{loc}}(0,\infty; L^2(\Omega))$.

(H14) The function g_1 arising in (H2) satisfies $g_1 \in L^\infty_{\text{loc}}(\Omega_\infty)$ and for every $T > 0$ there is a positive constant $K_{3,T}$ such that $|\partial g/\partial\theta| \leqslant K_{3,T}$ a.e. in $\Omega \times (0,T) \times \mathbb{R}$.

(H15) For every $T > 0$ there are positive constants $K_{4,T}, \ldots, K_{7,T}$ and non-decreasing functions $k_{5,T}, k_{6,T}\colon [0,\infty) \to [0,\infty)$ such that for all $\varepsilon, \varepsilon_1, \varepsilon_2, w, w_1, w_2 \in C[0,\infty)$ the following holds:

i) We have for all $t \in [0,T]$:

$$\begin{gathered}|\mathcal{H}_2[\varepsilon,w](t)| + |\mathcal{H}_4[\varepsilon,w](t)| \leqslant K_{4,T},\\ \max_{1\leqslant i\leqslant 4} |\mathcal{H}_i[\varepsilon_1,w_1](t) - \mathcal{H}_i[\varepsilon_2,w_2](t)| \leqslant K_{5,T}(|\varepsilon_1-\varepsilon_2|_{[0,t]} + |w_1-w_2|_{[0,t]}).\end{gathered}$$

ii) If $\varepsilon, \varepsilon_1, \varepsilon_2, w, w_1, w_2 \in W^{1,1}_{\text{loc}}(0,\infty)$ then the inequality in (H4) ii) with $k_1(|w|_{[0,t]})$ replaced by $K_{6,T}$ holds for a.e. $t \in (0,T)$ and

$$\max_{1\leqslant i\leqslant 4} |(\mathcal{H}_i[\varepsilon,w])_t(t)| \leqslant K_{7,T}(|\varepsilon_t(t)| + |w_t(t)|) \quad \text{for a.e. } t \in (0,T),$$

$$|\mathcal{F}_1[\varepsilon,w])_t(t)| \leqslant k_{5,T}(|\varepsilon|_{[0,t]} + |w|_{[0,t]})(|\varepsilon_t(t)| + |w_t(t)|) \quad \text{for a.e. } t \in (0,T), \tag{2.39}$$

$$\begin{aligned}|\mathcal{F}_1[\varepsilon_1,w_1](t) - \mathcal{F}_1[\varepsilon_2,w_2](t) &\leqslant k_{6,T}(|\varepsilon_1|_{[0,t]} + |\varepsilon_2|_{[0,t]} + |w_1|_{[0,t]} + |w_2|_{[0,t]})\\ &\quad\times \Big(|\varepsilon_1(0) - \varepsilon_2(0)| + |w_1(0) - w_2(0)|\\ &\quad + \int_0^t (|\varepsilon_{1,t}(\tau) - \varepsilon_{2,t}(\tau)| + |w_{1,t}(\tau) - w_{2,t}(\tau)|)\,\mathrm{d}\tau\Big)\\ &\qquad \forall t \in [0,T].\end{aligned} \tag{2.40}$$

One can extend Theorem 2.1 in [25] to the following result:

Theorem 3. *Assume that* (H1)–(H3), (H4) i), *and* (H13)–(H15) *are valid. Then the system* (1.1)–(1.7) *has a unique strong solution* (u,θ,w) *such that* (2.22)–(2.24) *hold. This solution also satisfies* (2.25).

The original existence result in [25] has been formulated with a stronger version of the assumption (H15), where $k_{5,T}(\cdots)$ in (2.39) and $k_{6,T}(\cdots)$ in (2.40) are replaced by positive constants. Combining this stronger assumption with (H4) i) and the continuity of $\mathcal{F}_1$ on $C[0,T] \times C[0,T]$ (see (H3)), it follows that $\mathcal{H}_1$ and $\mathcal{H}_3$ have to be uniformly bounded. However, uniform boundedness is not satisfied in many important situations, e.g., if $\mathcal{H}_1$, defined as in (2.11), is modelling a linear elasticity with a bounded hysteretic modification as in Remark 2.2. Using the assumption (H15) allows to apply the existence result above also in this situation. In [24], the authors of [25] consider a hypothesis analogous to (H15) for a modified version of the system (1.1)–(1.7).

We now sketch the proof of Theorem 3: We observe that, in the global existence proof in [25], the stronger versions of (2.39) and (2.40) are applied *after* the uniform estimates for u_x and w have been derived. To perform the a priori estimates, it suffices to use just (2.39) and (2.40). Moreover, (2.39) and (2.40) also suffice for the local existence result in [25, Section 3], as can be seen from a careful examination of the proof. Details can be found in the forthcoming paper [12]. Therein, it is also shown that one can replace the boundedness of $\mathcal{H}_2$ and $\mathcal{H}_4$, as assumed in (H15) i), by the hypothesis for $\mathcal{F}_2$ in (H6) i). One is then able to consider the case where one assumes (H11) for $\mathcal{H}_2$ consisting of Prandtl-Ishlinskii operators depending only on ε. In this case, $\mathcal{H}_2$ is unbounded, see Remark 2.6.

Remark 2.9. For nonnegative functions $h_1, \dots, h_4 \in C^2(\mathbb{R})$ with $h_1'', h_3'' \in L^\infty(\mathbb{R})$, $h_2', h_4' \in W^{1,\infty}(\mathbb{R})$, and operators $\tilde{\mathcal{H}}_1, \dots, \tilde{\mathcal{H}}_4$ as in Remark 2.6 with nonnegative weight functions $\varphi_1, \dots, \varphi_4 \in L^1(0,\infty)$ satisfying (2.21) one can use well-known properties of the stop operator (see, e.g., [3], [14], [15], [36]) to show that (H15) holds.

3. Uniform a priori estimates

In this section, it will be assumed that (H1)–(H8) are satisfied and that a solution (u, θ, w) to (1.1)–(1.7) is given, such that (2.22)–(2.25) hold. To prepare the proof of the asymptotic results in the next section, some a priori estimates are derived that are uniform with respect to time.

Before this is done, we consider the energy balance and derive an immediate consequence:

Remark 3.1. Multiplying (1.1) by u_t and adding the result to the balance law (1.3) for the internal energy, we get the balance law for the energy

$$(3.1)\quad \left(C_V\theta + \frac{\varrho}{2}u_t^2 + \mathcal{F}_1[u_x, w]\right)_t - \kappa\theta_{xx} = (u_t(\mu u_{tx} + \sigma))_x + g + u_t f \quad \text{a.e. in } \Omega_\infty.$$

For $t > 0$, we integrate this equation over $\Omega \times (0, t)$, and use Green's formula, (1.6), (1.7), (H1), and (H5), to show that

$$C_V \overline{\theta}(t) + \frac{\varrho}{2}\|u_t(\cdot, t)\|^2_{L^2(\Omega)} = I_0 + I_1(t) \quad \forall\, t \geqslant 0 \tag{3.2}$$

holds for the $\overline{\theta}$ defined in (2.29),

$$I_0 := C_V \|\theta_0\|_{L^1(\Omega)} + \frac{\varrho}{2}\|u_1\|^2_{L^2(\Omega)} + \int_\Omega \mathcal{F}_{1,0}(x)\,\mathrm{d}x > 0, \tag{3.3}$$

$$\begin{aligned} I_1(t) := {} & \int_0^t \int_\Omega (g(x, \tau, \theta(x, \tau)) + u_t(x, \tau) f(x, \tau))\,\mathrm{d}x\,\mathrm{d}\tau \\ & - \int_\Omega (\mathcal{F}_1[u_x, w](x, t))\,\mathrm{d}x \quad \forall\, t \geqslant 0. \end{aligned} \tag{3.4}$$

In the sequel, for $1 \leqslant p < \infty$, the notation $\|\cdot\|_p$ will be used as an abbreviation for the $L^p(\Omega)$-norm, and $\|\cdot\|_\infty$ will denote the $C(\overline{\Omega})$-norm, i.e., the maximum norm on $\overline{\Omega}$. Moreover, C_i, for $i \in \mathbb{N}$, will always denote generic positive constants, independent of time, space, and the considered solution.

Thanks to (2.22)–(2.25) and (H3), we can assume without losing generality that σ and ψ are continuous (maybe unbounded) functions on $\overline{\Omega}_\infty = \overline{\Omega} \times [0, \infty)$, such that (1.2) and (1.5) hold for all $(x, t) \in \overline{\Omega}_\infty$. Because of (1.7), (2.3), (2.4), we can apply the assumption (H8) for $\varepsilon(\cdot) := u_x(x, \cdot)$ and $w(\cdot) := w(x, \cdot)$. For the sake of notational convenience, we assume in the remaining part of this section without losing generality that $\varrho = \mu = C_V = \kappa = \nu = 1$.

In the following estimates, some ideas from [25], [33], [35] are used.

Lemma 3.2. *There are two positive constants C_1, C_2 such that*

$$\sup_{0 \leqslant t}(\|\theta(\cdot, t)\|_1 + \|u_t(\cdot, t)\|_2 + \|\mathcal{F}_1[u_x, w](\cdot, t)\|_1) \leqslant C_1, \tag{3.5}$$

$$\int_0^\infty (\|g(\cdot, t, \theta(\cdot, t))\|_1 + \|g(\cdot, t, \theta(\cdot, t))\|_1^2)\,\mathrm{d}t \leqslant C_2. \tag{3.6}$$

Proof. Let

$$\Psi(t) := \int_\Omega (\mathcal{F}_1[u_x, w](x, t) - f_\infty(x) u(x, t) + K_1)\,\mathrm{d}x \quad \forall\, t \geqslant 0. \tag{3.7}$$

Now, we get from (3.2) by using (2.29), (2.25), (3.3), (3.4), Hölder's inequality, Young's inequality, (H1), (H2), (H5), and (H7) that for all $t \geqslant 0$

$$(3.8)\qquad \Big(\|\theta(\cdot,t)\|_1 + \frac{1}{2}\|u_t(\cdot,t)\|_2^2 + \Psi(t)\Big)$$
$$< C_3 + \int_0^t (\|g_2(\cdot,t)\|_\infty \|\theta(\cdot,\tau)\|_1 + \|g_1(\cdot,\tau)\|_1)\,\mathrm{d}\tau$$
$$+ \frac{1}{2}\int_0^t (\|f(\cdot,\tau) - f_\infty\|_2 + \|f(\cdot,\tau) - f_\infty\|_2\,\|u_t(\cdot,\tau)\|_2^2)\,\mathrm{d}\tau.$$

By (3.7), Hölder's inequality, (1.6), (H3), and (H7), we have

$$\Psi(t) \geqslant K_0\|\mathcal{F}_1[u_x,w](\cdot,t)\|_1 \quad \forall\, t \geqslant 0.$$

Hence, because of (3.8), we can apply Gronwall's Lemma, (H2), and (H7) to show that (3.5) and (3.6) are satisfied. □

To prepare the following estimates, we now consider the transformation due to Andrews [1], which is also used, e.g., in [32], [33], [25], and introduce functions $p, q, \tilde\sigma\colon \overline{\Omega}_\infty \to \mathbb{R}$ that are defined by

$$(3.9)\qquad p(x,t) := \int_1^x u_t(\xi,t)\,\mathrm{d}\xi, \quad q(x,t) := u_x(x,t) - p(x,t) \quad \forall\,(x,t) \in \overline{\Omega}_\infty,$$
$$(3.10)\qquad \tilde\sigma(x,t) := \sigma(x,t) + F(x,t) \quad \forall\,(x,t) \in \overline{\Omega}_\infty,$$

with F as in (H7). Recalling (1.1)–(1.7) and (H7), we see that

$$(3.11)\qquad p_t - p_{xx} = \tilde\sigma \quad \text{a.e. in } \Omega_\infty,$$
$$(3.12)\qquad p(1,t) = p_x(0,t) = 0 \quad \text{a.e. in } (0,T),$$
$$(3.13)\qquad p(x,0) = \int_1^x u_1(\xi)\,\mathrm{d}\xi \quad \text{a.e. in } \Omega,$$
$$(3.14)\qquad q_t = -\tilde\sigma \quad \text{a.e. in } \Omega,$$
$$(3.15)\qquad q(x,0) = u_{0,x}(x) - \int_1^x u_1(\xi)\,\mathrm{d}\xi \quad \text{a.e. in } \Omega.$$

Lemma 3.3. *There are positive constant C_4, C_5 such that*

$$(3.16)\qquad \sup_{0\leqslant t}(\|p_x(\cdot,t)\|_2 + \|p(\cdot,t)\|_\infty) \leqslant C_4,$$
$$(3.17)\qquad \sup_{0\leqslant t}(\|u_x(\cdot,t)\|_\infty + \|w(\cdot,t)\|_\infty + \|u(\cdot,t)\|_\infty + \|q(\cdot,t)\|_\infty) \leqslant C_5.$$

Proof. In the light of the estimate for u_t in (3.5) and the definition of p in (3.9), we see that (3.16) holds. Considering (H8) for $\varepsilon_\Delta := 2C_4 + 1$, we get $\varepsilon_- < \varepsilon_{0,\min}$,

$\varepsilon_{0,\max} < \varepsilon_+$, $w_- < w_{0,\min}$, and $w_+ > w_{0,\max}$ such that the remaining conditions in (H8) are satisfied. Now,

$$u_x(x,t) \in [\varepsilon_- - 2C_4, \varepsilon_+ + 2C_4], \quad w(x,t) \in [w_-, w_+] \quad \forall\,(x,t) \in \overline{\Omega}_\infty \tag{3.18}$$

is proved by contradiction. Suppose that (3.18) does not hold. Then there is some $\delta \in (0, \min\{w_\Delta, 1\})$ such that $u_x \leqslant \varepsilon_- - 2C_4 - \delta$ and/or $u_x \geqslant \varepsilon_+ + 2C_4 + \delta$ and/or $w \leqslant w_- - \delta$ and/or $w \geqslant w_+ + \delta$ somewhere in $\overline{\Omega}_\infty$. We have $u_x(x,0) = u_{0,x}(x) \in [\varepsilon_-, \varepsilon_+]$ and $w(x,0) = w_0(x) \in [w_-, w_+]$ for all $x \in \overline{\Omega}$ because of (2.3) and (2.4). Since (2.22) and (2.24) yield that w and u_x are continuous on $\overline{\Omega}_\infty$, we get $x_1 \in \overline{\Omega}$, $t_1 > 0$ such that

$$\begin{cases} u_x(x_1,t_1) \in \{\varepsilon_- - 2C_4 - \delta, \varepsilon_+ + 2C_4 + \delta\} \\ \text{and/or } w(x_1,t_1) \in \{w_+ + \delta, w_- - \delta\}, \end{cases} \tag{3.19}$$

$$\varepsilon_- - 2C_4 - \delta < u_x(x,t) < \varepsilon_+ + 2C_4 + \delta \quad \forall\, t \in [0,t_1), \ x \in \overline{\Omega}, \tag{3.20}$$

$$\varepsilon_- - 2C_4 - \delta \leqslant u_x(x,t_1) \leqslant \varepsilon_+ + 2C_4 + \delta \quad \forall\, x \in \overline{\Omega}, \tag{3.21}$$

$$w_- - \delta < w(x,t) < w_+ + \delta \quad \forall\, t \in [0,t_1), \ x \in \overline{\Omega}, \tag{3.22}$$

$$w_- - \delta \leqslant w(x,t_1) \leqslant w_+ + \delta \quad \forall\, x \in \overline{\Omega}. \tag{3.23}$$

Hence, we see that (2.5) with $\varepsilon := u_x(x,\cdot)$ and (2.6) with $w := w(x,\cdot)$ hold for all $x \in \overline{\Omega}$ and $t \leqslant t_1$, and it remains only to check the first condition in (H8) i)–iv) if one wants to apply one the corresponding inequalities (2.7)–(2.10). Since u_x and w are uniformly continuous on $\overline{\Omega} \times [0,t_1]$, there is some open neighborhood $U \subset \overline{\Omega}$ of x_1 such that

$$|u_x(x,t) - u_x(x_1,t)| + |w(x,t) - w(x_1,t)| \leqslant \frac{\delta}{8} \quad \forall\, x \in U, \ t' \in [0,t_1]. \tag{3.24}$$

Now, we consider the case $u_x(x_1,t_0) = \varepsilon_+ + 2C_4 + \delta$. Since u_x is continuous on $\overline{\Omega} \times [0,t_1]$ and $u_x(x_1,0) \leqslant \varepsilon_+$, we get some $t_0 \in (0,t_1)$ such that

$$\varepsilon_+ + \frac{\delta}{2} = u_x(x_1,t_0), \quad \varepsilon_+ + \frac{\delta}{2} < u_x(x_1,t) < \varepsilon_+ + 2C_4 + \delta \quad \forall\, t \in (t_0,t_1). \tag{3.25}$$

Combining this with (3.24), we conclude that $u_x(x,t) \geqslant \varepsilon_+$ for all $x \in U$, $t \in (t_0,t_1)$. In the light of (2.7) in (H8) i), we see that

$$\|F\|_{L^\infty(\Omega_\infty)} \leqslant \mathcal{H}_1[u_x,w](x,t), \quad 0 \leqslant \mathcal{H}_2[u_x,w](x,t) \quad \forall\, x \in U, \ t \in (t_0,t_1). \tag{3.26}$$

Applying (1.2) and the fact that $\theta > 0$ on Ω_∞ by (2.25), we observe that $\sigma \geqslant -F$ a.e. in $U \times (t_0,t_1)$. Thanks to (3.14) and (3.10), we deduce that $q_t \leqslant 0$ a.e. in $U \times (t_0,t_1)$. This leads to

$$\int_U (q(x,t_1) - q(x,t_0))\,\mathrm{d}x\,\mathrm{d}\tau = \int_U\int_{t_0}^{t_1} q_t(x,t)\,\mathrm{d}t\,\mathrm{d}x \leqslant 0.$$

On the other hand, using (3.9), (3.16), (3.24), (3.25), and $u_x(x_1, t_0) = \varepsilon_+ + \frac{1}{2}\delta$, we conclude that

$$\begin{aligned}\int_U (q(x,t_1) - q(x,t_0))\,\mathrm{d}x &\geqslant \int_U (u_x(x,t_1) - C_4 - (u_x(x,t_0) + C_4))\,\mathrm{d}x \\ &\geqslant \int_U \Big(u_x(x_1,t_1) - \frac{\delta}{8} - \Big(u_x(x_1,t_0) + \frac{\delta}{8}\Big) - 2C_4\Big)\,\mathrm{d}x \\ &\geqslant \int_U \frac{\delta}{4}\,\mathrm{d}x > 0.\end{aligned}$$

Hence, we have derived a contradiction. By an analogous argument, we get a contradiction if $u_x(x_1, t_1) = \varepsilon_- - 2C_4 - \delta$.

Now, we will deal with the case of $w(x_1, t_1) = w_+ + \delta$. Applying the continuity of w, we get some $t_0 \in (0, t_1)$ such that

$$w(x_1,t_0) = w_+ + \frac{\delta}{2}, \quad w_+ + \frac{\delta}{2} < w(x_1,t) < w_+ + \delta \quad \forall\, t \in (t_0,t_1). \tag{3.27}$$

Combining this with (3.24), we see that $w(x,t) \geqslant w_+$ for all $x \in U$, $t \in (t_0, t_1)$. Therefore, we conclude from (2.9) in (H8) iii) that

$$\mathcal{H}_3[u_x,w](x,t) \geqslant 0, \quad \mathcal{H}_4[u_x,w](x,t) \geqslant 0 \quad \forall\, x \in U, \;\; t \in (t_0,t_1). \tag{3.28}$$

Since $\theta > 0$ a.e. on Ω_∞ by (2.25), we deduce now from (1.5) and (1.4) that $w_t \leqslant 0$ a.e. in $U \times (t_0, t_1)$. This leads to

$$\int_U (w(x,t_1) - w(x,t_0))\,\mathrm{d}x = \int_U\int_{t_0}^{t_1} w_t(x,t)\,\mathrm{d}t\,\mathrm{d}x \leqslant 0.$$

Since $w(x_1, t_1) = w_+ + \delta$, (3.27), and (3.24) yield that the integral on the left-hand side has to be positive, we have derived a contradiction. An analogous argument to get a contradiction can be used if $w(x_1, t_1) = w_- - \delta$.

