

Applications of Mathematics

Shanghai Jia; Deli Li; Tang Liu; Shu Hua Zhang

Richardson extrapolation and defect correction of mixed finite element methods for integro-differential equations in porous media

Applications of Mathematics, Vol. 53 (2008), No. 1, 13–39

Persistent URL: <http://dml.cz/dmlcz/134697>

Terms of use:

© Institute of Mathematics AS CR, 2008

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

RICHARDSON EXTRAPOLATION AND DEFECT CORRECTION
OF MIXED FINITE ELEMENT METHODS FOR
INTEGRO-DIFFERENTIAL EQUATIONS IN POROUS MEDIA*

SHANGHUI JIA, Beijing, DELI LI, Thunder Bay, TANG LIU, Tianjin,

SHUHUA ZHANG, Tianjin

(Received May 9, 2006)

Abstract. Asymptotic error expansions in the sense of L^∞ -norm for the Raviart-Thomas mixed finite element approximation by the lowest-order rectangular element associated with a class of parabolic integro-differential equations on a rectangular domain are derived, such that the Richardson extrapolation of two different schemes and an interpolation defect correction can be applied to increase the accuracy of the approximations for both the vector field and the scalar field by the aid of an interpolation postprocessing technique, and the key point in deriving them is the establishment of the error estimates for the mixed regularized Green's functions with memory terms presented in R. Ewing et al., *Int. J. Numer. Anal. Model 2* (2005), 301–328. As a result of all these higher order numerical approximations, they can be used to generate a posteriori error estimators for this mixed finite element approximation.

Keywords: integro-differential equations, mixed finite element methods, mixed regularized Green's functions, asymptotic expansions, interpolation defect correction, interpolation postprocessing, a posteriori error estimators

MSC 2000: 76S05, 45K05, 65M12, 65M60, 65R20

1. INTRODUCTION

The aim of this paper is to discuss the asymptotic behavior of the mixed finite ele-

*This project was supported in part by the Special Funds for Major State Basic Research Project (2007CB8149), the National Natural Science Foundation of China (10471103 and 10771158), the Social Science Foundation of the Ministry of Education of China (Numerical methods for convertible bonds, 06JA630047), the NSERC, Tianjin Natural Science Foundation (07JCYBJC14300), and Tianjin University of Finance and Economics.

ment approximation for a parabolic integro-differential equation with the Neumann boundary condition: Find $u = u(x, t)$ such that

$$\begin{aligned}
 (1.1) \quad & u_t = \nabla \cdot \boldsymbol{\sigma} + cu + f && \text{in } \Omega \times J, \\
 & \boldsymbol{\sigma} = A(t) \cdot \nabla u - \int_0^t B(t, s) \cdot \nabla u(s) \, ds && \text{in } \Omega \times J, \\
 & \frac{\partial u}{\partial \mathbf{n}} = 0 && \text{on } \partial\Omega \times J, \\
 & u = u_0(x) && x \in \Omega, \, t = 0,
 \end{aligned}$$

where Ω is an open bounded domain in \mathbb{R}^2 with Lipschitz boundary $\partial\Omega$, and \mathbf{n} is the outward unit normal vector along $\partial\Omega$, $J = (0, T)$ with $T > 0$, $A(t) = A(x, t)$ and $B(t, s) = B(x, t, s)$ are two 2×2 matrices with A being positive definite, c , f and u_0 are known smooth functions.

At present there is an extensive literature available for numerical approximations of the problem (1.1). See, for instance, [6], [7], [17], [19], [20], [22]–[24], [27], [28], [30]–[33] for the finite element method and the finite difference method. Recently, mixed finite element methods have been investigated for (1.1) in [10]–[13], and [16], in which optimal and superconvergent estimates in the L^2 -norm and the L^∞ -norm as well as Richardson extrapolation in L^2 -norm have been obtained by means of a mixed Ritz-Volterra projection, regularized Green’s functions and an interpolation postprocessing method.