Hence, we have derived a contradiction for all cases we have to consider by (3.19). Therefore, we have proved (3.18). Recalling (1.6) and (3.9), we get also uniform bounds for u and q, and (3.17) is proved. □

Remark 3.4. Because of (3.17), we have uniform bounds for u_x and w. Thanks to (H6), (3.5), (1.2), (1.5), and (1.4), we see that there are positive constants C_6,

$C_7, \dots, C_9$ such that

$$\max_{1 \leqslant i \leqslant 4} \sup_{0 \leqslant t} (\|\mathcal{H}_i[u_x, w](\cdot, t)\|_\infty) \leqslant C_6, \tag{3.29}$$

$$|\sigma| + |w_t| \leqslant C_7(1 + \theta) \quad \text{a.e. in } \Omega_\infty, \tag{3.30}$$

$$0 \leqslant \sup_{0 \leqslant t} \int_0^1 (-\mathcal{F}_2[u_x, w](x, t))\, \mathrm{d}x \leqslant C_8, \tag{3.31}$$

$$\max_{1 \leqslant i \leqslant 4} |(\mathcal{H}_i[u_x, w])_t| + |(\mathcal{F}_1[u_x, w])_t| \leqslant C_9\bigl(|u_{xt}| + \sqrt{w_t(\mathcal{G}[w])_t}\bigr) \quad \text{a.e. in } \Omega_\infty. \tag{3.32}$$

Since (3.17) and (H4) ii) yield that $0 \leqslant w_t(\mathcal{G}[w])_t \leqslant C_{10} w_t^2$ a.e. in Ω_∞, we deduce that

$$\max_{1 \leqslant i \leqslant 4} |(\mathcal{H}_i[u_x, w])_t| + |(\mathcal{F}_1[u_x, w])_t| \leqslant C_{11}(|u_{xt}| + |w_t|) \quad \text{a.e. in } \Omega_\infty. \tag{3.33}$$

We apply (H4) i), (1.2), (1.5), (1.4), and (H4) ii) to conclude that, a.e. on Ω_∞, it holds that

$$\begin{aligned} (\mathcal{F}_1[u_x, w])_t - \sigma(x,t) u_{xt} &= (\mathcal{G}[w])_t \mathcal{H}_3[u_x, w] - |\mathcal{D}_1[u_x, w]| - \theta \mathcal{H}_2[u_x, w] u_{xt} \\ &= -|(\mathcal{G}[w])_t w_t| - |\mathcal{D}_1[u_x, w]| \\ &\quad - \theta(\mathcal{H}_2[u_x, w] u_{xt} + (\mathcal{G}[w])_t \mathcal{H}_4[u_x, w]). \end{aligned} \tag{3.34}$$

Lemma 3.5. *We have a positive constant C_{12} such that*

$$\begin{aligned} &\int_0^\infty \left(\left\| \frac{\theta_x}{\theta}(\cdot, t) \right\|_2^2 + \left\| \frac{u_{xt}}{\sqrt{\theta}}(\cdot, t) \right\|_2^2 + \left\| \frac{(\mathcal{G}[w])_t w_t}{\theta}(\cdot, t) \right\|_1 \right) \mathrm{d}t \\ &\qquad + \int_0^\infty \|\mathcal{D}_2[u_x, w](\cdot, t)\|_1 \, \mathrm{d}t + \sup_{0 \leqslant t} \|\ln \theta(\cdot, t))\|_1 \leqslant C_{12}. \end{aligned} \tag{3.35}$$

Proof. Testing (1.3) by $-1/\theta$ and using (1.6), (3.34), (H2), and (H4) i), we observe that

$$\begin{aligned} &-\frac{\partial}{\partial t} \int_\Omega \ln \theta(x,t)\, \mathrm{d}x + \int_\Omega \left(\left(\frac{\theta_x(x,t)}{\theta(x,t)} \right)^2 + \frac{u_{xt}^2(x,t)}{\theta(x,t)} \right) \mathrm{d}x \\ &\quad \leqslant -\frac{\partial}{\partial t} \int_\Omega \mathcal{F}_2[u_x, w](x,t)\, \mathrm{d}x - \int_\Omega \frac{|(\mathcal{G}[w])_t(x,t) w_t(x,t)| + |\mathcal{D}_1[u_x, w](x,t)|}{\theta(x,t)} \mathrm{d}x \\ &\qquad + \int_\Omega (-|\mathcal{D}_2[u_x, w](x,t)| + |g_2(x,t)|)\, \mathrm{d}x. \end{aligned}$$

Now, we integrate this equation over time and observe that (3.35) follows by applying (3.31), (H2), (H5), (3.5), and the inequality $|\ln s| \leqslant s - \ln s$ for all $s > 0$, which can be proved by elementary analysis. $\square$

Lemma 3.6. *We have a positive constant C_{13} such that*

$$\int_0^\infty \big(\|u_{xt}(\cdot,t)\|_1^2 + \|u_t(\cdot,t)\|_\infty^2 + \|p(\cdot,t)\|_\infty^2 + \|(\mathcal{G}[w])_t(\cdot,t)\|_1^2 \\ + \|(\mathcal{F}_1[u_x,w])_t\|_1^2 + \big\|(\sqrt{\theta})_x(\cdot,1)\big\|_1^2\big)\,\mathrm{d}t \leqslant C_{13}. \tag{3.36}$$

Proof. Since $\theta > 0$ a.e. on Ω_∞, we can apply Schwarz's inequality and (3.5) to show that for all $t > 0$

$$\|u_{xt}(\cdot,t)\|_1 = \int_\Omega \frac{|u_{xt}(x,t)|}{\sqrt{\theta(x,t)}}\sqrt{\theta(x,t)}\,\mathrm{d}x \leqslant C_{14}\left\|\frac{u_{xt}}{\sqrt{\theta}}(\cdot,t)\right\|_2. \tag{3.37}$$

Recalling now (3.35) leads to the estimate for u_{xt} in (3.36). Using that, by (1.6) and (2.22), $u_t(y,t) = \int_0^y u_{xt}(x,t)\,\mathrm{d}x$ for all $y \in \overline{\Omega}$, we get the estimate for u_t. Combining this estimate with (3.9) leads to the estimate for p.

Applying (3.32), (H4) ii), (3.35), and Young's inequality, we deduce that

$$\int_0^\infty \left(\left\|\frac{(\mathcal{G}[w])_t}{\sqrt{\theta}}(\cdot,t)\right\|_2^2 + \left\|\frac{(\mathcal{F}_1[u_x,w])_t}{\sqrt{\theta}}(\cdot,t)\right\|_2^2\right)\mathrm{d}t \leqslant C_{15}.$$

Considering now (3.37) with u_{xt} replaced by $(\mathcal{G}[w])_t$, we get the estimate for $(\mathcal{G}[w])_t$ in (3.36), and the estimate for $(\mathcal{F}_1[u_x,w])_t$ is derived analogously. Thanks to Schwarz's inequality, we have

$$\big\|(\sqrt{\theta})_x(\cdot,t)\big\|_1 = \int_\Omega \frac{|\theta_x(x,t)|}{\sqrt{\theta(x,t)}}\,\mathrm{d}x \leqslant \left\|\frac{|\theta_x|}{\theta}(\cdot,t)\right\|_2 \big\|\sqrt{\theta}(\cdot,t)\big\|_2.$$

In the light of (3.5) and (3.35), we see that also the estimate for $\sqrt{\theta}_x$ in (3.36) is established. □

Lemma 3.7. *For $\overline{\theta}$ and I_1 as in (2.29) and (3.4) there are positive constant C_{16}, C_{17}, and C_{18} such that*

$$|I_1(t)| \leqslant C_{16}, \quad C_{17} < \overline{\theta}(t) < C_{18} \quad \forall\, t \geqslant 0, \tag{3.38}$$

$$\|\theta(\cdot,t) - \overline{\theta}(t)\|_\infty \leqslant \|\theta_x(\cdot,t)\|_1 \leqslant \|\theta_x(\cdot,t)\|_2 \quad \forall\, t \geqslant 0. \tag{3.39}$$

Proof. Combining (3.4), (3.6), (1.7), (3.5), and Hölder's inequality, we see that

$$|I_1(s)| \leqslant C_{19} + \left|\int_\Omega (u(x,s) - u_0(x))f_\infty(x)\,\mathrm{d}x\right| + \int_0^s \|f(\cdot,t) - f_\infty(t)\|_2\,\|u_t(\cdot,t)\|_2\,\mathrm{d}t.$$

Recalling (3.17), (3.5), (H7), and (H1), we get the uniform bound for I_1 in (3.38). Since $s \mapsto -\ln s$ is a convex function on $(0,\infty)$, we get by (2.25) and Jensen's inequality that

$$-\ln \int_\Omega \theta(x,t)\,\mathrm{d}x \leqslant -\int_\Omega \ln(\theta(x,t))\,\mathrm{d}x \quad \forall\, t \geqslant 0.$$

Invoking now (3.35), (2.29), and (3.5), we get (3.38). The first inequality in (3.39) follows from the definition in (2.29), and the second by applying Schwarz's inequality and $\int_\Omega 1\,\mathrm{d}x = 1$. □

Lemma 3.8. *We have a positive constant C_{20} such that*

$$\begin{aligned}(3.40) \qquad \int_0^\infty \Big(\|\theta_x(\cdot,t)\|_2^2 + \Big\|\frac{\partial}{\partial x}((u_t)^2)(\cdot,t)\Big\|_2^2 + \Big(\frac{\partial I_1(t)}{\partial t}\Big)^2\Big)\,\mathrm{d}t \\ + \sup_{0\leqslant t}(\|u_t(\cdot,t)\|_4 + \|\theta(\cdot,t)\|_2) \leqslant C_{20}.\end{aligned}$$

Proof. We test (3.1) by $\theta + \frac{1}{2}u_t^2$ and (1.1) by αu_t^3 where $\alpha > 0$ will be fixed later. Summing the resulting equations and using (1.6) and (3.4), we observe that for all $t \geqslant 0$

$$\begin{aligned}(3.41) \qquad \frac{1}{2}\frac{\partial}{\partial t}\Big\|\theta(\cdot,t) + \frac{1}{2}u_t^2(\cdot,t)\Big\|_2^2 + \|\theta_x(\cdot,t)\|_2^2 + \frac{\alpha}{4}\frac{\partial}{\partial t}\|u_t(\cdot,t)\|_4^4 \\ + (1+3\alpha)\|u_t(\cdot,t)u_{tx}(\cdot,t)\|_2^2 \\ \leqslant \overline{\theta}(t)\frac{\partial I_1(t)}{\partial t} + I_2(t) + I_3(t) + I_4(t),\end{aligned}$$

with

$$\begin{aligned}(3.42) \qquad I_2(t) := \int_\Omega (-(\mathcal{F}_1[u_x,w])_t(x,t) + g(x,t,\theta(x,t)) + u_t(x,t)f(x,t)) \\ \times (\theta(x,t) - \overline{\theta}(t))\,\mathrm{d}x,\end{aligned}$$

$$\begin{aligned}(3.43) \qquad I_3(t) := -\int_\Omega \Big(\frac{1}{2}(\mathcal{F}_1[u_x,w])_t u_t^2 + 2\theta_x u_t u_{tx} + u_t\sigma\theta_x \\ + (1+3\alpha)u_t^2 u_{tx}\sigma\Big)\,\mathrm{d}x,\end{aligned}$$

$$(3.44) \qquad I_4(t) := \int_\Omega (g + (1+2\alpha)u_t f)\frac{1}{2}u_t^2\,\mathrm{d}x.$$

In the sequel, the generic constants C_i will be independent of α. We estimate the left-hand side of (3.42) by using Hölder's inequality, (H7), (3.39), and Young's inequality, resulting in

$$
\begin{aligned}
(3.45)\quad I_2(t) &\leqslant (\|(\mathcal{F}_1[u_x,w])_t(\cdot,t)\|_1 + \|g(\cdot,t,\theta(\cdot,t))\|_1 + \|u_t(\cdot,t)\|_\infty \|f(\cdot,t)\|_1) \\
&\quad \times \|\theta(\cdot,t) - \overline{\theta}(t)\|_\infty \\
&\leqslant C_{21}(\|(\mathcal{F}_1[u_x,w])_t(\cdot,t)\|_1^2 + \|g(\cdot,t,\theta(\cdot,t))\|_1^2 + \|u_t(\cdot,t)\|_\infty^2) \\
&\quad + \frac{1}{6}\|\theta_x(\cdot,t)\|_2^2.
\end{aligned}
$$

Invoking (3.43), (3.33), (3.30), Hölder's inequality, and Young's inequality, we deduce that

$$
\begin{aligned}
(3.46)\quad I_3(t) &\leqslant C_{22}((1+\alpha)\|u_{tx}(\cdot,t)u_t^2(\cdot,t)\|_1 + \|u_t^2(\cdot,t)\|_1 + \|u_t^2(\cdot,t)\theta(\cdot,t)\|_1) \\
&\quad + 2\|\theta_x(\cdot,t)u_t(\cdot,t)u_{tx}(\cdot,t)\|_1 + C_{23}\|u_t(\cdot,t)\theta_x(\cdot,t)\|_1 \\
&\quad + C_{24}\|\theta_x(\cdot,t)u_t(\cdot,t)\theta(\cdot,t)\|_1 + (1+\alpha)C_{25}\|u_t^2(\cdot,t)u_{tx}(\cdot,t)\theta(\cdot,t)\|_1 \\
&\leqslant C_{26}\|u_t(\cdot,t)u_{tx}(\cdot,t)\|_2^2 + C_{27}(1+\alpha^2)\|u_t(\cdot,t)\|_2^2 \\
&\quad + C_{28}(1+\alpha^2)\|u_t(\cdot,t)\|_\infty^2\|\theta(\cdot,t)\|_2^2 + \frac{1}{6}\|\theta_x(\cdot,t)\|_2^2.
\end{aligned}
$$

Using (3.44), (H2), Hölder's inequality, (3.5), (H7), (3.39), (3.38), and Young's inequality, we conclude that

$$
\begin{aligned}
(3.47)\quad 2I_4(t) &\leqslant \|g_1(\cdot,t)\|_2\|u_t(\cdot,t)\|_2\|u_t(\cdot,t)\|_\infty \\
&\quad + (\overline{\theta}(t) + \|\theta(\cdot,t) - \overline{\theta}(t)\|_\infty)\|g_2(\cdot,t)\|_\infty\|u_t(\cdot,t)\|_2^2 \\
&\quad + (1+2\alpha)\|u_t(\cdot,t)\|_2\|f(\cdot,t)\|_2\|u_t(\cdot,t)\|_\infty^2 \\
&\leqslant \frac{1}{6}\|\theta_x\|_2^2 + C_{29}(\|g_1(\cdot,t)\|_2^2 + \|g_2(\cdot,t)\|_\infty + \|g_2(\cdot,t)\|_\infty^2) \\
&\quad + C_{30}(1+\alpha^2)\|u_t(\cdot,t)\|_\infty^2.
\end{aligned}
$$

Because of (3.2) and Young's inequality, we have

$$
(3.48)\quad \overline{\theta}(t)\frac{\partial I_1(t)}{\partial t} \leqslant I_0\frac{\partial I_1(t)}{\partial t} + \frac{1}{2}\frac{\partial}{\partial t}(I_1(t))^2 + \frac{1}{4}\|u_t(\cdot,t)\|_2^4 + \frac{1}{4}\left(\frac{\partial I_1(t)}{\partial t}\right)^2.
$$

From (3.4), we get by using Hölder's inequality, Young's inequality, (H7), and (H2) that

$$
(3.49)\quad \left(\frac{\partial I_1(t)}{\partial t}\right)^2 \leqslant C_{31}(\|g(\cdot,t,\theta(\cdot,t))\|_1^2 + \|u_t(\cdot,t)\|_2^2 + \|\mathcal{F}_1[u_x,w])_t(\cdot,t)\|_1^2).
$$

Now, we integrate the sum of (3.41) and (3.49) over time, and use (1.7), (H1), (3.45)–(3.49), (3.6), (H2), (3.5), (3.36), (3.38), and $\theta > 0$ a.e. on Ω to show that

$$\frac{1}{2}\|\theta(\cdot,s)\|_2^2 + \frac{\alpha}{4}\|u_t(\cdot,s)\|_4^4 \\ + \int_0^s \left(\frac{1}{2}\|\theta_x(\cdot,t)\|_2^2 + (1+3\alpha)\|u_t(\cdot,t)u_{tx}(\cdot,t)\|_2^2 + \frac{3}{4}\left(\frac{\partial I_1(t)}{\partial t}\right)^2\right) dt \\ \leqslant C_{32}\left(1+\alpha^2 + \int_0^s (\|u_t(\cdot,t)u_{tx}(\cdot,t)\|_2^2 + (1+\alpha^2)\|u_t(\cdot,t)\|_\infty^2\|\theta(\cdot,t)\|_2^2)\,dt\right)$$

holds for all $s > 0$. Next, we define $\alpha := C_{32}$, apply Gronwall's Lemma, and recall (3.36) to show that (3.40) is satisfied. □

Lemma 3.9. *There are positive constants C_{33}, C_{34} such that*

$$\int_0^\infty (\|u_{xt}(\cdot,t)\|_2^2 + \|(\mathcal{G}[w])_t(\cdot,t)w_t(\cdot,t)\|_1 + \|\mathcal{D}_1[u_x,w](\cdot,t)\|_1)\,dt \leqslant C_{33}, \tag{3.50}$$

$$\int_0^\infty (\|p_{xx}(\cdot,t)\|_2^2 + \|(p+q)_t(\cdot,t)\|_2^2 + \|u_t(\cdot,t)\|_\infty^2 + \|(\mathcal{F}_1[u_x,w])_t(\cdot,t)\|_2^2 \\ + \sum_{i=1}^4 \|(\mathcal{H}_i[u_x,w])_t(\cdot,t)\|_2^2 + \|(\mathcal{G}[w])_t(\cdot,t)\|_2^2)\,dt \leqslant C_{34}. \tag{3.51}$$

Proof. Integrating (1.3) over Ω, and applying (1.6), (2.29), (3.34), and (H4) ii), we derive

$$\|u_{xt}(\cdot,t)\|_2^2 \leqslant \frac{\partial\overline{\theta}(t)}{\partial t} + \|g(\cdot,t,\theta(\cdot,t))\|_1 - \|(\mathcal{G}[w])_t(\cdot,t)w_t(\cdot,t)\|_1 - \|\mathcal{D}_1[u_x,w](\cdot,t)\|_1 \\ - \int_\Omega (\theta(x,t)-\overline{\theta}(t))(\mathcal{H}_2[u_x,w](x,t)u_{xt}(x,t) + (G[w])_t(x,t)\mathcal{H}_4[u_w,w](x,t))\,dx \\ - \overline{\theta}(t)\frac{\partial}{\partial t}\int_\Omega \mathcal{F}_2[u_x,w](x,t)\,dx.$$

We multiply this inequality by $1/\overline{\theta}(t)$ and use (3.29), Hölder's inequality, and Young's inequality to prove

$$\frac{1}{\overline{\theta}(t)}(\|u_{xt}(\cdot,t)\|_2^2 + \|(G[w])_t(\cdot,t)w_t(\cdot,t)\|_1 + \|\mathcal{D}_1[u_x,w](\cdot,t)\|_1) \\ \leqslant \frac{\partial \ln\overline{\theta}(t)}{\partial t} + \frac{1}{\overline{\theta}(t)}\|g(\cdot,t,\theta(\cdot,t))\|_1 - \frac{\partial}{\partial t}\int_\Omega \mathcal{F}_2[u_x,w](x,t)\,dx \\ + \frac{C_{35}}{\overline{\theta}(t)}(\|\theta(\cdot,t)-\overline{\theta}(t)\|_\infty^2 + \|u_{xt}(\cdot,t)\|_1^2 + \|(G[w])_t(\cdot,t)\|_1^2).$$

Integrating this inequality over time, and using (3.6), (3.38), (3.39), (3.36), and (3.40), we observe that (3.50) is proved. The estimates in (3.51) follow by applying (3.9), (1.6), (3.32), (H4), and (3.17). □

Lemma 3.10. *There is a positive constant C_{36} such that*

$$\int_0^\infty (\|\tilde{\sigma}(\cdot,\tau)\|_2^2 + \|p_t(\cdot,\tau)\|_2^2)\,\mathrm{d}\tau \leqslant C_{36}. \tag{3.52}$$

Proof. Let $J(x,t)\colon \Omega_\infty \to \mathbb{R}$ be defined by

$$\begin{aligned} J(x,t) := \tilde{\sigma}(x,t) + \mathcal{H}_2[u_x,w](x,t)\Big(\overline{\theta}(t) - \theta(x,t) + \frac{1}{2}\|u_t(\cdot,t)\|_2^2 \\ + \int_1^x (f_\infty(\xi) - f(\xi,t))\,\mathrm{d}\xi\Big) \quad \text{a.e. in } \Omega. \end{aligned} \tag{3.53}$$

Using (3.11) two times, we get

$$\begin{aligned} (\tilde{\sigma}(x,t))^2 &= p_t(x,t)\tilde{\sigma}(x,t) - p_{xx}(x,t)\tilde{\sigma}(x,t) \\ &= p_t(x,t)J(x,t) + (\tilde{\sigma}(x,t) + p_{xx}(x,t))(\tilde{\sigma}(x,t) - J(x,t)) \\ &\quad - p_{xx}(x,t)\tilde{\sigma}(x,t). \end{aligned}$$

Integrating this equation over Ω, and using Young's inequality, (3.53), (3.29), and (3.39), we observe that

$$\begin{aligned} \frac{1}{2}\|\tilde{\sigma}(\cdot,t)\|_2^2 \leqslant \frac{\partial}{\partial t}\int_\Omega p(x,t)J(x,t)\,\mathrm{d}x - \int_\Omega p(x,t)\frac{\partial J(x,t)}{\partial t}\,\mathrm{d}x \\ + C_{37}(\|p_{xx}(\cdot,t)\|_2^2 + \|\theta_x(\cdot,t)\|_2 + \|u_t(\cdot,t)\|_2^4 \\ + \|f(\cdot,t) - f_\infty(\cdot,t)\|_1^2). \end{aligned} \tag{3.54}$$

Applying (3.53), (3.10), (1.2), (H7), and (3.2), we observe that

$$J(x,t) = \mathcal{H}_1[u_x,w](x,t) + \mathcal{H}_2[u_x,w](x,t)(I_1(t) + I_0) + \int_1^x f_\infty(\xi)\,\mathrm{d}\xi. \tag{3.55}$$

Hence, using (3.29), (3.38), (H7), Hölder's inequality, Young's inequality, (3.36), (3.51), and (3.40), we get uniform bounds for J and, for all $s \geqslant 0$,

$$-\int_0^s\int_\Omega p(x,t)\frac{\partial J(x,t)}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^s \left(\|p(\cdot,t)\|_\infty^2 + \left\|\frac{\partial J(\cdot,t)}{\partial t}\right\|_2^2\right)\mathrm{d}t \leqslant C_{38}.$$

Integrating now (3.54) with respect to time and using (3.16), (3.51), (3.40), (3.5), (3.36), and (H7), we have shown the estimate for $\tilde{\sigma}$ in (3.52). Combining this estimate with (3.11) and (3.51), we get the estimate for p_t. □

Lemma 3.11. *Let $\zeta \in L^2_{\mathrm{loc}}(0,\infty;H^2(\Omega)) \cap H^1_{\mathrm{loc}}(0,\infty;L^2(\Omega))$ be the solution to the parabolic initial-boundary value problem*

$$\zeta_t - \zeta_{xx} = \tilde{\sigma}_t \quad \text{a.e. in } \Omega_\infty, \tag{3.56}$$

$$\zeta_x(0,t) = \zeta(1,t) = 0 \quad \forall\, t \geqslant 0, \;\; \zeta(\cdot,0) \equiv 0. \tag{3.57}$$

Then we have a positive constant C_{39} such that, for all $t \geqslant 0$,

$$\|\zeta(\cdot,t)\|_\infty^2 \leqslant C_{39}\left(1 + \max_{0\leqslant\tau\leqslant t} \|\theta(\cdot,\tau)\|_\infty^{3/2} + \left(\int_0^t \|\theta_t(\cdot,\tau)\|_2^2\,\mathrm{d}\tau\right)^{3/4}\right). \tag{3.58}$$

Proof. Multiplying (3.56) by ζ, integrating over $\Omega \times (0,T)$, performing partial integrations, and using (3.57), we get for all $t > 0$

$$\begin{aligned}\frac{1}{2}\|\zeta(\cdot,t)\|_2^2 &+ \int_0^t \|\zeta_x(\cdot,\tau)\|_2^2\,\mathrm{d}\tau \\ &= \int_0^t\int_\Omega \tilde{\sigma}_t(x,\tau)\zeta(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ &= \int_\Omega \tilde{\sigma}(x,t)\zeta(x,t)\,\mathrm{d}x - \int_0^t\int_\Omega \tilde{\sigma}(x,\tau)\zeta_t(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned} \tag{3.59}$$

Because of (3.10), (3.30), (3.40), and (H7), we have a uniform upper bound for $\|\tilde{\sigma}(\cdot,t)\|_2$. Hence, we get from (3.59) by applying Hölder's inequality, Young's inequality, and (3.52) that

$$\frac{1}{4}\|\zeta(\cdot,t)\|_2^2 + \int_0^t \|\zeta_x(\cdot,\tau)\|_2^2\,\mathrm{d}\tau \leqslant C_{40}\left(\int_0^t \|\zeta_t(\cdot,\tau)\|_2^2\,\mathrm{d}\tau\right)^{1/2}. \tag{3.60}$$

Formally, we test (3.56) with ζ_t, use (3.57), integrate over time, and apply Young's inequality to deduce that

$$\int_0^t \|\zeta_t(\cdot,\tau)\|_2^2\,\mathrm{d}\tau + \|\zeta_x(\cdot,t)\|_2^2 \leqslant \frac{1}{2}\int_0^t \|\zeta_t(\cdot,\tau)\|_2^2\,\mathrm{d}\tau + \frac{1}{2}\int_0^t \|\tilde{\sigma}_t(\cdot,\tau)\|_2^2\,\mathrm{d}\tau. \tag{3.61}$$

For a rigorous derivation of this inequality, one has to consider (3.56) with $\tilde{\sigma}_t$ replaced by some smooth approximation, perform this computation for the corresponding solutions, and consider afterwards the limit.

Inserting (3.60) into the left-hand side of (3.61) and using (3.10), (1.2), (3.36), Hölder's inequality, Young's inequality, (3.29), (3.51), (H6), and (H7), we observe that

$$(3.62)\qquad \frac{1}{2\cdot 4^2 C_{40}^2}\|\zeta(\cdot,t)\|_2^4 + \|\zeta_x(\cdot,t)\|_2^2$$
$$\leqslant \frac{1}{2}\int_0^t \|(\mathcal{H}_1[u_x,w] + \theta\mathcal{H}_2[u_x,w] + F)_t(\cdot,\tau)\|_2^2\,\mathrm{d}\tau$$
$$\leqslant C_{41} + C_{42}\max_{0\leqslant\tau\leqslant t}\|\theta(\cdot,\tau)\|_\infty^2 + C_{43}\int_0^t \|\theta_t(\cdot,\tau)\|_2^2\,\mathrm{d}\tau.$$

Thanks to the Gagliardo-Nirenberg inequality (see below) and Young's inequality, we conclude that

$$\|\zeta(\cdot,t)\|_\infty^2 \leqslant (C_{44}\|\zeta_x(\cdot,t)\|_2^{1/2}\|\zeta(\cdot,t)\|_2^{1/2} + C_{45}\|\zeta(\cdot,t)\|_2)^2$$
$$\leqslant C_{46}(1 + \|\zeta_x(\cdot,t)\|_2^{3/2} + \|\zeta(\cdot,t)\|_2^3).$$

Now, we apply (3.62) and Young's inequality to prove that (3.58) holds. □

The following version of the Gagliardo-Nirenberg inequality is a special case, more general formulations can be found, e.g., in [3], [39].

Lemma 3.12 (Gagliardo-Nirenberg inequality). *For all $p \geqslant 1$ there are positive constants C_{47}, C_{48} such that*

$$(3.63)\qquad \|v\|_\infty \leqslant C_{47}\|v_x\|_2^{2/(p+2)}\|v\|_p^{p/(p+2)} + C_{48}\|v\|_p \quad \forall\, v\in H^1(\Omega).$$

Lemma 3.13. *There is a positive constant C_{49} such that*

$$(3.64)\qquad \|u_{xt}(\cdot,t)\|_\infty^2 \leqslant C_{49}\biggl(1 + \max_{0\leqslant\tau\leqslant t}\|\theta(\cdot,\tau)\|_\infty^2 + \Bigl(\int_0^t \|\theta_t(\cdot,\tau)\|_2^2\,\mathrm{d}\tau\Bigr)^{3/4}\biggr).$$

Proof. Let $z_1, z_2\colon \overline{\Omega}_\infty \to \mathbb{R}$ be the solutions to the parabolic initial-boundary value problems

$$(3.65)\qquad z_{i,t} - z_{i,xx} = 0 \quad \text{a.e. in } \Omega_\infty \quad \forall\, i\in\{1,2\},$$
$$(3.66)\qquad z_i(1,t) = z_{i,x}(0,t) = 0 \quad \text{for a.e. } t>0 \quad \forall\, i\in\{1,2\},$$
$$(3.67)\qquad z_1(x,0) = u_{1,x}(x,0), \quad z_2(x,0) = \tilde{\sigma}(x,0) \quad \text{a.e. in } \Omega.$$

Let $z_3\colon \overline{\Omega}_\infty \to \mathbb{R}$ be defined by

$$(3.68)\qquad z_3(x,t) = \int_1^x\int_0^y z_1(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}y + \int_0^t (z_2(x,\tau) + \zeta(x,\tau))\,\mathrm{d}\tau \quad \forall\,(x,t)\in\Omega_\infty.$$

Recalling (3.65), (3.66), (3.67), (3.56), (3.57), and (H1), we observe that

$$(3.69)\qquad z_{3,t} = z_1 + z_2 + \zeta, \quad z_{3,xx} = z_1 + z_2 + \zeta - \tilde{\sigma} \quad \text{a.e. in } \Omega_\infty,$$

$$z_3(1,t) = 0 = z_{3,x}(0,t) \quad \text{for a.e. } t \geqslant 0, \quad z_3(x,0) = \int_1^x u_1(\xi)\,\mathrm{d}\xi \quad \forall x \in \Omega.$$

Hence, we see that z_3 is a solution to the linear parabolic initial-boundary value problem considered in (3.11)–(3.13). Since p is the unique solution to this problem, we have $p = z_3$ a.e. on Ω_∞. Therefore, recalling $u_{xt} = p_{xx}$ and (3.69), we have

$$(3.70)\qquad u_{xt} = z_{3,xx} = z_1 + z_2 + \zeta - \tilde{\sigma} \quad \text{a.e. in } \Omega_\infty.$$

Using (3.67), (H1), (3.10), (1.2), (1.6), (H6), and (H7), we get uniform bounds for $z_1(\cdot,0)$ and $z_2(\cdot,0)$. Applying the maximum principle for linear parabolic equations, we get uniform bounds for z_1 and z_2. Because of (3.10), (H7), and (3.30), we have

$$\tilde{\sigma} \leqslant C_{50} + C_{51}\theta \quad \text{a.e. in } \Omega_\infty.$$

Thus, applying (3.70), (3.58), and Young's inequality yields that (3.64) holds. □

Lemma 3.14. *There is a positive constant C_{52} such that*

$$(3.71)\qquad \sup_{0\leqslant\tau\leqslant t} \|\theta_x(\cdot,\tau)\|_2 + \int_0^t \|\theta_t(\cdot,\tau)\|_2^2\,\mathrm{d}\tau \leqslant C_{52}.$$

Proof. Testing (1.3) by θ_t, using (1.6), (H2), Young's inequality, Hölder's inequality, and (3.30), we see that

$$(3.72)\qquad \begin{aligned} &\frac{1}{2}\|\theta_t(\cdot,t)\|_2^2 + \frac{1}{2}\frac{\partial}{\partial t}\|\theta_x(\cdot,t)\|_2^2 \\ &\quad\leqslant \frac{1}{2}\|u_{xt}^2(\cdot,t) + \sigma u_{xt}(\cdot,t) - (\mathcal{F}_1[u_x,t])_t(\cdot,t) + g(\cdot,t,\theta(\cdot,t))\|_2^2 \\ &\quad\leqslant C_{53}\|u_{xt}(\cdot,t)\|_2^2(\|u_{xt}(\cdot,t)\|_\infty^2 + 1 + \|\theta(\cdot,t)\|_\infty^2) \\ &\qquad + C_{54}(\mathcal{F}_1[u_x,t])_t(\cdot,t)\|_2^2 + C_{55}\|g_1(\cdot,t)\|_2^2 \\ &\qquad + C_{56}\|g_2(\cdot,t)\|_2^2\|\theta(\cdot,t)\|_\infty^2. \end{aligned}$$

Integrating this equation over time, using (1.7), (H1), (H2), Hölder's inequality, (3.50), (3.51), and (3.64), we see that

$$(3.73)\qquad \begin{aligned} &\int_0^s \|\theta_t(\cdot,t)\|_2^2\,\mathrm{d}t + \|\theta_x(\cdot,s)\|_2^2 \\ &\quad\leqslant C_{57} + C_{58}\max_{0\leqslant t\leqslant s}(\|u_{xt}(\cdot,t)\|_\infty^2 + \|\theta(\cdot,t)\|_\infty^2) \\ &\quad\leqslant C_{59} + C_{60}\left(\int_0^s \|\theta_t(\cdot,t)\|_2^2\,\mathrm{d}t\right)^{\frac{3}{4}} + C_{61}\max_{0\leqslant t\leqslant s}\|\theta(\cdot,t)\|_\infty^2. \end{aligned}$$

Thanks to the Gagliardo Nirenberg inequality and (3.5), we have

$$\|\theta(\cdot,t)\|_\infty \leqslant C_{62}\|\theta_x(\cdot,t)\|_2^{2/3}\|\theta(\cdot,t)\|_1^{1/3} + C_{63}\|\theta(\cdot,t)\|_1 \leqslant C_{64} + C_{65}\|\theta_x(\cdot,t)\|_2^{2/3}.$$

Using this inequality to estimate the right-hand side of (3.73), and applying Young's inequality afterwards, we see that (3.71) holds. □

Lemma 3.15. *There are positive constants C_{66}, C_{67} such that*

$$\sup_{0\leqslant t}(\|\theta(\cdot,t)\|_\infty + \|u_{xt}(\cdot,t)\|_\infty + \|\sigma(\cdot,t)\|_\infty + \|w_t(\cdot,t)\|_\infty) \leqslant C_{66}, \tag{3.74}$$

$$\int_0^\infty (\|\sigma_t(\cdot,t)\|_2^2 + \|\psi_t(\cdot,t)\|_2^2 + \|\tilde\sigma_t(\cdot,t)\|_2^2)\,\mathrm{d}t \leqslant C_{67}, \tag{3.75}$$

$$\int_0^\infty (|\mathcal{D}_1[u_x(x,\cdot),w(x,\cdot)](t)| + |\mathcal{D}_2[u_x(x,\cdot),w(x,\cdot)](t)|)\,\mathrm{d}t < \infty \quad \text{for a.e. } x\in\Omega. \tag{3.76}$$

Proof. Using (3.39) and (3.71), we get the estimate for θ in (3.75) and applying in addition (3.64) and (3.30) leads to the remaining estimates in (3.74). Invoking (1.2), (1.5), (3.51), (3.74), (3.71), and (3.29), we get the estimates for σ_t and ψ_t. Utilizing also (3.10), (H7), and (3.36), we derive the estimates for $\tilde\sigma_t$. Combining (3.35) and (3.50) and using Fubini's theorem, we see that (3.76) holds. □

4. Proof of the asymptotic results

As in the preceding section, it will be assumed that (H1)–(H8) are satisfied, and that a solution (u,θ,w) to (1,1)–(1.7) is given, such that (2.22)–(2.25) holds.

For proving the asymptotic results in Theorem 1 and in Theorem 2 with an argumentation similar to [33, Section 4], the following modification of [34, Lemma 3.1] will be used. In the original formulation, it was assumed that the inequality in (4.1) holds for all t in the interval considered, but the proof in [34] can also be used if this inequality holds only for a.e. t in the interval considered.

Lemma 4.1. *Suppose that y and h are nonnegative functions on $(0,\infty)$, with y' locally integrable, such that there are positive constants $A_1,\dots,A_4$ satisfying*

$$y'(t) \leqslant A_1y^2(t) + A_2 + h(t) \quad \text{for a.e. } t\in(0,\infty), \tag{4.1}$$

$$\int_0^\infty y(t)\,\mathrm{d}t \leqslant A_3, \quad \int_0^\infty h(t)\,\mathrm{d}t \leqslant A_4. \tag{4.2}$$

Then we have $\lim_{t\to\infty} y(t) = 0$.

Lemma 4.2. *We have* (2.27) *and*

$$\lim_{t\to\infty} \|p_x(\cdot,t)\|_2 = \lim_{t\to\infty} \|u_t(\cdot,t)\|_2 = 0, \tag{4.3}$$

$$\lim_{t\to\infty} \|\tilde{\sigma}(\cdot,t)\|_2 = \lim_{t\to\infty} \|q_t\|_2 = 0. \tag{4.4}$$

Proof. Testing (3.11) with $-p_{xx}$, applying (3.12) and Young's inequality, we see that

$$\frac{1}{2}\frac{\partial}{\partial t}\|p_x(\cdot,t)\|_2^2 + \|p_{xx}(\cdot,t)\|_2^2 \leqslant \frac{1}{2}\|p_{xx}(\cdot,t)\|_2^2 + \frac{1}{2}\|\tilde{\sigma}(\cdot,t)\|_2^2 \quad \text{for a.e. } t \in (0,\infty).$$

Since $u_t = p_x$ a.e. in Ω_∞, we see by recalling (3.36) and (3.52) that we can apply Lemma 4.1 to show that (4.3) holds. We have, by Young's inequality,

$$\frac{\partial}{\partial t}\|\tilde{\sigma}(\cdot,t)\|_2^2 = 2\int_\Omega \tilde{\sigma}(x,t)\tilde{\sigma}_t(x,t)\,\mathrm{d}x \leqslant \|\tilde{\sigma}(\cdot,t)\|_2^2 + \|\tilde{\sigma}_t(\cdot,t)\|_2^2 \quad \text{for a.e. } t \in (0,\infty).$$

Invoking (3.52), (3.75), and Lemma 4.1, we get the convergence result for $\tilde{\sigma}$ in (4.4). Since (3.14), (3.10), and (H7) yield that $q_t = -\tilde{\sigma}$, we also have the result for q_t in (4.4). Combining (4.4), (3.10), (H7), and the definition on F_∞ in (2.29), we get (2.27). □

Lemma 4.3. *We have*

$$\lim_{t\to\infty} \|p_t(\cdot,t)\|_2 = \lim_{t\to\infty} \|p_{xx}(\cdot,t)\|_2 = \lim_{t\to\infty} \|u_{xt}(\cdot,t)\|_2 = \lim_{t\to\infty} \|u_t(\cdot,t)\|_\infty = 0. \tag{4.5}$$

Proof. Differentiating (3.11) with respect to t, testing it afterwards by p_t, and applying (3.11) and Young's inequality, we see that

$$\frac{\partial}{\partial t}\|p_t(\cdot,t)\|_2^2 + \|p_{xt}(\cdot,t)\|_2^2 \leqslant \frac{1}{2}\|p_t(\cdot,t)\|_2^2 + \frac{1}{2}\|\tilde{\sigma}_t(\cdot,t)\|_2^2 \quad \text{for a.e. } t \in (0,\infty).$$

Using (3.52), (3.75), and Lemma 4.1, we get the convergence result for p_t in (4.5). By (3.11), we can combine this with (4.4) to prove the convergence result for p_{xx} in (4.5). Recalling also (3.9), we get the convergence result for u_{xt} and using (1.6), we obtain the result for u_t. □

Lemma 4.4. *We have*

$$\lim_{t\to\infty} \|\theta_x(\cdot,t)\|_2 = \lim_{t\to\infty} \|\theta(\cdot,t) - \overline{\theta}(t)\|_\infty = 0. \tag{4.6}$$

Moreover, we have some constant $\theta_* > 0$ such that (2.30) *holds.*

Proof. Combining (3.72) with (3.74), we get for a.e. $t \in (0,\infty)$

$$\frac{1}{2}\frac{\partial}{\partial t}\|\theta_x(\cdot,t)\|_2^2 \leqslant C_{68}(\|u_{xt}(\cdot,t)\|_2^2 + \|(\mathcal{F}_1[u_x,t])_t(\cdot,t)\|_2^2 + \|g_1(\cdot,t)\|_2^2 + \|g_2(\cdot,t)\|_2^2).$$

Because of (3.40), (3.50), (3.51), and (H2), we can now use Lemma 4.1 to get the convergence result for θ_x. Recalling (3.39), we obtain the result for $\theta - \overline{\theta}$. Combining this with (3.38), we get some $t_0 > 0$ such that

$$\theta(x,t) > \frac{1}{2}C_{17} \quad \forall\, x \in \overline{\Omega},\ \ t \geqslant t_0.$$

Moreover, (2.23) and (2.25) yield that θ is continuous and positive on $\overline{\Omega} \times [0,t_0]$, and therefore also bounded from below by a positive constant C' on this set. Setting $\theta_* := \min(\frac{1}{2}C_{17}, C')$, we see that (2.30) holds. □

This completes the proof of Theorem 1.

Now, the additional convergence results in Theorem 2 will be proved.

Lemma 4.5. *If $\mathcal{G}$ is the identity operator, then we have* (2.32) *and*

$$\lim_{t\to\infty} \|w_t(\cdot,t)\|_2 = \lim_{t\to\infty} \|\psi(\cdot,t)\|_2 = 0. \tag{4.7}$$

Proof. Testing the time derivative of (1.4) by w_t and using Young's inequality, we see that for a.e. $t \in (0,\infty)$

$$\frac{\partial}{\partial t}\|w_t(\cdot,t)\|_2^2 \leqslant \int_\Omega w_t(x,t)\psi_t(x,t)\,\mathrm{d}x \leqslant \frac{1}{2}\|w_t(\cdot,t)\|_2^2 + \frac{1}{2}\|\psi_t(\cdot,t)\|_2^2.$$

By assumption, we have $w_t = (\mathcal{G}[w])_t$, and can therefore apply (3.51), (3.75), Lemma 4.1, and (1.4) to show that (4.7) holds. Using now (H6) iii) and (4.5), we get also (2.32). □

Lemma 4.6. *Assume that $\mathcal{H}_1 \equiv \mathcal{H}_3 \equiv \mathcal{F}_1 \equiv 0$, $g \equiv 0$, and $f \equiv 0$. Then, we have*

$$\theta(\cdot,t) \to \|\theta_0\|_1 + \frac{\varrho}{2C_V}\|u_1\|_2^2 \text{ as } t \to \infty, \quad \text{in } L^\infty(\Omega) \tag{4.8}$$

and (2.34). *If $\mathcal{G}$ is the identity operator then we have* (2.35).

Proof. Thanks to the assumptions, (3.4), (3.10), (1.2), (H7), and (H5), we see that $I_1 \equiv 0$, that I_0/C_V is equal to the right-hand side of (4.8), and that $\tilde{\sigma} = \theta\mathcal{H}_2[u_x, w]$. Invoking (3.2), (4.5), (4.6), (4.4), and (H1), we get (4.8) and (2.34). If $\mathcal{G}$ is the identity operator then it follows from (4.7), $\psi = \theta\mathcal{H}_4[u_x, w]$, and (4.8) that (2.35) holds. □

Lemma 4.7. *If* (H9) *holds then there is a $u_\infty \in W^{1,\infty}(\Omega)$ such that* (2.36)–(2.37) *hold.*

Proof. Owing to (3.76) and (H9), we have a function $\varepsilon_\infty\colon \Omega \to \mathbb{R}$ such that

$$u_x(x,t) \to \varepsilon_\infty(x) \text{ as } t \to \infty, \quad \text{for a.e. } x \in \Omega. \tag{4.9}$$

Invoking (3.17), compactness, and properties of weak-star and weak convergence, we see that $u_x(\cdot,t) \to \varepsilon_\infty$ as $t \to \infty$ weakly-star in L^∞_Ω. Defining now $u_\infty(x) := \int_0^x \varepsilon_\infty(\xi)$ and using (1.6), we conclude that $u_\infty \in W^{1,\infty}(\Omega)$ and (2.36)–(2.37) hold. □

Lemma 4.8. *If* (H10) *holds then there is a $w_\infty \in L^\infty(\Omega)$ such that* (2.38) *holds.*

Proof. Thanks to (3.76), (H10), (3.17), compactness, and properties of weak convergence, we get a $w_\infty \in L^\infty(\Omega)$ such that (2.38) holds. □

Hence, Theorem 2 is proved.

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ASYMPTOTIC BEHAVIOUR FOR A PHASE-FIELD MODEL WITH HYSTERESIS IN ONE-DIMENSIONAL THERMO-VISCO-PLASTICITY*

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Abstract. The asymptotic behaviour for $t \to \infty$ of the solutions to a one-dimensional model for thermo-visco-plastic behaviour is investigated in this paper. The model consists of a coupled system of nonlinear partial differential equations, representing the equation of motion, the balance of the internal energy, and a phase evolution equation, determining the evolution of a phase variable. The phase evolution equation can be used to deal with relaxation processes. Rate-independent hysteresis effects in the strain-stress law and also in the phase evolution equation are described by using the mathematical theory of hysteresis operators.