In the present paper we study two numerical approaches of higher accuracy—the Richardson extrapolation method of two different forms in the L^∞ -norm and an interpolation defect correction method in the L^2 -norm and the L^∞ -norm. As an efficient numerical method to increase the accuracy of approximations, Richardson extrapolation has been demonstrated in [29] for the difference method, in [3], [5], [8], [15], [19], [21]–[25], [26], [34], [35], [37], [38] for the (Galerkin and Petrov-Galerkin) finite element method and the mixed finite element method, in [18] and [36] for the collocation method and the boundary element method, respectively.

The defect correction of (Galerkin and Petrov-Galerkin) finite elements by means of an interpolation postprocessing technique is another numerical method to obtain approximations of higher accuracy, which has been probed for a wide variety of models. See, for example, [2], [5], [19], [21], [24], and the references cited therein.

As we have done in [14], we employ the analysis for the “short side” in the FE-right triangle plus the sharp integral estimates of the “hypotenuse” (see, for example, [19] and [20]) to present an immediate analysis for the asymptotic expansion of the error between the mixed finite element solution and the corresponding interpolation function of the exact solution to (1.1) in the sense of L^∞ -norm on the basis of the

estimates for the mixed regularized Green's functions with memory terms introduced in [13]. Thus, the asymptotic expansion of the error in the mixed finite element solution is derived, by which the Richardson extrapolation of two different types and the interpolation defect correction can be applied to generate mixed finite element approximations of higher accuracy. In addition, by means of these approximations with higher precision, a class of a posteriori error estimators are constructed for this mixed finite element method.

This paper is organized as follows. In Section 2, the approximate subspace and the variational formula of (1.1) are provided. Also, the asymptotic expansion of the Raviart-Thomas projection is presented in this section for the future need. Section 3 is devoted to investigating the asymptotic expansion of the error between the mixed finite element solution and the Raviart-Thomas projection of the exact solution to (1.1) in the L^∞ -norm. Finally, Section 4 deals with an interpolation defect correction approximation in the L^2 - and the L^∞ -norm based on the results given in Section 3. Furthermore, at each end of Sections 3 and 4, a posteriori error estimators are furnished as by-products of these numerical solutions with higher convergence rates.

2. THE ASYMPTOTIC EXPANSION

In this section we first give the formula of the mixed finite element method for the parabolic integro-differential equation (1.1). For the sake of simplicity of analysis, we take the domain Ω to be a rectangle in this paper.

Let

$$W := L^2(\Omega) \quad \text{and} \quad \mathbf{V} := H(\text{div}, \Omega) = \{\boldsymbol{\sigma} \in (L^2(\Omega))^2 : \nabla \cdot \boldsymbol{\sigma} \in L^2(\Omega)\}$$

be the standard L^2 -space on Ω with norm $\|\cdot\|_0$ and the Hilbert space equipped with the norm

$$\|\boldsymbol{\sigma}\|_{\mathbf{V}} := (\|\boldsymbol{\sigma}\|_0^2 + \|\nabla \cdot \boldsymbol{\sigma}\|_0^2)^{1/2},$$

respectively. Moreover, set

$$\mathbf{V}_0 := \{\boldsymbol{\sigma} \in \mathbf{V} : \boldsymbol{\sigma} \cdot \mathbf{n} = 0 \text{ on } x \in \partial\Omega\}.$$