Keywords: phase-field system, phase transition, hysteresis operator, thermo-visco-plasticity, asymptotic behaviour

MSC 2000: 74N30, 35B40, 47J40, 34C55, 35K60, 74K05

1. Introduction

In this paper, an initial-boundary value problem for a system of partial differential equations involving hysteresis operators is considered, and the asymptotic behaviour of the solutions to this system is investigated. The system has been derived in [25] to model one-dimensional thermo-visco-plastic developments connected with solid-solid phase transitions taking also into account the hysteresis effects appearing on the macroscopic scale as a consequence of effects on the micro- and/or mesoscale.

To model such developments, one is considering the evolution of several quantities: the displacement u, the absolute temperature θ, and a phase variable w, which is

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usually a so-called *generalized freezing index*, see [21]. For a wire of unit length, the evolution of these fields is determined by the following system:

$$\varrho u_{tt} - \mu u_{xxt} = \sigma_x + f(x,t) \quad \text{a.e. in } \Omega_\infty, \tag{1.1}$$

$$\sigma = \mathcal{H}_1[u_x, w] + \theta\mathcal{H}_2[u_x, w] \quad \text{a.e. in } \Omega_\infty, \tag{1.2}$$

$$(C_V\theta + \mathcal{F}_1[u_x, w])_t - \kappa\theta_{xx} = \mu u_{xt}^2 + \sigma u_{xt} + g(x,t,\theta) \quad \text{a.e. in } \Omega_\infty, \tag{1.3}$$

$$\nu w_t = -\psi \quad \text{a.e. in } \Omega_\infty, \tag{1.4}$$

$$\psi = \mathcal{H}_3[u_x, w] + \theta\mathcal{H}_4[u_x, w] \quad \text{a.e. in } \Omega_\infty, \tag{1.5}$$

$$u(0,t) = 0, \;\; \mu u_{xt}(1,t) + \sigma(1,t) = 0, \;\; \theta_x(0,t) = \theta_x(1,t) = 0 \quad \text{a.e. in } (0,\infty), \tag{1.6}$$

$$u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad \theta(\cdot,0) = \theta_0, \quad w(\cdot,0) = w_0 \quad \text{a.e. in } \Omega, \tag{1.7}$$

with $\Omega_\infty := \Omega \times (0,\infty)$ and $\Omega := [0,1]$.

The equation (1.1) is the equation of motion, (1.3) is the balance of internal energy, and (1.4) is the phase evolution equation. By the constitutive law (1.2), the elastoplastic stress σ is determined, and the constitutive law (1.4) defines the thermodynamic force ψ. The boundary condition (1.6) means that the wire is fixed at $x = 0$, stress-free at $x = 1$, and thermally insulated at both ends. Here, x denotes the space variable, t denotes the time, and the indices x and t denote the differentiation with respect to space and time, respectively.

The mass density ϱ, the viscosity μ, the specific heat C_V, the heat conductivity κ, and the kinetic relaxation coefficient ν are supposed to be positive constants. The initial data for the displacement, the velocity, the temperature, and the phase variable considered in (1.7) are denoted by u_0, u_1, θ_0, and w_0, respectively. Finally, the nonlinearities $\mathcal{H}_i$, $1 \leqslant i \leqslant 4$, and $\mathcal{F}_1$ are hysteresis operators (see below), where one needs to take into account $u_x(x,\cdot)|_{[0,t]}$ and $w(x,\cdot)|_{[0,t]}$ to compute $\mathcal{H}_i[u_x, w](x,t)$ and $\mathcal{F}_1[u_x, w](x,t)$.

These operators are supposed to reflect some *memory* in the material on the macroscale, resulting from effects in the micro/mesoscale. Such effects can lead to *hysteresis loops*, as they are for example observed in the macroscopic strain-stress relation (ε-σ, where $\varepsilon = u_x$ is the linearized strain) determined from measurements in uniaxial load-deformation of materials like *shape memory alloys*, see, e.g., [2], [4], [6], [7], [8], [9], [10], [30], [31], [38]. The curves show a strong dependence on the temperature, but many of them are *rate-independent*, i.e., they are independent of the speed with which they are traversed.

There are other approaches to model hysteretic behaviour by considering systems similar to parts of (1.1)–(1.5), where the operators $\mathcal{F}_1$ and $\mathcal{H}_i$, for $1 \leqslant i \leqslant 4$, are superposition operators. These models are derived by considering a free energy, which is a superposition operator, involving a potential which has (one or more) concave parts. The concave parts of the potential correspond to unstable physical states, and these instabilities are supposed to produce the observed hysteresis effects. Such approaches have successfully been used and investigated in a number of papers, see, e.g., [3], [5], [7], [9], [33], [37], [39] and the references therein, but the modelling by non-convex free energies has its limits, since a non-convex part of the potential alone does not ensure that hysteresis loops are present, see, e.g., [29]. Moreover, the simple superposition operator cannot represent all the complicated hysteresis curves that are observed in experiments.

Hence, to describe such structures, the more general *hysteresis operators* have been introduced and used in a number of papers, see, e.g., the monographs [3], [14], [15], [36] on this subject and the references therein. For a final time $T > 0$, an operator $\mathcal{H}\colon C[0,T] \to \mathrm{Map}[0,T] := \{v\colon [0,T] \to \mathbb{R}\}$ is a *hysteresis operator* if it is rate-independent and causal according to the following definitions. The operator $\mathcal{H}$ is called *rate-independent*, if for every $v \in C[0,T]$ and every continuous increasing (not necessarily strictly increasing) function $\alpha\colon [0,T] \to [0,T]$ with $\alpha(0) = 0$ and $\alpha(T) = T$ it holds that $\mathcal{H}[v \circ \alpha](t) = \mathcal{H}[v](\alpha(t))$ for all $t \in [0,T]$.

An operator $\mathcal{H}\colon D(\mathcal{H})$ $(\subseteq \mathrm{Map}[0,T]) \to \mathrm{Map}[0,T]$ is said to be *causal*, if for every $v_1, v_2 \in D(\mathcal{H})$ and every $t \in [0,T]$ we have the implication

$$v_1(\tau) = v_2(\tau) \quad \forall\, \tau \in [0,t] \Rightarrow \mathcal{H}[v_1](t) = \mathcal{H}[v_2](t). \tag{1.8}$$

An example of a hysteresis operator is the *stop operator*, which is also called *Prandtl's normalized elastic-perfectly plastic element.* To define the stop operator, we consider some yield limit $r > 0$, an initial stress $\sigma_r^0 \in [-r,r]$, and a final time $T > 0$. For each input function $\varepsilon \in W^{1.1}(0,T)$, we have (see, e.g., [3], [14], [15], [36]) a unique solution $\sigma_r \in W^{1,1}(0,T)$ to the variational inequality

$$\sigma_r(t) \in [-r,r] \quad \forall\, t \in [0,T], \quad \sigma_r(0) = \sigma_r^0, \tag{1.9}$$

$$(\varepsilon_t(t) - \sigma_{r,t}(t))(\sigma_r(t) - \eta) \geqslant 0 \quad \forall\, \eta \in [-r,r], \quad \text{a.e. in } (0,T). \tag{1.10}$$

This defines the stop operator $\mathcal{S}_r\colon [-r,r] \times W^{1,1}(0,T) \to W^{1,1}(0,T)\colon (\sigma_r^0, \varepsilon) \mapsto \sigma_r$. An example for the evolution of the input and the output for the stop operator is presented in Fig. 1, showing the input-output relation of $\mathcal{S}_2[0,\varepsilon]$ for an input function ε which initially increases from 0 to 5, then decreases to -6, then increases to 0, then decreases to -3, and finally increases to 6.

Connected with the stop operator $\mathcal{S}_r$ is another important hysteresis operator, the so-called *play operator* $\mathcal{P}_r$ defined by

$$\mathcal{P}_r\colon [-r,r] \times W^{1,1}(0,T) \to W^{1,1}(0,T)\colon (\sigma_r^0, \varepsilon) \mapsto \varepsilon - \mathcal{S}_r[\sigma_r^0, \varepsilon]. \tag{1.11}$$

It is well-known, see, e.g., [3], [14], [15], that the stop and the play operator can be extended to Lipschitz continuous operators on $[-r,r] \times C[0,T]$. Moreover, using the

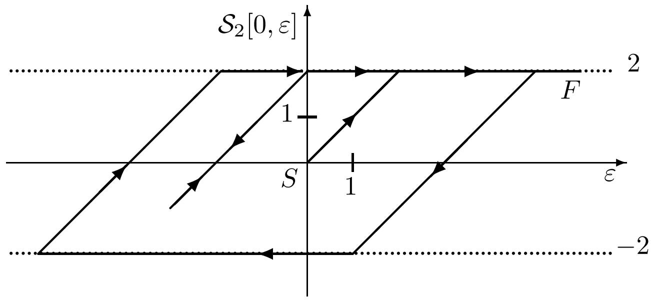

Figure 1. An example for the evolution of $(\varepsilon(t), \mathcal{S}_2[0,\varepsilon](t))$, starting in $S = (0,0)$ and finishing in $F = (6,2)$.

notation of [3, Chapter 2.5], one has for all $\sigma_r^0 \in [-r,r]$ that $\frac{1}{2}\mathcal{S}_r^2[\sigma_r^0,\cdot]$ is the *clockwise admissible potential* and $r\mathcal{P}_r[\sigma_r^0,\cdot]$ is the corresponding *dissipation operator* for the operator $\mathcal{S}_r[\sigma_r^0,\cdot]$, i.e., for all $\varepsilon \in W^{1,1}(0,T)$ it holds that

$$\left(\frac{1}{2}\mathcal{S}_r^2[\sigma_r^0,\varepsilon]\right)_t + |(r\mathcal{P}_r[\sigma_r^0,\varepsilon])_t| = \mathcal{S}_r[\sigma_r^0,\varepsilon]\varepsilon_t \quad \text{a.e. in } (0,T). \tag{1.12}$$

Let $\mathrm{Map}[0,\infty) := \{v\colon [0,\infty) \to \mathbb{R}\}$. An operator $\mathcal{H}\colon D(\mathcal{H})$ $(\subset \mathrm{Map}[0,\infty) \times \mathrm{Map}[0,\infty) \to \mathrm{Map}[0,\infty)$ is said to be *causal*, if for every $(\varepsilon_1,w_1),(\varepsilon_2,w_2) \in D(\mathcal{H})$ and every $t \geqslant 0$ we have the implication

$$\varepsilon_1(\tau) = \varepsilon_2(\tau),\ \ w_1(\tau) = w_2(\tau) \quad \forall\,\tau \in [0,t] \Rightarrow \mathcal{H}[\varepsilon_1,w_1](t) = \mathcal{H}[\varepsilon_2,w_2](t).$$

Moreover, the operator $\mathcal{H}$ generates an operator $\overline{\mathcal{H}}$ mapping (ε,w) with $\varepsilon,w\colon \Omega \times [0,\infty) \to \mathbb{R}$ such that $(\varepsilon(x,\cdot),w(x,\cdot)) \in D(\mathcal{H})$ for a.e. $x \in \Omega$ to the function on $\Omega \times [0,\infty)$ defined by $\overline{\mathcal{H}}[\varepsilon,w](x,t) = \mathcal{H}[\varepsilon(x,\cdot),w(x,\cdot)](t)$ for all $t \geqslant 0$ and for a.e. $x \in \Omega$. In the sequel, we will no longer distinguish between $\mathcal{H}$ and the generated operator $\overline{\mathcal{H}}$.

The hysteresis phenomena described by hysteresis operators are often related to changes between different configurations within the wire. In the system above, these configurations are described by the phase parameter w, and the evolution of these configurations is described by the phase evolution equation (1.4). By considering such an equation, one can take into account relaxation processes that appear in addition to the rate independent hysteresis loops, which are modelled by the hysteresis operators.

Let us recall some results for systems with hysteresis operators similar to the one above. In [11], [17], [20], [21], [23], [26], [27], a multi-dimensional phase transition is considered without taking mechanical effects into account. This corresponds to investigating (1.3)–(1.5) without a dependence on u or σ. The one-dimensional thermoelastoplastic hysteresis without considering relaxation processes in the phase transition, i.e., (1.1)–(1.3) with no dependence on w, has been studied in [16], [18].

For the complete system (1.1)–(1.7) above with an additional Ginzburg term u_{xxxx} on the left-hand side of (1.1) and boundary condition $u = u_{xx} = 0$ on $\partial\Omega$ for u, the global existence and uniqueness of a solution has been shown in [24].

The system (1.1)–(1.7) has been derived and investigated in [25]. Therein, the existence, uniqueness, and regularity of a strong solution has been proved (see Theorem 3 in Section 2.3), and it has also been shown that the Clausius-Duhem inequality and therefore the second principle of thermodynamics is satisfied for the solution.

In the present work, we are dealing with the asymptotic behaviour for $t \to \infty$ of the system under consideration. After discussing the assumptions in Section 2.1, the results are presented in Theorem 1 and Theorem 2 in Section 2.2. The a priori estimates derived in Section 3 are used in Section 4 to prove these theorems.

2. Asymptotic behaviour of solutions

2.1. Assumptions

The assumptions used in the investigation of the asymptotic behaviour of the solution to (1.1)–(1.7) are now presented and discussed. Let $C[0,\infty)$ denote the set of all continuous functions from $[0,\infty)$ to $\mathbb{R}$, including also the unbounded ones. For $t \geqslant 0$, the seminorm $|\cdot|_{[0,t]}$ on $C[0,\infty)$ and on $C[0,T]$ for $T \geqslant t$ is defined by

$$|f|_{[0,t]} = \max_{0\leqslant s\leqslant t} |f(s)|. \tag{2.1}$$

We will use the following assumptions:

(H1) We have $u_0 \in H^2(\Omega)$, $u_1 \in W^{1,\infty}(\Omega)$, $\theta_0 \in H^1(\Omega)$, $w_0 \in H^1(\Omega)$, and there is some $\delta > 0$ such that $\theta_0(x) \geqslant \delta$ for all $x \in \overline{\Omega}$. Moreover, the compatibility condition $u_0(0) = u_1(0) = 0$ is satisfied.

(H2) We assume that $g\colon \Omega \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that there are functions $g_1, g_2\colon \Omega_\infty \to [0,\infty)$, with

$$\begin{gathered} g_1 \in L^1(\Omega_\infty) \cap L^2(\Omega_\infty), \quad g_2 \in L^1(0,\infty; L^\infty(\Omega)) \cap L^2(0,\infty; L^\infty(\Omega)), \\ |g(x,t,s) - g_1(x,t)| \leqslant g_2(x,t)s, \quad g(x,t,-s) = g_1(x,t) \quad \forall\,(x,t) \in \Omega_\infty,\ s \geqslant 0. \end{gathered}$$

(H3) The operators $\mathcal{H}_1, \dots, \mathcal{H}_4$, $\mathcal{F}_1\colon C[0,\infty) \times C[0,\infty) \to C[0,\infty)$ are causal and map $W^{1,1}_{\mathrm{loc}}(0,\infty) \times W^{1,1}_{\mathrm{loc}}(0,\infty)$ into $W^{1,1}_{\mathrm{loc}}(0,\infty)$. The operators map $C[0,T] \times C[0,T]$ continuously into $C[0,T]$ for all $T > 0$, and for all $\varepsilon, w \in C[0,\infty)$

$$\mathcal{F}_1[\varepsilon, w](t) \geqslant 0 \quad \forall\, t \geqslant 0.$$

(H4) There exist causal operators $\mathcal{F}_2\colon W^{1,1}_{\mathrm{loc}}(0,\infty)\times W^{1,1}_{\mathrm{loc}}(0,\infty)\to W^{1,1}_{\mathrm{loc}}(0,\infty)$, $\mathcal{D}_1,\mathcal{D}_2\colon W^{1,1}_{\mathrm{loc}}(0,\infty)\times W^{1,1}_{\mathrm{loc}}(0,\infty)\to L^1_{\mathrm{loc}}(0,\infty)$, $\mathcal{G}\colon W^{1,1}_{\mathrm{loc}}(0,\infty)\to W^{1,1}_{\mathrm{loc}}(0,\infty)$, and a non-decreasing function k_1 such that for all $\varepsilon,w\in W^{1,1}_{\mathrm{loc}}(0,\infty)$

i) $|\mathcal{D}_1[\varepsilon,w]| = \varepsilon_t\mathcal{H}_1[\varepsilon,w]+(\mathcal{G}[w])_t\mathcal{H}_3[\varepsilon,w]-(\mathcal{F}_1[\varepsilon,w])_t$ a.e. in $(0,\infty)$,

$|\mathcal{D}_2[\varepsilon,w]| = \varepsilon_t\mathcal{H}_2[\varepsilon,w]+(\mathcal{G}[w])_t\mathcal{H}_4[\varepsilon,w]-(\mathcal{F}_2[\varepsilon,w])_t$ a.e. in $(0,\infty)$

ii) $|(\mathcal{G}[w])_t(t)|^2\leqslant k_1(|w|_{[0,t]})w_t(t)(\mathcal{G}[w])_t(t)$ for a.e. $t\in(0,\infty)$.

(H5) We have $\mathcal{F}_{1,0},\mathcal{F}_{2,0}\in L^1(\Omega)$ such that for all $\varepsilon,w\in W^{1,1}_{\mathrm{loc}}(0,\infty;L^2(\Omega))$ with $\varepsilon(\cdot,0)=u_{0,x}$ and $w(\cdot,0)=w_0$ a.e. on Ω it holds that

$$\mathcal{F}_1[\varepsilon,w](\cdot,0)=\mathcal{F}_{1,0},\quad \mathcal{F}_2[\varepsilon,w](\cdot,0)=\mathcal{F}_{2,0}\quad\text{a.e. in }\Omega.$$

(H6) There are non-decreasing functions $k_2,k_3,k_4\colon[0,\infty)\to[0,\infty)$ such that for all $\varepsilon,w\in C[0,\infty)$

i) $\max\limits_{1\leqslant i\leqslant 4}|\mathcal{H}_i[\varepsilon,w](t)|\leqslant k_2(|\varepsilon|_{[0,t]}+|w|_{[0,t]})\quad\forall\, t\geqslant 0.$

ii) $-\mathcal{F}_2[\varepsilon,w](t)\leqslant k_3(|\varepsilon|_{[0,t]}+|w|_{[0,t]})(1+\mathcal{F}_1[\varepsilon,w](t))\quad\forall\, t\geqslant 0.$

iii) If $\varepsilon,w\in W^{1,1}_{\mathrm{loc}}(0,\infty)$ then

$$\begin{aligned}&\max_{1\leqslant i\leqslant 4}|(\mathcal{H}_i[\varepsilon,w])_t(t)|+|(\mathcal{F}_1[\varepsilon,w])_t(t)|\\&\qquad\leqslant k_4(|\varepsilon|_{[0,t]}+|w|_{[0,t]})\big(|\varepsilon_t(t)|+\sqrt{w_t(t)(\mathcal{G}[w])_t(t)}\big)\quad\text{for a.e. } t\in(0,\infty).\end{aligned}$$

(H7) We have $f\in L^\infty(0,\infty;L^2(\Omega)$ and there exist functions $f_\infty\in L^2(\Omega)$, $F\in L^2(0,\infty;H^1(\Omega)\cap H^1(0,\infty;L^2(\Omega)\cap L^\infty(\Omega_\infty)$, and positive constants K_0, K_1 such that

$$f-f_\infty\in L^1(0,\infty;L^2(\Omega),\quad F(x,t)=\int_1^x f(\xi,t)\,\mathrm{d}\xi\quad\text{for a.e. }(x,t)\in\Omega_\infty,$$

$$\|f_\infty\|_{L^1(\Omega)}|\varepsilon(t)|\leqslant(1-K_0)|\mathcal{F}_1[\varepsilon,w](t)|+K_1\quad\forall\,\varepsilon,w\in C[0,\infty),\ t\geqslant 0.\tag{2.2}$$

For the formulation of the remaining assumptions, we use the following notations, which are well defined by (H1):

$$\varepsilon_{0,\min}:=\min\{u_{0,x}(x)\colon x\in\overline{\Omega}\},\qquad \varepsilon_{0,\max}:=\max\{u_{0,x}(x)\colon x\in\overline{\Omega}\},\tag{2.3}$$

$$w_{0,\min}:=\min\{w_0(x)\colon x\in\overline{\Omega}\},\qquad w_{0,\max}:=\max\{w_0(x)\colon x\in\overline{\Omega}\}.\tag{2.4}$$

(H8) For each $\varepsilon_\Delta>0$, there exists $\varepsilon_-\leqslant\varepsilon_{0,\min}$, $\varepsilon_+\geqslant\varepsilon_{0,\max}$, $w_\Delta>0$, $w_-\leqslant w_{0,\min}$, and $w_+\geqslant w_{0,\max}$ such that for all $\varepsilon,w\in C[0,\infty)$ and all $t\geqslant 0$,

i) If $\varepsilon(t) \geqslant \varepsilon_+$,

$$\varepsilon_{0,\min} \leqslant \varepsilon(0) \leqslant \varepsilon_{0,\max}, \qquad \varepsilon_- - \varepsilon_\Delta \leqslant \varepsilon(\tau) \leqslant \varepsilon_+ + \varepsilon_\Delta \quad \forall \tau \in [0,t], \tag{2.5}$$

$$w_{0,\min} \leqslant w(0) \leqslant w_{0,\max}, \quad w_- - w_\Delta \leqslant w(\tau) \leqslant w_+ + w_\Delta \quad \forall \tau \in [0,t], \tag{2.6}$$

hold then we have

$$\mathcal{H}_1[\varepsilon, w](t) \geqslant \|F\|_{L^\infty(\Omega_\infty)}, \quad \mathcal{H}_2[\varepsilon, w](t) \geqslant 0. \tag{2.7}$$

ii) If $\varepsilon(t) \leqslant \varepsilon_-$, (2.5), and (2.6) hold then we have

$$\mathcal{H}_1[\varepsilon, w](t) \leqslant -\|F\|_{L^\infty(\Omega_\infty)}, \quad \mathcal{H}_2[\varepsilon, w](t) \leqslant 0. \tag{2.8}$$

iii) If $w(t) \geqslant w_+$, (2.5), and (2.6) hold then we have

$$\mathcal{H}_3[\varepsilon, w](t) \geqslant 0, \quad \mathcal{H}_4[\varepsilon, w](t) \geqslant 0. \tag{2.9}$$

iv) If $w(t) \leqslant w_-$, (2.5), and (2.6) hold then we have

$$\mathcal{H}_3[\varepsilon, w](t) \leqslant 0, \quad \mathcal{H}_4[\varepsilon, w](t) \leqslant 0. \tag{2.10}$$

(H9) For every $\varepsilon, w \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ with ε and w bounded and

$$\int_0^\infty (|\mathcal{D}_1[\varepsilon, w](t)| + |\mathcal{D}_2[\varepsilon, w](t)|)\,\mathrm{d}t < \infty,$$

there exists $\varepsilon_\infty \in \mathbb{R}$ such that $\lim\limits_{t\to\infty} \varepsilon(t) = \varepsilon_\infty$.

(H10) For every ε, w as in (H9), there exists $w_\infty \in \mathbb{R}$ such that $\lim\limits_{t\to\infty} w(t) = w_\infty$.

Before the asymptotic results will be presented in Section 2.2, the above assumptions are discussed, starting with considerations concerning relations to the physical background.

Remark 2.1. Thanks to (H1), there is a positive lower bound for the initial temperature and the lower bound for g in (H2) ensures that this function does not model any further cooling at absolute zero. Considering the free energy $\mathcal{F}$, the entropy $\mathcal{S}$, and the internal energy $\mathcal{U}$ as in [25], i.e.

$$\begin{aligned}
\mathcal{F}[\varepsilon, w, \theta] &:= C_V\theta(1-\ln(\theta)) + \mathcal{F}_1[\varepsilon, w] + \theta\mathcal{F}_2[\varepsilon, w],\\
\mathcal{S}[\varepsilon, w, \theta] &:= C_V\theta - \mathcal{F}_2[\varepsilon, w],\\
\mathcal{U}[\varepsilon, w, \theta] &:= C_V\theta + \mathcal{F}_1[\varepsilon, w],
\end{aligned}$$

the lower bound for $\mathcal{F}_1$ in (H3) yields that the internal energy is nonnegative. Moreover, the nonnegativity of the expressions on the right-hand sides of the equations in (H5) i) is combined with (H5) ii) to prove that the system (1.1)–(1.7) is thermodynamically consistent, see [25, Remark 3]. The functions $\mathcal{D}_1$ and $\mathcal{D}_2$ arising in (H4) are related to the energy dissipation during a hysteresis loop.

Remark 2.2. There are cases where the operators $\mathcal{H}_i$ are decoupled. For example, the model for phase transition without mechanical effects as studied in [11], [17], [20], [21], [23], [26] can be combined with the model considered in [16], [18], that is the thermoelastoplastic hysteresis model without relaxation processes. In that case, if one does not take into account any direct coupling between phase transitions and mechanical effects, but only a coupling via the energy balance, one ends up with the system (1.1)–(1.7) with $\mathcal{H}_1$ and $\mathcal{H}_2$ depending only on u_x, and $\mathcal{H}_3$ and $\mathcal{H}_4$ depending only on w. Moreover, one is sometimes dealing with hysteresis operators arising as the sum of a superposition operator and some well-known hysteresis operator. Hence, we will investigate decoupled $\mathcal{H}_i$ of this form. Considering causal operators $\tilde{\mathcal{H}}_1, \ldots, \tilde{\mathcal{H}}_4 \colon C[0,\infty) \to C[0,\infty)$ and nonnegative functions $h_1, \ldots, h_4 \in C^2(\mathbb{R})$, we can define the operators $\mathcal{H}_1, \ldots, \mathcal{H}_4$ by setting, for all $\varepsilon, w \in C[0,\infty)$ and all $t \geqslant 0$,

$$\mathcal{H}_i[\varepsilon, w](t) := \begin{cases} h_i'(\varepsilon(t)) + \tilde{\mathcal{H}}_i[\varepsilon](t) & \text{for } i = 1,2, \\ h_i'(w(t)) + \tilde{\mathcal{H}}_i[w](t) & \text{for } i = 3,4. \end{cases} \tag{2.11}$$

For $1 \leqslant i \leqslant 4$, we assume that we have a *clockwise admissible potential* and the corresponding *dissipation operator* for $\tilde{\mathcal{H}}_i$, i.e. (see [3, Chapter 2.5]), we assume that we have a causal operator $\tilde{\mathcal{F}}_i \colon C[0,\infty) \to C[0,\infty)$ which is mapping $W^{1,1}_{\mathrm{loc}}(0,\infty)$ in $W^{1,1}_{\mathrm{loc}}(0,\infty)$ and a causal operator $\tilde{\mathcal{D}}_i \colon W^{1,1}_{\mathrm{loc}}(0,\infty) \to L^1_{\mathrm{loc}}(0,\infty)$ with

$$|\tilde{\mathcal{D}}_i[v]| = v_t \tilde{\mathcal{H}}_i[v] - (\tilde{\mathcal{F}}_i[v])_t \quad \text{a.e. in } (0,\infty), \quad \forall\, v \in W^{1,1}_{\mathrm{loc}}[0,\infty). \tag{2.12}$$

Then (H4) holds with $\mathcal{G}$ being the identity and $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{D}_1$, $\mathcal{D}_2$ defined by

$$\mathcal{F}_j[\varepsilon, w](t) := h_j(\varepsilon(t)) + \tilde{\mathcal{F}}_j[\varepsilon](t) + h_{j+2}(w(t)) + \tilde{\mathcal{F}}_{j+2}[w](t), \tag{2.13}$$

$$\mathcal{D}_j[\varepsilon, w](t) := |\tilde{\mathcal{D}}_j[\varepsilon](t)| + |\tilde{\mathcal{D}}_{j+2}[w](t)|, \tag{2.14}$$

for all $\varepsilon, w \in C[0,\infty)$, $t \geqslant 0$, and $j \in \{1,2\}$.

If $h_1(r) = h_1^* r^2$ with some positive constant h_1^* then the corresponding operator $\mathcal{H}_1$ models a linear elasticity with a hysteretic modification.

Remark 2.3. A sufficient condition for (H8) to be satisfied is that the two following assumptions (H11) and (H12) hold. These assumptions are especially useful, if the operators $\mathcal{H}_1, \ldots, \mathcal{H}_4$ are decoupled as in the Remarks 2.2, 2.5–2.6. The notation of an *outward pointing* operator used in these assumptions is introduced and discussed in [13].

The more general formulation in (H8) is helpful, if the operators are coupled, e.g., if they are derived from multi-dimensional stop or Prandtl-Ishlinskii operators (see, e.g., [15], [21], [22], [23]).

(H11) For each $\varepsilon_\Delta > 0$, there exists $\varepsilon_- \leqslant \varepsilon_{0,\min}$ and $\varepsilon_+ \geqslant \varepsilon_{0,\max}$ such that for all $w \in C[0,\infty)$ with $w_{0,\min} \leqslant w(0) \leqslant w_{0,\max}$ the operator mapping $\varepsilon \in C[0,\infty)$ to $\mathcal{H}_1[\varepsilon, w] \in C[0,\infty)$ is *pointing outwards with bound* $\|F\|_{L^\infty(\Omega_\infty)}$ *in the* ε_Δ*-neighbourhood of* $[\varepsilon_-, \varepsilon_+]$ *for initial values in* $[\varepsilon_{0,\min}, \varepsilon_{0,\max}]$ and that the same holds for $\mathcal{H}_2$ just with bound 0, that is to say for all $\varepsilon \in C[0,\infty)$ and all $t \geqslant 0$ holds:
 i) If $\varepsilon(t) \geqslant \varepsilon_+$ and (2.5) hold then we have (2.7).
 ii) If $\varepsilon(t) \leqslant \varepsilon_-$ and (2.5) hold then we have (2.8).

(H12) There are $w_\Delta > 0$, $w_- \leqslant w_{0,\min}$, and $w_+ \geqslant w_{0,\max}$ such that for all $\varepsilon \in C[0,\infty)$ with $\varepsilon_{0,\min} \leqslant \varepsilon(0) \leqslant \varepsilon_{0,\max}$ the operators $C[0,\infty) \ni w \mapsto \mathcal{H}_3[\varepsilon, w]$ and $C[0,\infty) \ni w \mapsto \mathcal{H}_4[\varepsilon, w]$ are pointing outwards with bound 0 in the w_Δ-neighbourhood of $[w_-, w_+]$ for initial values in $[w_{0,\min}, w_{0,\max}]$, that is to say for all $w \in C[0,\infty)$ and $t \geqslant 0$ it holds that:
 i) If $w(t) \geqslant w_+$ and (2.6) hold then we have (2.9).
 ii) If $w(t) \leqslant w_-$ and (2.6) hold then we have (2.10).

Remark 2.4. If we use $\tilde{\mathcal{H}}_3 = \tilde{\mathcal{H}}_4 \equiv 0$ in Remark 2.2 then $\mathcal{H}_3$ and $\mathcal{H}_4$ are superposition operators and the assumption (H12) holds if and only if there are $w_\Delta > 0$, $w_- \leqslant w_{0,\min}$, and $w_+ \geqslant w_{0,\max}$ such that
- For all $s \in [w_+, w_+ + w_\Delta]$ holds $h_3'(s) \geqslant 0$, $h_4'(s) \geqslant 0$.
- For all $s \leqslant [w_- - w_\Delta, w_-]$ holds $h_3'(s) \leqslant 0$, $h_4'(s) \leqslant 0$.

A similar condition has been used in [1], [32], [33]. If this condition is directly adapted to hysteresis operators, one ends up with an assumption similar to (H12), but with the condition (2.6) replaced by $w_- - w_\Delta \leqslant w(t) \leqslant w_+ + w_\Delta$ only. This assumption is stronger than (H12) and will be denoted by (H12+). There are important hysteresis operators satisfying (H12), but not (H12+).

In a similar way, one can consider a stronger version (H11+) of (H11), where $\varepsilon_- - \varepsilon_\Delta \leqslant \varepsilon(t) \leqslant \varepsilon_+ + \varepsilon_\Delta$ is used instead of (2.5).

Remark 2.5. If for the functions and operators in Remark 2.2 there are positive constants $K_{2,1}, \ldots, K_{2,4}$ such that

$$|\tilde{\mathcal{H}}_i[v](t)| \leqslant K_{2,i} \quad \forall\, t \geqslant 0, \ v \in C[0,\infty), \ 1 \leqslant i \leqslant 4, \tag{2.15}$$

$$\pm \lim_{r \to \pm\infty} h_1'(r) > K_{2,1} + \|F\|_{L^\infty(\Omega_\infty)}, \tag{2.16}$$

$$\pm \lim_{r \to \pm\infty} h_j'(r) > K_{2,j} \quad \forall\, 2 \leqslant j \leqslant 4, \tag{2.17}$$

then the assumptions (H11+) and (H12+) are satisfied. Hence, (H11), (H12), and (H8) hold. Moreover, the condition (2.2) in (H7) is satisfied if the other assumptions in (H7) hold.

R e m a r k 2.6. For $1 \leqslant i \leqslant 4$, we consider a nonnegative weight function $\varphi_i \in L^1(0,\infty)$ and a function $\sigma_i^0 \in W^{1,\infty}(0,\infty)$ such that $\sigma_i^0(r) \in [-r,r]$ for all $r \geqslant 0$, $|(\sigma_i^0)_r| \leqslant 1$ a.e. on $(0,\infty)$, and $\sigma_r^0(r') = 0$ for all $r' \geqslant R_i$ for some $R_i > 0$. Moreover, we consider yield limits $r_{i,j} \in \mathbb{R}$, initial values $\sigma_{i,j}^0 \in [-r_{i,j}, r_{i,j}]$, and weights $\varphi_{i,j} > 0$. Now, we define $\tilde{\mathcal{H}}_i\colon C[0,\infty) \to C[0,\infty)$ as the *Prandtl-Ishlinskii operator*

$$\tilde{\mathcal{H}}_i[v] := \int_0^\infty \varphi_i(r)\mathcal{S}_r[\sigma_i^0(r), v]\,\mathrm{d}r + \sum_j \varphi_{i,j}\mathcal{S}_{r_{i,j}}[\sigma_{i,j}^0, v] \quad \forall\, v \in C[0,\infty). \tag{2.18}$$

The more general definition of this operator involing a Stieljes integral, see, e.g. [15], would allow to write this sum as one integral. A clockwise admissible potential for this operator is defined by $\tilde{\mathcal{F}}_i\colon C[0,\infty) \to C[0,\infty)$ with

$$\tilde{\mathcal{F}}_i[v] := \frac{1}{2}\int_0^\infty \varphi_i(r)\mathcal{S}_r^2[\sigma_i^0(r), v]\,\mathrm{d}r + \frac{1}{2}\sum_j \varphi_{i,j}\mathcal{S}_{r_{i,j}}^2[\sigma_{i,j}^0, v] \tag{2.19}$$

for all $v \in C[0,\infty)$ since Proposition 2.5.5 in [3] and (1.12) yield that (2.12) holds for

$$\tilde{\mathcal{D}}_i[v] := \left|\frac{\partial}{\partial t}\int_0^\infty r\varphi_i(r)\mathcal{P}_r[\sigma_r^0, v]\,\mathrm{d}r\right| + \sum_j \varphi_{i,j}|(r\mathcal{P}_r[\sigma_{i,j}^0, v])_t| \tag{2.20}$$

for all $v \in W^{1,1}_{\mathrm{loc}}[0,\infty)$. Defining now $\mathcal{H}_i$ and $\mathcal{F}_i$ as in Remark 2.2, and using well-known properties of the stop operator one can show that (H3)–(H6) hold.

Since for oscillations that are smaller then the yield limit of a play operator, the operator stays constant after the first oscillation, we can apply (2.14) and (2.20) to deduce that (H9) holds, if and only if for all $s > 0$ the function $\varphi_1 + \varphi_2$ does not vanish a.e. on $[0,s]$. For (H10), we get an analogous condition, just with $\varphi_1 + \varphi_2$ replaced by $\varphi_3 + \varphi_4$. If one wants to ensure as in Remark 2.2 that (H11) and (H12) are satisfied, one has to require that 2.15 holds, which is equivalent to the condition

$$\int_0^\infty r\varphi_i(r)\,\mathrm{d}r + \sum_j \varphi_{i,j} r_{i,j} < K_{2,i} < +\infty \quad \forall\, 1 \leqslant i \leqslant 4. \tag{2.21}$$

If this condition is satisfied, we see that (H11) and (H12) hold for appropriate functions h_i, but this argumentation can not be applied if $\mathcal{H}_i = \tilde{\mathcal{H}}_i$ for some $i \in \{1,\dots,4\}$.

In [13], it is proved that (H12) holds for $\mathcal{H}_3 := \tilde{\mathcal{H}}_3$ and $\mathcal{H}_4 := \tilde{\mathcal{H}}_4$, independently of (2.21). Moreover, it is shown there that for $\mathcal{H}_1 := \tilde{\mathcal{H}}_1$ the condition in (H11) holds if and only if $\int_0^\infty r\varphi_1(r)\,\mathrm{d}r = \infty$, and that an analogous equivalence holds for $\mathcal{H}_2 := \tilde{\mathcal{H}}_2$.

2.2. The asymptotic result

The following two theorems are the main result of this paper:

Theorem 1. *Assume that* (H1)–(H8) *are satisfied. Moreover, assume that* (u,θ,w) *is a solution to* (1.1)–(1.7) *such that*

$$u \in H^2_{\mathrm{loc}}(0,\infty;L^2(\Omega)) \cap H^1_{\mathrm{loc}}(0,\infty;H^2(\Omega)), \tag{2.22}$$

$$\theta \in H^1_{\mathrm{loc}}(0,\infty;L^2(\Omega)) \cap L^2_{\mathrm{loc}}(0,\infty;H^2(\Omega)), \tag{2.23}$$

$$w \in H^2_{\mathrm{loc}}(0,\infty;L^2(\Omega)) \cap H^1_{\mathrm{loc}}(0,\infty;H^2(\Omega)), \tag{2.24}$$

$$\theta(x,t) > 0 \quad \forall\, x \in \overline{\Omega},\ \ t \geqslant 0. \tag{2.25}$$

Then, it holds that

$$\lim_{t\to\infty} \|u_{xt}(\cdot,t)\|_{L^2(\Omega)} = 0, \quad \lim_{t\to\infty} \|u_t(\cdot,t)\|_{C(\overline{\Omega})} = 0, \tag{2.26}$$

$$\sigma(\cdot,t) \to -F_\infty \ \text{ as } t \to \infty, \quad \text{in } L^2(\Omega), \tag{2.27}$$

$$\lim_{t\to\infty} \|\theta_x(\cdot,t)\|_{L^2(\Omega)} = 0, \quad \lim_{t\to\infty} \|\theta(\cdot,t) - \overline{\theta}(t)\|_{C(\overline{\Omega})} = 0, \tag{2.28}$$

with

$$F_\infty(x) := \int_1^x f_\infty(\xi)\,\mathrm{d}\xi, \quad \overline{\theta}(t) := \int_\Omega \theta(y,t)\,\mathrm{d}y \quad \forall\, x \in \overline{\Omega},\ t \geqslant 0. \tag{2.29}$$

In addition, we have a constant $\theta_* > 0$ *such that*

$$\theta(x,t) \geqslant \theta_* \quad \forall\, x \in \overline{\Omega},\ t \geqslant 0. \tag{2.30}$$

Remark 2.7. We see that (2.26) yields that for $t \to \infty$ the viscous part of the stress tends to zero, and by (2.27) the stress tends to $-F_\infty$, which is the potential corresponding to the limit f_∞ for $t \to \infty$ of the applied force f. Moreover, by (2.28), we see that the temperature becomes more and more uniform in space. It is an open questions whether one can show convergence for θ, u_x, or w under the general assumptions of the theorem or if oscillations can appear up to $t \to \infty$.

Also in [33], where the system (1.1)–(1.3) with $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{F}_1$ just being nonlinear superposition operators of u_x has been considered, convergence for θ and u_x could only been proved by using additional assumptions. Corresponding additional conditions are required here in part b) and c) of Theorem 2 below, and allow to show the convergence of the temperature for $t \to \infty$. If, in addition, $\mathcal{H}_2$ and $\mathcal{H}_4$ are special operators, like, e.g. stop operators, one could also show some convergence for u and w, by adapting the argument in [33, Lemma 4.5] to the more general situation considered here.

Now, convergence results are presented that can be proved using additional hypotheses.

Theorem 2. *Assume that the assumptions of Theorem 1 are satisfied.*

a) *If $\mathcal{G}$ is the identity operator, then we have*

$$\lim_{t\to\infty} \|w_t(\cdot,t)\|_{L^2(\Omega)} = 0, \quad \lim_{t\to\infty} \|\psi(\cdot,t)\|_{L^2(\Omega)} = 0, \tag{2.31}$$

$$\lim_{t\to\infty} \|(\mathcal{F}_1[u_x,w])_t(\cdot,t)\|_{L^2(\Omega)} = \sum_{i=1}^{4} \lim_{t\to\infty} \|(\mathcal{H}_i[u_x,w])_t(\cdot,t)\|_{L^2(\Omega)} = 0. \tag{2.32}$$

b) *If $\mathcal{H}_1 \equiv \mathcal{H}_3 \equiv \mathcal{F}_1 \equiv 0$, $g \equiv 0$, and $f \equiv 0$, then we have*

$$\theta(\cdot,t) \to \|\theta_0\|_{L^1(\Omega)} + \frac{\varrho}{2C_V}\|u_1\|^2_{L(\Omega)} \quad \text{as } t\to\infty, \quad \text{in } L^\infty(\Omega), \tag{2.33}$$

$$\lim_{t\to\infty} \|\mathcal{H}_2[u_x,w](\cdot,t)\|_{L^2(\Omega)} = 0. \tag{2.34}$$

c) *If $\mathcal{H}_1 \equiv \mathcal{H}_3 \equiv \mathcal{F}_1 \equiv 0$, $g \equiv 0$, $f \equiv 0$, and $\mathcal{G}$ is the identity operator, then we have*

$$\lim_{t\to\infty} \|\mathcal{H}_4[u_x,w](\cdot,t)\|_{L^2(\Omega)} = 0. \tag{2.35}$$

d) *If* (H9) *holds then there exists a $u_\infty \in W^{1,\infty}(\Omega)$ such that*

$$u(\cdot,t) \to u_\infty \quad \text{as } t\to\infty, \quad \text{weakly-star in } W^{1,\infty}(\Omega), \tag{2.36}$$

$$u_x(\cdot,t) \to u_{\infty,x} \quad \text{as } t\to\infty, \quad \text{a.e. in } \Omega. \tag{2.37}$$

e) *If* (H10) *holds then there exists a $w_\infty \in L^\infty(\Omega)$ such that*

$$w(\cdot,t) \to w_\infty \quad \text{as } t\to\infty, \quad \text{weakly-star in } L^\infty(\Omega) \quad \text{and a.e. in } \Omega. \tag{2.38}$$

Remark 2.8. If (H8) does not hold then one can still prove the results in Theorem 1 and some of the results in Theorem 2, if some other additional assumptions are satisfied.

i) If (H4) and (H6) with $k_1,\dots,k_4$ replaced by positive constants hold then one can still show the results in Theorem 1 and the results in Theorem 2 a)–c) hold.
ii) If (H11), (H4) ii) with k_1 replaced by a positive constant, and (H6) without the $|w|_{[0,t]}$-term in the evaluation of k_2, k_3, k_4 hold then one can prove that the results in Theorem 1 and the results in Theorem 2 a)–d) hold.
iii) If (H12) and (H6) without the $|\varepsilon|_{[0,t]}$-term in the evaluation of k_2, k_3, k_4 hold then one can prove the results in Theorem 1 and the results in Theorem 2 a)–c) and e) hold.