Then, from [10] we recall that the weak mixed formulation of (1.1) is given by finding $(u, \boldsymbol{\sigma}) \in W \times \mathbf{V}_0$ such that

$$\begin{aligned} (2.1) \quad & (u_t, w) - (\nabla \cdot \boldsymbol{\sigma}, w) - (cu, w) = (f, w), \quad w \in W, \\ & (\alpha \boldsymbol{\sigma}, \mathbf{v}) + \int_0^t (M(t, s) \boldsymbol{\sigma}(s), \mathbf{v}) \, ds + (\nabla \cdot \mathbf{v}, u) = 0, \quad \mathbf{v} \in \mathbf{V}_0, \\ & u(0, x) = u_0(x) \quad \text{in } L^2(\Omega), \end{aligned}$$

where $\alpha = A^{-1}(t)$, $M(t, s) = R(t, s)A^{-1}(s)$, and $R(t, s)$ is the resolvent of the matrix $A^{-1}(t)B(t, s)$ and is presented by

$$R(t, s) = A^{-1}(t)B(t, s) + \int_s^t A^{-1}(t)B(t, \tau)R(\tau, s) d\tau, \quad t > s \geq 0.$$

Let $T_{h,k}$ be a finite element partition of $\bar{\Omega}$ into uniform rectangles and $\mathbf{V}_{h,k} \times W_{h,k} \subset \mathbf{V} \times W$ denote a pair of finite element spaces associated with this partition which satisfy the inf-sup stability condition of Babuška and Brezzi (see, for instance, [1] and [4]), where h and k are the mesh sizes in x - and y -axis, respectively. Some spaces like that have been constructed and analyzed for rectangular elements. However, here our analysis is concentrated on the Raviart-Thomas space of the lowest order; i.e.,

$$(2.2) \quad \begin{aligned} \mathbf{V}_{h,k} &:= \{\mathbf{v}_{h,k} \in \mathbf{V} : \mathbf{v}_{h,k}|_e \in Q_{1,0}(e) \times Q_{0,1}(e), e \in T_{h,k}\}, \\ W_{h,k} &:= \{w_{h,k} \in W : w_{h,k}|_e \in Q_0(e), e \in T_{h,k}\}, \end{aligned}$$

where $Q_{m,n}(e)$ stands for the space of polynomials of degree no more than m and n in x and y on e , respectively. The extension to other stable rectangular element spaces can also be made.

The corresponding semi-discrete version of (2.1) seeks a pair $(u_{h,k}, \boldsymbol{\sigma}_{h,k}) \in W_{h,k} \times \mathbf{V}_{0,h,k} \subset W \times \mathbf{V}_0$ to satisfy

$$(2.3) \quad \begin{aligned} (u_{h,k,t}, w_{h,k}) - (\nabla \cdot \boldsymbol{\sigma}_{h,k}, w_{h,k}) - (cu_{h,k}, w_{h,k}) &= (f, w_{h,k}), \quad w_{h,k} \in W_{h,k}, \\ (\alpha \boldsymbol{\sigma}_{h,k}, \mathbf{v}_{h,k}) + \int_0^t (M(t, s) \boldsymbol{\sigma}_{h,k}(s), \mathbf{v}_{h,k}) ds + (u_{h,k}, \nabla \cdot \mathbf{v}_{h,k}) &= 0, \\ \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}, \quad u_{h,k}(0, x) &= u_{0,h,k}, \end{aligned}$$

where $u_{0,h,k} \in W_{h,k}$ is some appropriately chosen approximation of the initial data $u_0(x)$. Furthermore, $\boldsymbol{\sigma}_{h,k}(0, x)$ is also chosen to satisfy (2.3) at $t = 0$; i.e.,

$$(2.4) \quad (\alpha(0) \boldsymbol{\sigma}_{h,k}(0), \mathbf{v}_{h,k}) + (u_{0,h,k}, \nabla \cdot \mathbf{v}_{h,k}) = 0, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}.$$

In addition, from (2.1) and (2.3) one derives the following mixed finite element error equation:

$$(2.5) \quad \begin{aligned} (u_t - u_{h,k,t}, w_{h,k}) - (\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_{h,k}), w_{h,k}) - (c(u - u_{h,k}), w_{h,k}) &= 0, \\ w_{h,k} &\in W_{h,k}, \\ (\alpha(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{h,k}), \mathbf{v}_{h,k}) + \int_0^t (M(t, s)(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{h,k})(s), \mathbf{v}_{h,k}) ds + (u - u_{h,k}, \nabla \cdot \mathbf{v}_{h,k}) &= 0, \\ \mathbf{v}_{h,k} &\in \mathbf{V}_{0,h,k}. \end{aligned}$$