2.3. Existence of solutions

Before proving the asymptotic result, it will be recalled that there is a solution to the problem under consideration satisfying the regularity and positivity demands presented in Theorem 1, at least if some additional assumptions are satisfied. These assumptions will be

(H13) $f \in H^1_{\mathrm{loc}}(0,\infty; L^2(\Omega))$.

(H14) The function g_1 arising in (H2) satisfies $g_1 \in L^\infty_{\mathrm{loc}}(\Omega_\infty)$ and for every $T > 0$ there is a positive constant $K_{3,T}$ such that $|\partial g/\partial\theta| \leqslant K_{3,T}$ a.e. in $\Omega \times (0,T) \times \mathbb{R}$.

(H15) For every $T > 0$ there are positive constants $K_{4,T}, \ldots, K_{7,T}$ and non-decreasing functions $k_{5,T}, k_{6,T} \colon [0,\infty) \to [0,\infty)$ such that for all $\varepsilon, \varepsilon_1, \varepsilon_2, w, w_1, w_2 \in C[0,\infty)$ the following holds:

i) We have for all $t \in [0,T]$:

$$|\mathcal{H}_2[\varepsilon,w](t)| + |\mathcal{H}_4[\varepsilon,w](t)| \leqslant K_{4,T},$$
$$\max_{1\leqslant i\leqslant 4} |\mathcal{H}_i[\varepsilon_1,w_1](t) - \mathcal{H}_i[\varepsilon_2,w_2](t)| \leqslant K_{5,T}(|\varepsilon_1-\varepsilon_2|_{[0,t]} + |w_1-w_2|_{[0,t]}).$$

ii) If $\varepsilon, \varepsilon_1, \varepsilon_2, w, w_1, w_2 \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ then the inequality in (H4) ii) with $k_1(|w|_{[0,t]})$ replaced by $K_{6,T}$ holds for a.e. $t \in (0,T)$ and

$$\max_{1\leqslant i\leqslant 4} |(\mathcal{H}_i[\varepsilon,w])_t(t)| \leqslant K_{7,T}(|\varepsilon_t(t)| + |w_t(t)|) \quad \text{for a.e. } t \in (0,T),$$

$$|\mathcal{F}_1[\varepsilon,w])_t(t)| \leqslant k_{5,T}(|\varepsilon|_{[0,t]} + |w|_{[0,t]})(|\varepsilon_t(t)| + |w_t(t)|) \quad \text{for a.e. } t \in (0,T), \tag{2.39}$$

$$\begin{aligned} |\mathcal{F}_1[\varepsilon_1,w_1](t) - \mathcal{F}_1[\varepsilon_2,w_2](t) \\ \leqslant k_{6,T}(|\varepsilon_1|_{[0,t]} + |\varepsilon_2|_{[0,t]} + |w_1|_{[0,t]} + |w_2|_{[0,t]}) \\ \times \biggl(|\varepsilon_1(0) - \varepsilon_2(0)| + |w_1(0) - w_2(0)| \\ + \int_0^t (|\varepsilon_{1,t}(\tau) - \varepsilon_{2,t}(\tau)| + |w_{1,t}(\tau) - w_{2,t}(\tau)|)\,\mathrm{d}\tau \biggr) \\ \forall\, t \in [0,T]. \end{aligned} \tag{2.40}$$

One can extend Theorem 2.1 in [25] to the following result:

Theorem 3. *Assume that* (H1)–(H3), (H4) i), *and* (H13)–(H15) *are valid. Then the system* (1.1)–(1.7) *has a unique strong solution* (u,θ,w) *such that* (2.22)–(2.24) *hold. This solution also satisfies* (2.25).

The original existence result in [25] has been formulated with a stronger version of the assumption (H15), where $k_{5,T}(\cdots)$ in (2.39) and $k_{6,T}(\cdots)$ in (2.40) are replaced by positive constants. Combining this stronger assumption with (H4) i) and the continuity of $\mathcal{F}_1$ on $C[0,T] \times C[0,T]$ (see (H3)), it follows that $\mathcal{H}_1$ and $\mathcal{H}_3$ have to be uniformly bounded. However, uniform boundedness is not satisfied in many important situations, e.g., if $\mathcal{H}_1$, defined as in (2.11), is modelling a linear elasticity with a bounded hysteretic modification as in Remark 2.2. Using the assumption (H15) allows to apply the existence result above also in this situation. In [24], the authors of [25] consider a hypothesis analogous to (H15) for a modified version of the system (1.1)–(1.7).

We now sketch the proof of Theorem 3: We observe that, in the global existence proof in [25], the stronger versions of (2.39) and (2.40) are applied *after* the uniform estimates for u_x and w have been derived. To perform the a priori estimates, it suffices to use just (2.39) and (2.40). Moreover, (2.39) and (2.40) also suffice for the local existence result in [25, Section 3], as can be seen from a careful examination of the proof. Details can be found in the forthcoming paper [12]. Therein, it is also shown that one can replace the boundedness of $\mathcal{H}_2$ and $\mathcal{H}_4$, as assumed in (H15) i), by the hypothesis for $\mathcal{F}_2$ in (H6) i). One is then able to consider the case where one assumes (H11) for $\mathcal{H}_2$ consisting of Prandtl-Ishlinskii operators depending only on ε. In this case, $\mathcal{H}_2$ is unbounded, see Remark 2.6.

Remark 2.9. For nonnegative functions $h_1, \ldots, h_4 \in C^2(\mathbb{R})$ with $h_1'', h_3'' \in L^\infty(\mathbb{R})$, $h_2', h_4' \in W^{1,\infty}(\mathbb{R})$, and operators $\tilde{\mathcal{H}}_1, \ldots, \tilde{\mathcal{H}}_4$ as in Remark 2.6 with nonnegative weight functions $\varphi_1, \ldots, \varphi_4 \in L^1(0,\infty)$ satisfying (2.21) one can use well-known properties of the stop operator (see, e.g., [3], [14], [15], [36]) to show that (H15) holds.

3. Uniform a priori estimates

In this section, it will be assumed that (H1)–(H8) are satisfied and that a solution (u, θ, w) to (1.1)–(1.7) is given, such that (2.22)–(2.25) hold. To prepare the proof of the asymptotic results in the next section, some a priori estimates are derived that are uniform with respect to time.

Before this is done, we consider the energy balance and derive an immediate consequence:

Remark 3.1. Multiplying (1.1) by u_t and adding the result to the balance law (1.3) for the internal energy, we get the balance law for the energy

$$(3.1) \quad \left(C_V\theta + \frac{\varrho}{2}u_t^2 + \mathcal{F}_1[u_x, w]\right)_t - \kappa\theta_{xx} = (u_t(\mu u_{tx} + \sigma))_x + g + u_t f \quad \text{a.e. in } \Omega_\infty.$$

For $t > 0$, we integrate this equation over $\Omega \times (0, t)$, and use Green's formula, (1.6), (1.7), (H1), and (H5), to show that

$$C_V \overline{\theta}(t) + \frac{\varrho}{2} \|u_t(\cdot, t)\|^2_{L^2(\Omega)} = I_0 + I_1(t) \quad \forall\, t \geqslant 0 \tag{3.2}$$

holds for the $\overline{\theta}$ defined in (2.29),

$$I_0 := C_V \|\theta_0\|_{L^1(\Omega)} + \frac{\varrho}{2} \|u_1\|^2_{L^2(\Omega)} + \int_\Omega \mathcal{F}_{1,0}(x)\,\mathrm{d}x > 0, \tag{3.3}$$

$$\begin{aligned} I_1(t) := &\int_0^t \int_\Omega (g(x, \tau, \theta(x, \tau)) + u_t(x, \tau) f(x, \tau))\,\mathrm{d}x\,\mathrm{d}\tau \\ &- \int_\Omega (\mathcal{F}_1[u_x, w](x, t))\,\mathrm{d}x \quad \forall\, t \geqslant 0. \end{aligned} \tag{3.4}$$

In the sequel, for $1 \leqslant p < \infty$, the notation $\|\cdot\|_p$ will be used as an abbreviation for the $L^p(\Omega)$-norm, and $\|\cdot\|_\infty$ will denote the $C(\overline{\Omega})$-norm, i.e., the maximum norm on $\overline{\Omega}$. Moreover, C_i, for $i \in \mathbb{N}$, will always denote generic positive constants, independent of time, space, and the considered solution.

Thanks to (2.22)–(2.25) and (H3), we can assume without losing generality that σ and ψ are continuous (maybe unbounded) functions on $\overline{\Omega}_\infty = \overline{\Omega} \times [0, \infty)$, such that (1.2) and (1.5) hold for all $(x, t) \in \overline{\Omega}_\infty$. Because of (1.7), (2.3), (2.4), we can apply the assumption (H8) for $\varepsilon(\cdot) := u_x(x, \cdot)$ and $w(\cdot) := w(x, \cdot)$. For the sake of notational convenience, we assume in the remaining part of this section without losing generality that $\varrho = \mu = C_V = \kappa = \nu = 1$.

In the following estimates, some ideas from [25], [33], [35] are used.

Lemma 3.2. *There are two positive constants C_1, C_2 such that*

$$\sup_{0 \leqslant t} (\|\theta(\cdot, t)\|_1 + \|u_t(\cdot, t)\|_2 + \|\mathcal{F}_1[u_x, w](\cdot, t)\|_1) \leqslant C_1, \tag{3.5}$$

$$\int_0^\infty (\|g(\cdot, t, \theta(\cdot, t))\|_1 + \|g(\cdot, t, \theta(\cdot, t))\|_1^2)\,\mathrm{d}t \leqslant C_2. \tag{3.6}$$

Proof. Let

$$\Psi(t) := \int_\Omega (\mathcal{F}_1[u_x, w](x, t) - f_\infty(x) u(x, t) + K_1)\,\mathrm{d}x \quad \forall\, t \geqslant 0. \tag{3.7}$$

Now, we get from (3.2) by using (2.29), (2.25), (3.3), (3.4), Hölder's inequality, Young's inequality, (H1), (H2), (H5), and (H7) that for all $t \geqslant 0$

$$\begin{aligned}(3.8)\qquad &\Big(\|\theta(\cdot,t)\|_1 + \frac{1}{2}\|u_t(\cdot,t)\|_2^2 + \Psi(t)\Big)\\ &< C_3 + \int_0^t (\|g_2(\cdot,t)\|_\infty \|\theta(\cdot,\tau)\|_1 + \|g_1(\cdot,\tau)\|_1)\,\mathrm{d}\tau\\ &+ \frac{1}{2}\int_0^t (\|f(\cdot,\tau) - f_\infty\|_2 + \|f(\cdot,\tau) - f_\infty\|_2\,\|u_t(\cdot,\tau)\|_2^2)\,\mathrm{d}\tau.\end{aligned}$$

By (3.7), Hölder's inequality, (1.6), (H3), and (H7), we have

$$\Psi(t) \geqslant K_0\|\mathcal{F}_1[u_x,w](\cdot,t)\|_1 \quad \forall\, t \geqslant 0.$$

Hence, because of (3.8), we can apply Gronwall's Lemma, (H2), and (H7) to show that (3.5) and (3.6) are satisfied. $\square$

To prepare the following estimates, we now consider the transformation due to Andrews [1], which is also used, e.g., in [32], [33], [25], and introduce functions $p, q, \tilde{\sigma}\colon \overline{\Omega}_\infty \to \mathbb{R}$ that are defined by

$$(3.9)\qquad p(x,t) := \int_1^x u_t(\xi,t)\,\mathrm{d}\xi, \quad q(x,t) := u_x(x,t) - p(x,t) \quad \forall\,(x,t) \in \overline{\Omega}_\infty,$$

$$(3.10)\qquad \tilde{\sigma}(x,t) := \sigma(x,t) + F(x,t) \quad \forall\,(x,t) \in \overline{\Omega}_\infty,$$

with F as in (H7). Recalling (1.1)–(1.7) and (H7), we see that

$$(3.11)\qquad p_t - p_{xx} = \tilde{\sigma} \quad \text{a.e. in } \Omega_\infty,$$

$$(3.12)\qquad p(1,t) = p_x(0,t) = 0 \quad \text{a.e. in } (0,T),$$

$$(3.13)\qquad p(x,0) = \int_1^x u_1(\xi)\,\mathrm{d}\xi \quad \text{a.e. in } \Omega,$$

$$(3.14)\qquad q_t = -\tilde{\sigma} \quad \text{a.e. in } \Omega,$$

$$(3.15)\qquad q(x,0) = u_{0,x}(x) - \int_1^x u_1(\xi)\,\mathrm{d}\xi \quad \text{a.e. in } \Omega.$$

Lemma 3.3. *There are positive constant C_4, C_5 such that*

$$(3.16)\qquad \sup_{0\leqslant t}(\|p_x(\cdot,t)\|_2 + \|p(\cdot,t)\|_\infty) \leqslant C_4,$$

$$(3.17)\qquad \sup_{0\leqslant t}(\|u_x(\cdot,t)\|_\infty + \|w(\cdot,t)\|_\infty + \|u(\cdot,t)\|_\infty + \|q(\cdot,t)\|_\infty) \leqslant C_5.$$

P r o o f. In the light of the estimate for u_t in (3.5) and the definition of p in (3.9), we see that (3.16) holds. Considering (H8) for $\varepsilon_\Delta := 2C_4 + 1$, we get $\varepsilon_- < \varepsilon_{0,\min}$,

$\varepsilon_{0,\max} < \varepsilon_+$, $w_- < w_{0,\min}$, and $w_+ > w_{0,\max}$ such that the remaining conditions in (H8) are satisfied. Now,

$$u_x(x,t) \in [\varepsilon_- - 2C_4, \varepsilon_+ + 2C_4], \quad w(x,t) \in [w_-, w_+] \quad \forall\,(x,t) \in \overline{\Omega}_\infty \tag{3.18}$$

is proved by contradiction. Suppose that (3.18) does not hold. Then there is some $\delta \in (0, \min\{w_\Delta, 1\})$ such that $u_x \leqslant \varepsilon_- - 2C_4 - \delta$ and/or $u_x \geqslant \varepsilon_+ + 2C_4 + \delta$ and/or $w \leqslant w_- - \delta$ and/or $w \geqslant w_+ + \delta$ somewhere in $\overline{\Omega}_\infty$. We have $u_x(x,0) = u_{0,x}(x) \in [\varepsilon_-, \varepsilon_+]$ and $w(x,0) = w_0(x) \in [w_-, w_+]$ for all $x \in \overline{\Omega}$ because of (2.3) and (2.4). Since (2.22) and (2.24) yield that w and u_x are continuous on $\overline{\Omega}_\infty$, we get $x_1 \in \overline{\Omega}$, $t_1 > 0$ such that

$$\begin{cases} u_x(x_1,t_1) \in \{\varepsilon_- - 2C_4 - \delta, \varepsilon_+ + 2C_4 + \delta\} \\ \text{and/or } \ w(x_1,t_1) \in \{w_+ + \delta, w_- - \delta\}, \end{cases} \tag{3.19}$$

$$\varepsilon_- - 2C_4 - \delta < u_x(x,t) < \varepsilon_+ + 2C_4 + \delta \quad \forall\, t \in [0,t_1),\ \ x \in \overline{\Omega}, \tag{3.20}$$

$$\varepsilon_- - 2C_4 - \delta \leqslant u_x(x,t_1) \leqslant \varepsilon_+ + 2C_4 + \delta \quad \forall\, x \in \overline{\Omega}, \tag{3.21}$$

$$w_- - \delta < w(x,t) < w_+ + \delta \quad \forall\, t \in [0,t_1),\ \ x \in \overline{\Omega}, \tag{3.22}$$

$$w_- - \delta \leqslant w(x,t_1) \leqslant w_+ + \delta \quad \forall\, x \in \overline{\Omega}. \tag{3.23}$$

Hence, we see that (2.5) with $\varepsilon := u_x(x,\cdot)$ and (2.6) with $w := w(x,\cdot)$ hold for all $x \in \overline{\Omega}$ and $t \leqslant t_1$, and it remains only to check the first condition in (H8) i)–iv) if one wants to apply one the corresponding inequalities (2.7)–(2.10). Since u_x and w are uniformly continuous on $\overline{\Omega} \times [0,t_1]$, there is some open neighborhood $U \subset \overline{\Omega}$ of x_1 such that

$$|u_x(x,t) - u_x(x_1,t)| + |w(x,t) - w(x_1,t)| \leqslant \frac{\delta}{8} \quad \forall\, x \in U,\ \ t' \in [0,t_1]. \tag{3.24}$$

Now, we consider the case $u_x(x_1,t_0) = \varepsilon_+ + 2C_4 + \delta$. Since u_x is continuous on $\overline{\Omega} \times [0,t_1]$ and $u_x(x_1,0) \leqslant \varepsilon_+$, we get some $t_0 \in (0,t_1)$ such that

$$\varepsilon_+ + \frac{\delta}{2} = u_x(x_1,t_0), \quad \varepsilon_+ + \frac{\delta}{2} < u_x(x_1,t) < \varepsilon_+ + 2C_4 + \delta \quad \forall\, t \in (t_0,t_1). \tag{3.25}$$

Combining this with (3.24), we conclude that $u_x(x,t) \geqslant \varepsilon_+$ for all $x \in U$, $t \in (t_0,t_1)$. In the light of (2.7) in (H8) i), we see that

$$\|F\|_{L^\infty(\Omega_\infty)} \leqslant \mathcal{H}_1[u_x,w](x,t), \quad 0 \leqslant \mathcal{H}_2[u_x,w](x,t) \quad \forall\, x \in U,\ \ t \in (t_0,t_1). \tag{3.26}$$

Applying (1.2) and the fact that $\theta > 0$ on Ω_∞ by (2.25), we observe that $\sigma \geqslant -F$ a.e. in $U \times (t_0,t_1)$. Thanks to (3.14) and (3.10), we deduce that $q_t \leqslant 0$ a.e. in $U \times (t_0,t_1)$. This leads to

$$\int_U (q(x,t_1) - q(x,t_0))\,\mathrm{d}x\,\mathrm{d}\tau = \int_U\int_{t_0}^{t_1} q_t(x,t)\,\mathrm{d}t\,\mathrm{d}x \leqslant 0.$$

On the other hand, using (3.9), (3.16), (3.24), (3.25), and $u_x(x_1,t_0)=\varepsilon_+ + \frac{1}{2}\delta$, we conclude that

$$\begin{aligned}\int_U \big(q(x,t_1)-q(x,t_0)\big)\,\mathrm{d}x &\geqslant \int_U \big(u_x(x,t_1)-C_4-(u_x(x,t_0)+C_4)\big)\,\mathrm{d}x\\ &\geqslant \int_U \Big(u_x(x_1,t_1)-\frac{\delta}{8} \;-\Big(u_x(x_1,t_0)+\frac{\delta}{8}\Big)-2C_4\Big)\,\mathrm{d}x\\ &\geqslant \int_U \frac{\delta}{4}\,\mathrm{d}x>0.\end{aligned}$$

Hence, we have derived a contradiction. By an analogous argument, we get a contradiction if $u_x(x_1,t_1)=\varepsilon_- -2C_4-\delta$.

Now, we will deal with the case of $w(x_1,t_1)=w_+ +\delta$. Applying the continuity of w, we get some $t_0\in(0,t_1)$ such that

$$w(x_1,t_0)=w_+ +\frac{\delta}{2},\quad w_+ +\frac{\delta}{2}<w(x_1,t)<w_+ +\delta\quad\forall\, t\in(t_0,t_1). \tag{3.27}$$

Combining this with (3.24), we see that $w(x,t)\geqslant w_+$ for all $x\in U$, $t\in(t_0,t_1)$. Therefore, we conclude from (2.9) in (H8) iii) that

$$\mathcal{H}_3[u_x,w](x,t)\geqslant 0,\quad \mathcal{H}_4[u_x,w](x,t)\geqslant 0\quad \forall\, x\in U,\;\; t\in(t_0,t_1). \tag{3.28}$$

Since $\theta>0$ a.e. on Ω_∞ by (2.25), we deduce now from (1.5) and (1.4) that $w_t\leqslant 0$ a.e. in $U\times(t_0,t_1)$. This leads to

$$\int_U (w(x,t_1)-w(x,t_0))\,\mathrm{d}x=\int_U\int_{t_0}^{t_1} w_t(x,t)\,\mathrm{d}t\,\mathrm{d}x\leqslant 0.$$

Since $w(x_1,t_1)=w_+ +\delta$, (3.27), and (3.24) yield that the integral on the left-hand side has to be positive, we have derived a contradiction. An analogous argument to get a contradiction can be used if $w(x_1,t_1)=w_- -\delta$.

Hence, we have derived a contradiction for all cases we have to consider by (3.19). Therefore, we have proved (3.18). Recalling (1.6) and (3.9), we get also uniform bounds for u and q, and (3.17) is proved. □

Remark 3.4. Because of (3.17), we have uniform bounds for u_x and w. Thanks to (H6), (3.5), (1.2), (1.5), and (1.4), we see that there are positive constants C_6,

$C_7, \ldots, C_9$ such that

$$\max_{1\leqslant i\leqslant 4} \sup_{0\leqslant t}(\|\mathcal{H}_i[u_x,w](\cdot,t)\|_\infty) \leqslant C_6, \tag{3.29}$$

$$|\sigma| + |w_t| \leqslant C_7(1+\theta) \quad \text{a.e. in } \Omega_\infty, \tag{3.30}$$

$$0 \leqslant \sup_{0\leqslant t} \int_0^1 (-\mathcal{F}_2[u_x,w](x,t))\,\mathrm{d}x \leqslant C_8, \tag{3.31}$$

$$\max_{1\leqslant i\leqslant 4} |(\mathcal{H}_i[u_x,w])_t| + |(\mathcal{F}_1[u_x,w])_t| \leqslant C_9\big(|u_{xt}| + \sqrt{w_t(\mathcal{G}[w])_t}\big) \quad \text{a.e. in } \Omega_\infty. \tag{3.32}$$

Since (3.17) and (H4) ii) yield that $0 \leqslant w_t(\mathcal{G}[w])_t \leqslant C_{10}w_t^2$ a.e. in Ω_∞, we deduce that

$$\max_{1\leqslant i\leqslant 4} |(\mathcal{H}_i[u_x,w])_t| + |(\mathcal{F}_1[u_x,w])_t| \leqslant C_{11}(|u_{xt}| + |w_t|) \quad \text{a.e. in } \Omega_\infty. \tag{3.33}$$

We apply (H4) i), (1.2), (1.5), (1.4), and (H4) ii) to conclude that, a.e. on Ω_∞, it holds that

$$\begin{aligned}(\mathcal{F}_1[u_x,w])_t &- \sigma(x,t)u_{xt} \\ &= (\mathcal{G}[w])_t\mathcal{H}_3[u_x,w] - |\mathcal{D}_1[u_x,w]| - \theta\mathcal{H}_2[u_x,w]u_{xt} \\ &= -|(\mathcal{G}[w])_t w_t| - |\mathcal{D}_1[u_x,w]| \\ &\quad - \theta(\mathcal{H}_2[u_x,w]u_{xt} + (\mathcal{G}[w])_t\mathcal{H}_4[u_x,w]).\end{aligned} \tag{3.34}$$

Lemma 3.5. *We have a positive constant C_{12} such that*

$$\begin{aligned}\int_0^\infty \bigg(\bigg\|\frac{\theta_x}{\theta}(\cdot,t)\bigg\|_2^2 + \bigg\|\frac{u_{xt}}{\sqrt{\theta}}(\cdot,t)\bigg\|_2^2 + \bigg\|\frac{(\mathcal{G}[w])_t w_t}{\theta}(\cdot,t)\bigg\|_1\bigg)\,\mathrm{d}t \\ + \int_0^\infty \|\mathcal{D}_2[u_x,w](\cdot,t)\|_1\,\mathrm{d}t + \sup_{0\leqslant t}\|\ln\theta(\cdot,t))\|_1 \leqslant C_{12}.\end{aligned} \tag{3.35}$$

P r o o f. Testing (1.3) by $-1/\theta$ and using (1.6), (3.34), (H2), and (H4) i), we observe that

$$\begin{aligned}-\frac{\partial}{\partial t}\int_\Omega \ln\theta(x,t)\,\mathrm{d}x &+ \int_\Omega\bigg(\bigg(\frac{\theta_x(x,t)}{\theta(x,t)}\bigg)^2 + \frac{u_{xt}^2(x,t)}{\theta(x,t)}\bigg)\,\mathrm{d}x \\ &\leqslant -\frac{\partial}{\partial t}\int_\Omega \mathcal{F}_2[u_x,w](x,t)\,\mathrm{d}x - \int_\Omega \frac{|(\mathcal{G}[w])_t(x,t)w_t(x,t)| + |\mathcal{D}_1[u_x,w](x,t)|}{\theta(x,t)}\,\mathrm{d}x \\ &\quad + \int_\Omega(-|\mathcal{D}_2[u_x,w](x,t)| + |g_2(x,t)|)\,\mathrm{d}x.\end{aligned}$$

Now, we integrate this equation over time and observe that (3.35) follows by applying (3.31), (H2), (H5), (3.5), and the inequality $|\ln s| \leqslant s - \ln s$ for all $s > 0$, which can be proved by elementary analysis. $\square$

Lemma 3.6. *We have a positive constant* C_{13} *such that*

$$\begin{aligned}\int_0^\infty \big(\|u_{xt}(\cdot,t)\|_1^2 + \|u_t(\cdot,t)\|_\infty^2 + \|p(\cdot,t)\|_\infty^2 + \|(\mathcal{G}[w])_t(\cdot,t)\|_1^2 \\ + \|(\mathcal{F}_1[u_x,w])_t\|_1^2 + \big\|(\sqrt{\theta})_x(\cdot,1)\big\|_1^2\big)\,\mathrm{d}t \leqslant C_{13}.\end{aligned} \tag{3.36}$$

P r o o f. Since $\theta > 0$ a.e. on Ω_∞, we can apply Schwarz's inequality and (3.5) to show that for all $t > 0$

$$\|u_{xt}(\cdot,t)\|_1 = \int_\Omega \frac{|u_{xt}(x,t)|}{\sqrt{\theta(x,t)}}\sqrt{\theta(x,t)}\,\mathrm{d}x \leqslant C_{14}\left\|\frac{u_{xt}}{\sqrt{\theta}}(\cdot,t)\right\|_2. \tag{3.37}$$

Recalling now (3.35) leads to the estimate for u_{xt} in (3.36). Using that, by (1.6) and (2.22), $u_t(y,t) = \int_0^y u_{xt}(x,t)\,\mathrm{d}x$ for all $y \in \overline{\Omega}$, we get the estimate for u_t. Combining this estimate with (3.9) leads to the estimate for p.

Applying (3.32), (H4) ii), (3.35), and Young's inequality, we deduce that

$$\int_0^\infty \left(\left\|\frac{(\mathcal{G}[w])_t}{\sqrt{\theta}}(\cdot,t)\right\|_2^2 + \left\|\frac{(\mathcal{F}_1[u_x,w])_t}{\sqrt{\theta}}(\cdot,t)\right\|_2^2\right)\mathrm{d}t \leqslant C_{15}.$$

Considering now (3.37) with u_{xt} replaced by $(\mathcal{G}[w])_t$, we get the estimate for $(\mathcal{G}[w])_t$ in (3.36), and the estimate for $(\mathcal{F}_1[u_x,w])_t$ is derived analogously. Thanks to Schwarz's inequality, we have

$$\big\|(\sqrt{\theta})_x(\cdot,t)\big\|_1 = \int_\Omega \frac{|\theta_x(x,t)|}{\sqrt{\theta(x,t)}}\,\mathrm{d}x \leqslant \left\|\frac{|\theta_x|}{\theta}(\cdot,t)\right\|_2 \big\|\sqrt{\theta}(\cdot,t)\big\|_2.$$

In the light of (3.5) and (3.35), we see that also the estimate for $\sqrt{\theta}_x$ in (3.36) is established. $\square$

Lemma 3.7. *For* $\overline{\theta}$ *and* I_1 *as in* (2.29) *and* (3.4) *there are positive constant* C_{16}*,* C_{17}*, and* C_{18} *such that*

$$|I_1(t)| \leqslant C_{16}, \quad C_{17} < \overline{\theta}(t) < C_{18} \quad \forall\, t \geqslant 0, \tag{3.38}$$

$$\|\theta(\cdot,t) - \overline{\theta}(t)\|_\infty \leqslant \|\theta_x(\cdot,t)\|_1 \leqslant \|\theta_x(\cdot,t)\|_2 \quad \forall\, t \geqslant 0. \tag{3.39}$$

P r o o f. Combining (3.4), (3.6), (1.7), (3.5), and Hölder's inequality, we see that

$$|I_1(s)| \leqslant C_{19} + \left|\int_\Omega (u(x,s) - u_0(x))f_\infty(x)\,\mathrm{d}x\right| + \int_0^s \|f(\cdot,t) - f_\infty(t)\|_2\,\|u_t(\cdot,t)\|_2\,\mathrm{d}t.$$

Recalling (3.17), (3.5), (H7), and (H1), we get the uniform bound for I_1 in (3.38). Since $s \mapsto -\ln s$ is a convex function on $(0,\infty)$, we get by (2.25) and Jensen's inequality that

$$-\ln \int_\Omega \theta(x,t)\,\mathrm{d}x \leqslant -\int_\Omega \ln(\theta(x,t))\,\mathrm{d}x \quad \forall\, t \geqslant 0.$$

Invoking now (3.35), (2.29), and (3.5), we get (3.38). The first inequality in (3.39) follows from the definition in (2.29), and the second by applying Schwarz's inequality and $\int_\Omega 1\,\mathrm{d}x = 1$. □

Lemma 3.8. *We have a positive constant C_{20} such that*

$$\int_0^\infty \left(\|\theta_x(\cdot,t)\|_2^2 + \left\|\frac{\partial}{\partial x}((u_t)^2)(\cdot,t)\right\|_2^2 + \left(\frac{\partial I_1(t)}{\partial t}\right)^2\right)\mathrm{d}t + \sup_{0\leqslant t}(\|u_t(\cdot,t)\|_4 + \|\theta(\cdot,t)\|_2) \leqslant C_{20}. \tag{3.40}$$

Proof. We test (3.1) by $\theta + \frac{1}{2}u_t^2$ and (1.1) by αu_t^3 where $\alpha > 0$ will be fixed later. Summing the resulting equations and using (1.6) and (3.4), we observe that for all $t \geqslant 0$

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\left\|\theta(\cdot,t) + \frac{1}{2}u_t^2(\cdot,t)\right\|_2^2 &+ \|\theta_x(\cdot,t)\|_2^2 + \frac{\alpha}{4}\frac{\partial}{\partial t}\|u_t(\cdot,t)\|_4^4 \\ &+ (1+3\alpha)\|u_t(\cdot,t)u_{tx}(\cdot,t)\|_2^2 \\ &\leqslant \overline{\theta}(t)\frac{\partial I_1(t)}{\partial t} + I_2(t) + I_3(t) + I_4(t),\end{aligned} \tag{3.41}$$

with

$$I_2(t) := \int_\Omega (-(\mathcal{F}_1[u_x,w])_t(x,t) + g(x,t,\theta(x,t)) + u_t(x,t)f(x,t)) \times (\theta(x,t) - \overline{\theta}(t))\,\mathrm{d}x, \tag{3.42}$$

$$I_3(t) := -\int_\Omega \left(\frac{1}{2}(\mathcal{F}_1[u_x,w])_t u_t^2 + 2\theta_x u_t u_{tx} + u_t\sigma\theta_x + (1+3\alpha)u_t^2 u_{tx}\sigma\right)\mathrm{d}x, \tag{3.43}$$

$$I_4(t) := \int_\Omega (g + (1+2\alpha)u_t f)\frac{1}{2}u_t^2\,\mathrm{d}x. \tag{3.44}$$

In the sequel, the generic constants C_i will be independent of α. We estimate the left-hand side of (3.42) by using Hölder's inequality, (H7), (3.39), and Young's inequality, resulting in

$$
\begin{aligned}
(3.45)\quad I_2(t) &\leqslant (\|(\mathcal{F}_1[u_x,w])_t(\cdot,t)\|_1 + \|g(\cdot,t,\theta(\cdot,t))\|_1 + \|u_t(\cdot,t)\|_\infty\|f(\cdot,t)\|_1) \\
&\quad \times \|\theta(\cdot,t) - \overline{\theta}(t)\|_\infty \\
&\leqslant C_{21}(\|(\mathcal{F}_1[u_x,w])_t(\cdot,t)\|_1^2 + \|g(\cdot,t,\theta(\cdot,t))\|_1^2 + \|u_t(\cdot,t)\|_\infty^2) \\
&\quad + \frac{1}{6}\|\theta_x(\cdot,t)\|_2^2.
\end{aligned}
$$

Invoking (3.43), (3.33), (3.30), Hölder's inequality, and Young's inequality, we deduce that

$$
\begin{aligned}
(3.46)\quad I_3(t) &\leqslant C_{22}((1+\alpha)\|u_{tx}(\cdot,t)u_t^2(\cdot,t)\|_1 + \|u_t^2(\cdot,t)\|_1 + \|u_t^2(\cdot,t)\theta(\cdot,t)\|_1) \\
&\quad + 2\|\theta_x(\cdot,t)u_t(\cdot,t)u_{tx}(\cdot,t)\|_1 + C_{23}\|u_t(\cdot,t)\theta_x(\cdot,t)\|_1 \\
&\quad + C_{24}\|\theta_x(\cdot,t)u_t(\cdot,t)\theta(\cdot,t)\|_1 + (1+\alpha)C_{25}\|u_t^2(\cdot,t)u_{tx}(\cdot,t)\theta(\cdot,t)\|_1 \\
&\leqslant C_{26}\|u_t(\cdot,t)u_{tx}(\cdot,t)\|_2^2 + C_{27}(1+\alpha^2)\|u_t(\cdot,t)\|_2^2 \\
&\quad + C_{28}(1+\alpha^2)\|u_t(\cdot,t)\|_\infty^2\|\theta(\cdot,t)\|_2^2 + \frac{1}{6}\|\theta_x(\cdot,t)\|_2^2.
\end{aligned}
$$

Using (3.44), (H2), Hölder's inequality, (3.5), (H7), (3.39), (3.38), and Young's inequality, we conclude that

$$
\begin{aligned}
(3.47)\quad 2I_4(t) &\leqslant \|g_1(\cdot,t)\|_2\|u_t(\cdot,t)\|_2\|u_t(\cdot,t)\|_\infty \\
&\quad + (\overline{\theta}(t) + \|\theta(\cdot,t) - \overline{\theta}(t)\|_\infty)\|g_2(\cdot,t)\|_\infty\|u_t(\cdot,t)\|_2^2 \\
&\quad + (1+2\alpha)\|u_t(\cdot,t)\|_2\|f(\cdot,t)\|_2\|u_t(\cdot,t)\|_\infty^2 \\
&\leqslant \frac{1}{6}\|\theta_x\|_2^2 + C_{29}(\|g_1(\cdot,t)\|_2^2 + \|g_2(\cdot,t)\|_\infty + \|g_2(\cdot,t)\|_\infty^2) \\
&\quad + C_{30}(1+\alpha^2)\|u_t(\cdot,t)\|_\infty^2.
\end{aligned}
$$

Because of (3.2) and Young's inequality, we have

$$
(3.48)\quad \overline{\theta}(t)\frac{\partial I_1(t)}{\partial t} \leqslant I_0\frac{\partial I_1(t)}{\partial t} + \frac{1}{2}\frac{\partial}{\partial t}(I_1(t))^2 + \frac{1}{4}\|u_t(\cdot,t)\|_2^4 + \frac{1}{4}\left(\frac{\partial I_1(t)}{\partial t}\right)^2.
$$

From (3.4), we get by using Hölder's inequality, Young's inequality, (H7), and (H2) that

$$
(3.49)\quad \left(\frac{\partial I_1(t)}{\partial t}\right)^2 \leqslant C_{31}(\|g(\cdot,t,\theta(\cdot,t))\|_1^2 + \|u_t(\cdot,t)\|_2^2 + \|\mathcal{F}_1[u_x,w])_t(\cdot,t)\|_1^2).
$$

Now, we integrate the sum of (3.41) and (3.49) over time, and use (1.7), (H1), (3.45)–(3.49), (3.6), (H2), (3.5), (3.36), (3.38), and $\theta > 0$ a.e. on Ω to show that

$$\begin{aligned}\frac{1}{2}\|\theta(\cdot,s)\|_2^2 + \frac{\alpha}{4}\|u_t(\cdot,s)\|_4^4 \\ + \int_0^s\Big(\frac{1}{2}\|\theta_x(\cdot,t)\|_2^2 + (1+3\alpha)\|u_t(\cdot,t)u_{tx}(\cdot,t)\|_2^2 + \frac{3}{4}\Big(\frac{\partial I_1(t)}{\partial t}\Big)^2\Big)\,\mathrm{d}t \\ \leqslant C_{32}\Big(1+\alpha^2+\int_0^s(\|u_t(\cdot,t)u_{tx}(\cdot,t)\|_2^2 + (1+\alpha^2)\|u_t(\cdot,t)\|_\infty^2\|\theta(\cdot,t)\|_2^2)\,\mathrm{d}t\Big)\end{aligned}$$

holds for all $s > 0$. Next, we define $\alpha := C_{32}$, apply Gronwall's Lemma, and recall (3.36) to show that (3.40) is satisfied. □

Lemma 3.9. *There are positive constants C_{33}, C_{34} such that*

$$\int_0^\infty(\|u_{xt}(\cdot,t)\|_2^2 + \|(\mathcal{G}[w])_t(\cdot,t)w_t(\cdot,t)\|_1 + \|\mathcal{D}_1[u_x,w](\cdot,t)\|_1)\,\mathrm{d}t \leqslant C_{33}, \tag{3.50}$$

$$\begin{aligned}\int_0^\infty(\|p_{xx}(\cdot,t)\|_2^2 + \|(p+q)_t(\cdot,t)\|_2^2 + \|u_t(\cdot,t)\|_\infty^2 + \|(\mathcal{F}_1[u_x,w])_t(\cdot,t)\|_2^2 \\ + \sum_{i=1}^4\|(\mathcal{H}_i[u_x,w])_t(\cdot,t)\|_2^2 + \|(\mathcal{G}[w])_t(\cdot,t)\|_2^2)\,\mathrm{d}t \leqslant C_{34}.\end{aligned} \tag{3.51}$$

P r o o f. Integrating (1.3) over Ω, and applying (1.6), (2.29), (3.34), and (H4) ii), we derive

$$\begin{aligned}\|u_{xt}(\cdot,t)\|_2^2 \leqslant \frac{\partial\overline{\theta}(t)}{\partial t} + \|g(\cdot,t,\theta(\cdot,t))\|_1 - \|(\mathcal{G}[w])_t(\cdot,t)w_t(\cdot,t)\|_1 - \|\mathcal{D}_1[u_x,w](\cdot,t)\|_1 \\ - \int_\Omega(\theta(x,t)-\overline{\theta}(t))(\mathcal{H}_2[u_x,w](x,t)u_{xt}(x,t) + (G[w])_t(x,t)\mathcal{H}_4[u_w,w](x,t))\,\mathrm{d}x \\ - \overline{\theta}(t)\frac{\partial}{\partial t}\int_\Omega\mathcal{F}_2[u_x,w](x,t)\,\mathrm{d}x.\end{aligned}$$

We multiply this inequality by $1/\overline{\theta}(t)$ and use (3.29), Hölder's inequality, and Young's inequality to prove

$$\begin{aligned}\frac{1}{\overline{\theta}(t)}(\|u_{xt}(\cdot,t)\|_2^2 + \|(G[w])_t(\cdot,t)w_t(\cdot,t)\|_1 + \|\mathcal{D}_1[u_x,w](\cdot,t)\|_1) \\ \leqslant \frac{\partial\ln\overline{\theta}(t)}{\partial t} + \frac{1}{\overline{\theta}(t)}\|g(\cdot,t,\theta(\cdot,t))\|_1 - \frac{\partial}{\partial t}\int_\Omega\mathcal{F}_2[u_x,w](x,t)\,\mathrm{d}x \\ + \frac{C_{35}}{\overline{\theta}(t)}(\|\theta(\cdot,t)-\overline{\theta}(t)\|_\infty^2 + \|u_{xt}(\cdot,t)\|_1^2 + \|(G[w])_t(\cdot,t)\|_1^2).\end{aligned}$$

Integrating this inequality over time, and using (3.6), (3.38), (3.39), (3.36), and (3.40), we observe that (3.50) is proved. The estimates in (3.51) follow by applying (3.9), (1.6), (3.32), (H4), and (3.17). □

Lemma 3.10. *There is a positive constant C_{36} such that*

$$\int_0^\infty (\|\tilde{\sigma}(\cdot,\tau)\|_2^2 + \|p_t(\cdot,\tau)\|_2^2)\,\mathrm{d}\tau \leqslant C_{36}. \tag{3.52}$$

Proof. Let $J(x,t)\colon \Omega_\infty \to \mathbb{R}$ be defined by

$$\begin{aligned} J(x,t) := \tilde{\sigma}(x,t) + \mathcal{H}_2[u_x,w](x,t)\bigg(\overline{\theta}(t) - \theta(x,t) + \frac{1}{2}\|u_t(\cdot,t)\|_2^2 \\ + \int_1^x (f_\infty(\xi) - f(\xi,t))\,\mathrm{d}\xi\bigg) \quad \text{a.e. in } \Omega. \end{aligned} \tag{3.53}$$

Using (3.11) two times, we get

$$\begin{aligned} (\tilde{\sigma}(x,t))^2 &= p_t(x,t)\tilde{\sigma}(x,t) - p_{xx}(x,t)\tilde{\sigma}(x,t) \\ &= p_t(x,t)J(x,t) + (\tilde{\sigma}(x,t) + p_{xx}(x,t))(\tilde{\sigma}(x,t) - J(x,t)) \\ &\quad - p_{xx}(x,t)\tilde{\sigma}(x,t). \end{aligned}$$

Integrating this equation over Ω, and using Young's inequality, (3.53), (3.29), and (3.39), we observe that

$$\begin{aligned} \frac{1}{2}\|\tilde{\sigma}(\cdot,t)\|_2^2 \leqslant \frac{\partial}{\partial t}\int_\Omega p(x,t)J(x,t)\,\mathrm{d}x - \int_\Omega p(x,t)\frac{\partial J(x,t)}{\partial t}\,\mathrm{d}x \\ + C_{37}(\|p_{xx}(\cdot,t)\|_2^2 + \|\theta_x(\cdot,t)\|_2 + \|u_t(\cdot,t)\|_2^4 \\ + \|f(\cdot,t) - f_\infty(\cdot,t)\|_1^2). \end{aligned} \tag{3.54}$$

Applying (3.53), (3.10), (1.2), (H7), and (3.2), we observe that

$$J(x,t) = \mathcal{H}_1[u_x,w](x,t) + \mathcal{H}_2[u_x,w](x,t)(I_1(t) + I_0) + \int_1^x f_\infty(\xi)\,\mathrm{d}\xi. \tag{3.55}$$

Hence, using (3.29), (3.38), (H7), Hölder's inequality, Young's inequality, (3.36), (3.51), and (3.40), we get uniform bounds for J and, for all $s \geqslant 0$,

$$-\int_0^s\int_\Omega p(x,t)\frac{\partial J(x,t)}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^s \bigg(\|p(\cdot,t)\|_\infty^2 + \bigg\|\frac{\partial J(\cdot,t)}{\partial t}\bigg\|_2^2\bigg)\,\mathrm{d}t \leqslant C_{38}.$$

Integrating now (3.54) with respect to time and using (3.16), (3.51), (3.40), (3.5), (3.36), and (H7), we have shown the estimate for $\tilde{\sigma}$ in (3.52). Combining this estimate with (3.11) and (3.51), we get the estimate for p_t. □

Lemma 3.11. *Let $\zeta \in L^2_{\rm loc}(0,\infty; H^2(\Omega)) \cap H^1_{\rm loc}(0,\infty; L^2(\Omega))$ be the solution to the parabolic initial-boundary value problem*

$$\zeta_t - \zeta_{xx} = \tilde{\sigma}_t \quad \text{a.e. in } \Omega_\infty, \tag{3.56}$$

$$\zeta_x(0,t) = \zeta(1,t) = 0 \quad \forall\, t \geqslant 0, \;\; \zeta(\cdot,0) \equiv 0. \tag{3.57}$$

Then we have a positive constant C_{39} such that, for all $t \geqslant 0$,

$$\|\zeta(\cdot,t)\|_\infty^2 \leqslant C_{39}\biggl(1 + \max_{0\leqslant\tau\leqslant t} \|\theta(\cdot,\tau)\|_\infty^{3/2} + \Bigl(\int_0^t \|\theta_t(\cdot,\tau)\|_2^2 \,\mathrm{d}\tau\Bigr)^{3/4}\biggr). \tag{3.58}$$

P r o o f. Multiplying (3.56) by ζ, integrating over $\Omega \times (0,T)$, performing partial integrations, and using (3.57), we get for all $t > 0$

$$\begin{aligned} \frac{1}{2}\|\zeta(\cdot,t)\|_2^2 + \int_0^t \|\zeta_x(\cdot,\tau)\|_2^2 \,\mathrm{d}\tau \\ &= \int_0^t\int_\Omega \tilde{\sigma}_t(x,\tau)\zeta(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ &= \int_\Omega \tilde{\sigma}(x,t)\zeta(x,t)\,\mathrm{d}x - \int_0^t\int_\Omega \tilde{\sigma}(x,\tau)\zeta_t(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{3.59}$$

Because of (3.10), (3.30), (3.40), and (H7), we have a uniform upper bound for $\|\tilde{\sigma}(\cdot,t)\|_2$. Hence, we get from (3.59) by applying Hölder's inequality, Young's inequality, and (3.52) that

$$\frac{1}{4}\|\zeta(\cdot,t)\|_2^2 + \int_0^t \|\zeta_x(\cdot,\tau)\|_2^2 \,\mathrm{d}\tau \leqslant C_{40}\Bigl(\int_0^t \|\zeta_t(\cdot,\tau)\|_2^2 \,\mathrm{d}\tau\Bigr)^{1/2}. \tag{3.60}$$

Formally, we test (3.56) with ζ_t, use (3.57), integrate over time, and apply Young's inequality to deduce that

$$\int_0^t \|\zeta_t(\cdot,\tau)\|_2^2 \,\mathrm{d}\tau + \|\zeta_x(\cdot,t)\|_2^2 \leqslant \frac{1}{2}\int_0^t \|\zeta_t(\cdot,\tau)\|_2^2 \,\mathrm{d}\tau + \frac{1}{2}\int_0^t \|\tilde{\sigma}_t(\cdot,\tau)\|_2^2 \,\mathrm{d}\tau. \tag{3.61}$$

For a rigorous derivation of this inequality, one has to consider (3.56) with $\tilde{\sigma}_t$ replaced by some smooth approximation, perform this computation for the corresponding solutions, and consider afterwards the limit.

Inserting (3.60) into the left-hand side of (3.61) and using (3.10), (1.2), (3.36), Hölder's inequality, Young's inequality, (3.29), (3.51), (H6), and (H7), we observe that

$$\frac{1}{2\cdot 4^2 C_{40}^2}\|\zeta(\cdot,t)\|_2^4+\|\zeta_x(\cdot,t)\|_2^2 \leqslant \frac{1}{2}\int_0^t \|(\mathcal{H}_1[u_x,w]+\theta\mathcal{H}_2[u_x,w]+F)_t(\cdot,\tau)\|_2^2\,\mathrm{d}\tau \leqslant C_{41}+C_{42}\max_{0\leqslant\tau\leqslant t}\|\theta(\cdot,\tau)\|_\infty^2+C_{43}\int_0^t\|\theta_t(\cdot,\tau)\|_2^2\,\mathrm{d}\tau. \tag{3.62}$$

Thanks to the Gagliardo-Nirenberg inequality (see below) and Young's inequality, we conclude that

$$\begin{aligned}\|\zeta(\cdot,t)\|_\infty^2 &\leqslant (C_{44}\|\zeta_x(\cdot,t)\|_2^{1/2}\|\zeta(\cdot,t)\|_2^{1/2}+C_{45}\|\zeta(\cdot,t)\|_2)^2\\ &\leqslant C_{46}(1+\|\zeta_x(\cdot,t)\|_2^{3/2}+\|\zeta(\cdot,t)\|_2^3).\end{aligned}$$

Now, we apply (3.62) and Young's inequality to prove that (3.58) holds. □

The following version of the Gagliardo-Nirenberg inequality is a special case, more general formulations can be found, e.g., in [3], [39].

Lemma 3.12 (Gagliardo-Nirenberg inequality). *For all $p\geqslant 1$ there are positive constants C_{47}, C_{48} such that*

$$\|v\|_\infty \leqslant C_{47}\|v_x\|_2^{2/(p+2)}\|v\|_p^{p/(p+2)}+C_{48}\|v\|_p \quad \forall\, v\in H^1(\Omega). \tag{3.63}$$

Lemma 3.13. *There is a positive constant C_{49} such that*

$$\|u_{xt}(\cdot,t)\|_\infty^2 \leqslant C_{49}\biggl(1+\max_{0\leqslant\tau\leqslant t}\|\theta(\cdot,\tau)\|_\infty^2+\biggl(\int_0^t\|\theta_t(\cdot,\tau)\|_2^2\,\mathrm{d}\tau\biggr)^{3/4}\biggr). \tag{3.64}$$

P r o o f. Let $z_1,z_2\colon \overline{\Omega}_\infty\to\mathbb{R}$ be the solutions to the parabolic initial-boundary value problems

$$z_{i,t}-z_{i,xx}=0 \quad \text{a.e. in } \Omega_\infty \quad \forall\, i\in\{1,2\}, \tag{3.65}$$

$$z_i(1,t)=z_{i,x}(0,t)=0 \quad \text{for a.e. } t>0 \quad \forall\, i\in\{1,2\}, \tag{3.66}$$

$$z_1(x,0)=u_{1,x}(x,0),\quad z_2(x,0)=\tilde\sigma(x,0) \quad \text{a.e. in } \Omega. \tag{3.67}$$

Let $z_3\colon\overline{\Omega}_\infty\to\mathbb{R}$ be defined by

$$z_3(x,t)=\int_1^x\int_0^y z_1(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}y+\int_0^t(z_2(x,\tau)+\zeta(x,\tau))\,\mathrm{d}\tau \quad \forall\,(x,t)\in\Omega_\infty. \tag{3.68}$$

Recalling (3.65), (3.66), (3.67), (3.56), (3.57), and (H1), we observe that

$$(3.69)\qquad z_{3,t}=z_1+z_2+\zeta,\quad z_{3,xx}=z_1+z_2+\zeta-\tilde{\sigma}\quad\text{a.e. in }\Omega_\infty,$$
$$z_3(1,t)=0=z_{3,x}(0,t)\quad\text{for a.e. }t\geqslant 0,\quad z_3(x,0)=\int_1^x u_1(\xi)\,\mathrm{d}\xi\quad\forall\,x\in\Omega.$$

Hence, we see that z_3 is a solution to the linear parabolic initial-boundary value problem considered in (3.11)–(3.13). Since p is the unique solution to this problem, we have $p=z_3$ a.e. on Ω_∞. Therefore, recalling $u_{xt}=p_{xx}$ and (3.69), we have

$$(3.70)\qquad u_{xt}=z_{3,xx}=z_1+z_2+\zeta-\tilde{\sigma}\quad\text{a.e. in }\Omega_\infty.$$

Using (3.67), (H1), (3.10), (1.2), (1.6), (H6), and (H7), we get uniform bounds for $z_1(\cdot,0)$ and $z_2(\cdot,0)$. Applying the maximum principle for linear parabolic equations, we get uniform bounds for z_1 and z_2. Because of (3.10), (H7), and (3.30), we have

$$\tilde{\sigma}\leqslant C_{50}+C_{51}\theta\quad\text{a.e. in }\Omega_\infty.$$

Thus, applying (3.70), (3.58), and Young's inequality yields that (3.64) holds. □

Lemma 3.14. *There is a positive constant C_{52} such that*

$$(3.71)\qquad \sup_{0\leqslant\tau\leqslant t}\|\theta_x(\cdot,\tau)\|_2+\int_0^t\|\theta_t(\cdot,\tau)\|_2^2\,\mathrm{d}\tau\leqslant C_{52}.$$

Proof. Testing (1.3) by θ_t, using (1.6), (H2), Young's inequality, Hölder's inequality, and (3.30), we see that

$$\begin{aligned}(3.72)\quad &\frac{1}{2}\|\theta_t(\cdot,t)\|_2^2+\frac{1}{2}\frac{\partial}{\partial t}\|\theta_x(\cdot,t)\|_2^2\\ &\leqslant\frac{1}{2}\|u_{xt}^2(\cdot,t)+\sigma u_{xt}(\cdot,t)-(\mathcal{F}_1[u_x,t])_t(\cdot,t)+g(\cdot,t,\theta(\cdot,t))\|_2^2\\ &\leqslant C_{53}\|u_{xt}(\cdot,t)\|_2^2(\|u_{xt}(\cdot,t)\|_\infty^2+1+\|\theta(\cdot,t)\|_\infty^2)\\ &\quad+C_{54}(\mathcal{F}_1[u_x,t])_t(\cdot,t)\|_2^2+C_{55}\|g_1(\cdot,t)\|_2^2\\ &\quad+C_{56}\|g_2(\cdot,t)\|_2^2\|\theta(\cdot,t)\|_\infty^2.\end{aligned}$$

Integrating this equation over time, using (1.7), (H1), (H2), Hölder's inequality, (3.50), (3.51), and (3.64), we see that

$$\begin{aligned}(3.73)\quad &\int_0^s\|\theta_t(\cdot,t)\|_2^2\,\mathrm{d}t+\|\theta_x(\cdot,s)\|_2^2\\ &\leqslant C_{57}+C_{58}\max_{0\leqslant t\leqslant s}(\|u_{xt}(\cdot,t)\|_\infty^2+\|\theta(\cdot,t)\|_\infty^2)\\ &\leqslant C_{59}+C_{60}\Big(\int_0^s\|\theta_t(\cdot,t)\|_2^2\,\mathrm{d}t\Big)^{\frac{3}{4}}+C_{61}\max_{0\leqslant t\leqslant s}\|\theta(\cdot,t)\|_\infty^2.\end{aligned}$$

Thanks to the Gagliardo Nirenberg inequality and (3.5), we have

$$\|\theta(\cdot,t)\|_\infty \leqslant C_{62}\|\theta_x(\cdot,t)\|_2^{2/3}\|\theta(\cdot,t)\|_1^{1/3} + C_{63}\|\theta(\cdot,t)\|_1 \leqslant C_{64} + C_{65}\|\theta_x(\cdot,t)\|_2^{2/3}.$$

Using this inequality to estimate the right-hand side of (3.73), and applying Young's inequality afterwards, we see that (3.71) holds. □

Lemma 3.15. *There are positive constants C_{66}, C_{67} such that*

$$\sup_{0\leqslant t}(\|\theta(\cdot,t)\|_\infty + \|u_{xt}(\cdot,t)\|_\infty + \|\sigma(\cdot,t)\|_\infty + \|w_t(\cdot,t)\|_\infty) \leqslant C_{66}, \tag{3.74}$$

$$\int_0^\infty (\|\sigma_t(\cdot,t)\|_2^2 + \|\psi_t(\cdot,t)\|_2^2 + \|\tilde\sigma_t(\cdot,t)\|_2^2)\,\mathrm{d}t \leqslant C_{67}, \tag{3.75}$$

$$\int_0^\infty (|\mathcal{D}_1[u_x(x,\cdot),w(x,\cdot)](t)| + |\mathcal{D}_2[u_x(x,\cdot),w(x,\cdot)](t)|)\,\mathrm{d}t < \infty \quad \textit{for a.e. } x\in\Omega. \tag{3.76}$$

Proof. Using (3.39) and (3.71), we get the estimate for θ in (3.75) and applying in addition (3.64) and (3.30) leads to the remaining estimates in (3.74). Invoking (1.2), (1.5), (3.51), (3.74), (3.71), and (3.29), we get the estimates for σ_t and ψ_t. Utilizing also (3.10), (H7), and (3.36), we derive the estimates for $\tilde\sigma_t$. Combining (3.35) and (3.50) and using Fubini's theorem, we see that (3.76) holds. □

4. Proof of the asymptotic results

As in the preceding section, it will be assumed that (H1)–(H8) are satisfied, and that a solution (u,θ,w) to (1,1)–(1.7) is given, such that (2.22)–(2.25) holds.

For proving the asymptotic results in Theorem 1 and in Theorem 2 with an argumentation similar to [33, Section 4], the following modification of [34, Lemma 3.1] will be used. In the original formulation, it was assumed that the inequality in (4.1) holds for all t in the interval considered, but the proof in [34] can also be used if this inequality holds only for a.e. t in the interval considered.

Lemma 4.1. *Suppose that y and h are nonnegative functions on $(0,\infty)$, with y' locally integrable, such that there are positive constants $A_1,\dots,A_4$ satisfying*

$$y'(t) \leqslant A_1y^2(t) + A_2 + h(t) \quad \textit{for a.e. } t\in(0,\infty), \tag{4.1}$$

$$\int_0^\infty y(t)\,\mathrm{d}t \leqslant A_3, \quad \int_0^\infty h(t)\,\mathrm{d}t \leqslant A_4. \tag{4.2}$$

Then we have $\lim_{t\to\infty} y(t) = 0$.

Lemma 4.2. *We have* (2.27) *and*

$$\lim_{t\to\infty} \|p_x(\cdot,t)\|_2 = \lim_{t\to\infty} \|u_t(\cdot,t)\|_2 = 0, \tag{4.3}$$
$$\lim_{t\to\infty} \|\tilde\sigma(\cdot,t)\|_2 = \lim_{t\to\infty} \|q_t\|_2 = 0. \tag{4.4}$$

Proof. Testing (3.11) with $-p_{xx}$, applying (3.12) and Young's inequality, we see that

$$\frac{1}{2}\frac{\partial}{\partial t}\|p_x(\cdot,t)\|_2^2 + \|p_{xx}(\cdot,t)\|_2^2 \leqslant \frac{1}{2}\|p_{xx}(\cdot,t)\|_2^2 + \frac{1}{2}\|\tilde\sigma(\cdot,t)\|_2^2 \quad \text{for a.e. } t\in(0,\infty).$$

Since $u_t = p_x$ a.e. in Ω_∞, we see by recalling (3.36) and (3.52) that we can apply Lemma 4.1 to show that (4.3) holds. We have, by Young's inequality,

$$\frac{\partial}{\partial t}\|\tilde\sigma(\cdot,t)\|_2^2 = 2\int_\Omega \tilde\sigma(x,t)\tilde\sigma_t(x,t)\,\mathrm{d}x \leqslant \|\tilde\sigma(\cdot,t)\|_2^2 + \|\tilde\sigma_t(\cdot,t)\|_2^2 \quad \text{for a.e. } t\in(0,\infty).$$

Invoking (3.52), (3.75), and Lemma 4.1, we get the convergence result for $\tilde\sigma$ in (4.4). Since (3.14), (3.10), and (H7) yield that $q_t = -\tilde\sigma$, we also have the result for q_t in (4.4). Combining (4.4), (3.10), (H7), and the definition on F_∞ in (2.29), we get (2.27). □

Lemma 4.3. *We have*

$$\lim_{t\to\infty} \|p_t(\cdot,t)\|_2 = \lim_{t\to\infty} \|p_{xx}(\cdot,t)\|_2 = \lim_{t\to\infty} \|u_{xt}(\cdot,t)\|_2 = \lim_{t\to\infty} \|u_t(\cdot,t)\|_\infty = 0. \tag{4.5}$$

Proof. Differentiating (3.11) with respect to t, testing it afterwards by p_t, and applying (3.11) and Young's inequality, we see that

$$\frac{\partial}{\partial t}\|p_t(\cdot,t)\|_2^2 + \|p_{xt}(\cdot,t)\|_2^2 \leqslant \frac{1}{2}\|p_t(\cdot,t)\|_2^2 + \frac{1}{2}\|\tilde\sigma_t(\cdot,t)\|_2^2 \quad \text{for a.e. } t\in(0,\infty).$$

Using (3.52), (3.75), and Lemma 4.1, we get the convergence result for p_t in (4.5). By (3.11), we can combine this with (4.4) to prove the convergence result for p_{xx} in (4.5). Recalling also (3.9), we get the convergence result for u_{xt} and using (1.6), we obtain the result for u_t. □

Lemma 4.4. *We have*

$$\lim_{t\to\infty}\|\theta_x(\cdot,t)\|_2=\lim_{t\to\infty}\|\theta(\cdot,t)-\overline{\theta}(t)\|_\infty=0. \tag{4.6}$$

Moreover, we have some constant $\theta_*>0$ *such that* (2.30) *holds.*

P r o o f. Combining (3.72) with (3.74), we get for a.e. $t\in(0,\infty)$

$$\frac{1}{2}\frac{\partial}{\partial t}\|\theta_x(\cdot,t)\|_2^2\leqslant C_{68}(\|u_{xt}(\cdot,t)\|_2^2+\|(\mathcal{F}_1[u_x,t])_t(\cdot,t)\|_2^2+\|g_1(\cdot,t)\|_2^2+\|g_2(\cdot,t)\|_2^2).$$

Because of (3.40), (3.50), (3.51), and (H2), we can now use Lemma 4.1 to get the convergence result for θ_x. Recalling (3.39), we obtain the result for $\theta-\overline{\theta}$. Combining this with (3.38), we get some $t_0>0$ such that

$$\theta(x,t)>\frac{1}{2}C_{17}\quad\forall\,x\in\overline{\Omega},\ \ t\geqslant t_0.$$

Moreover, (2.23) and (2.25) yield that θ is continuous and positive on $\overline{\Omega}\times[0,t_0]$, and therefore also bounded from below by a positive constant C' on this set. Setting $\theta_*:=\min(\frac{1}{2}C_{17},C')$, we see that (2.30) holds. □

This completes the proof of Theorem 1.

Now, the additional convergence results in Theorem 2 will be proved.

Lemma 4.5. *If* $\mathcal{G}$ *is the identity operator, then we have* (2.32) *and*

$$\lim_{t\to\infty}\|w_t(\cdot,t)\|_2=\lim_{t\to\infty}\|\psi(\cdot,t)\|_2=0. \tag{4.7}$$

P r o o f. Testing the time derivative of (1.4) by w_t and using Young's inequality, we see that for a.e. $t\in(0,\infty)$

$$\frac{\partial}{\partial t}\|w_t(\cdot,t)\|_2^2\leqslant\int_\Omega w_t(x,t)\psi_t(x,t)\,\mathrm{d}x\leqslant\frac{1}{2}\|w_t(\cdot,t)\|_2^2+\frac{1}{2}\|\psi_t(\cdot,t)\|_2^2.$$

By assumption, we have $w_t=(\mathcal{G}[w])_t$, and can therefore apply (3.51), (3.75), Lemma 4.1, and (1.4) to show that (4.7) holds. Using now (H6) iii) and (4.5), we get also (2.32). □

Lemma 4.6. *Assume that* $\mathcal{H}_1 \equiv \mathcal{H}_3 \equiv \mathcal{F}_1 \equiv 0$, $g \equiv 0$, *and* $f \equiv 0$. *Then, we have*

$$\theta(\cdot, t) \to \|\theta_0\|_1 + \frac{\varrho}{2C_V}\|u_1\|_2^2 \ \textit{as}\ t \to \infty, \quad \textit{in}\ L^\infty(\Omega) \tag{4.8}$$

and (2.34). *If* $\mathcal{G}$ *is the identity operator then we have* (2.35).

Proof. Thanks to the assumptions, (3.4), (3.10), (1.2), (H7), and (H5), we see that $I_1 \equiv 0$, that I_0/C_V is equal to the right-hand side of (4.8), and that $\tilde{\sigma} = \theta\mathcal{H}_2[u_x, w]$. Invoking (3.2), (4.5), (4.6), (4.4), and (H1), we get (4.8) and (2.34). If $\mathcal{G}$ is the identity operator then it follows from (4.7), $\psi = \theta\mathcal{H}_4[u_x, w]$, and (4.8) that (2.35) holds. □

Lemma 4.7. *If* (H9) *holds then there is a* $u_\infty \in W^{1,\infty}(\Omega)$ *such that* (2.36)–(2.37) *hold.*

Proof. Owing to (3.76) and (H9), we have a function $\varepsilon_\infty\colon\ \Omega \to \mathbb{R}$ such that

$$u_x(x, t) \to \varepsilon_\infty(x) \ \text{as}\ t \to \infty, \quad \text{for a.e. } x \in \Omega. \tag{4.9}$$

Invoking (3.17), compactness, and properties of weak-star and weak convergence, we see that $u_x(\cdot, t) \to \varepsilon_\infty$ as $t \to \infty$ weakly-star in L^∞_Ω. Defining now $u_\infty(x) := \int_0^x \varepsilon_\infty(\xi)$ and using (1.6), we conclude that $u_\infty \in W^{1,\infty}(\Omega)$ and (2.36)–(2.37) hold. □

Lemma 4.8. *If* (H10) *holds then there is a* $w_\infty \in L^\infty(\Omega)$ *such that* (2.38) *holds.*

Proof. Thanks to (3.76), (H10), (3.17), compactness, and properties of weak convergence, we get a $w_\infty \in L^\infty(\Omega)$ such that (2.38) holds. □

Hence, Theorem 2 is proved.

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