We recall that the Raviart-Thomas projection

$$\mathbf{\Pi}_{h,k}^0 \times P_{h,k}^0: \mathbf{V} \times W \rightarrow \mathbf{V}_{0,h,k} \times W_{h,k}$$

is defined by the following conditions:

$$(2.6) \quad \int_{s_i} (\boldsymbol{\sigma} - \mathbf{\Pi}_{h,k}^0 \boldsymbol{\sigma}) \cdot \mathbf{n} \, ds = 0, \quad i = 1, 2, 3, 4,$$

$$\int_e (u - P_{h,k}^0 u) = 0,$$

where s_i ($i = 1, 2, 3, 4$) are the four edges of the rectangle $e \in T_{h,k}$ and \mathbf{n} is the outward normal direction on the s_i . This projection has the following properties [9]:

- (i) $P_{h,k}^0$ is the local $L^2(\Omega)$ projection;
- (ii) $\mathbf{\Pi}_{h,k}^0$ and $P_{h,k}^0$ satisfy

$$(2.7) \quad (\nabla \cdot (\boldsymbol{\sigma} - \mathbf{\Pi}_{h,k}^0 \boldsymbol{\sigma}), w_{h,k}) = 0, \quad w_{h,k} \in W_{h,k},$$

$$(\nabla \cdot \mathbf{v}_{h,k}, u - P_{h,k}^0 u) = 0, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{h,k};$$

(iii) there hold the approximation properties,

$$(2.8) \quad \|\boldsymbol{\sigma} - \mathbf{\Pi}_{h,k}^0 \boldsymbol{\sigma}\|_{0,p} \leq CH \|\boldsymbol{\sigma}\|_{1,p}, \quad 1 \leq p \leq \infty,$$

$$\|\nabla \cdot (\boldsymbol{\sigma} - \mathbf{\Pi}_{h,k}^0 \boldsymbol{\sigma})\|_{-s,p} \leq CH^{1+s} \|\nabla \cdot \boldsymbol{\sigma}\|_{1,p}, \quad 0 \leq s \leq 1, \quad 1 \leq p \leq \infty,$$

$$\|u - P_{h,k}^0 u\|_{-s,p} \leq CH^{1+s} \|u\|_{1,p}, \quad 0 \leq s \leq 1, \quad 1 \leq p \leq \infty,$$

where $H := \max\{h, k\}$.

Also, from [14] we recall the following two theorems to conclude the section.

Theorem 2.1. *Assume that $\boldsymbol{\sigma} \in \mathbf{V} \cap (W^{4,p}(\Omega))^2$ for some $p \in [1, \infty]$. Then, for sufficiently smooth functions $\alpha_{ij}(x, y)$ ($1 \leq i, j \leq 2$) we have*

$$\begin{aligned} (\boldsymbol{\alpha} \cdot (\boldsymbol{\sigma} - \mathbf{\Pi}_{h,k}^0 \boldsymbol{\sigma}), \mathbf{v}_{h,k}) &= -\frac{h^2}{3} \int_{\Omega} [\alpha_{11}(\sigma_1)_{xx} + \alpha_{12}(\sigma_2)_{xx}] v_{1,h,k} \\ &\quad + \frac{h^2}{3} \int_{\Omega} [(\alpha_{22})_x (\sigma_2)_x - \alpha_{21}(\sigma_1)_{xx}] v_{2,h,k} \\ &\quad + \frac{k^2}{3} \int_{\Omega} [(\alpha_{11})_y (\sigma_1)_y - \alpha_{12}(\sigma_2)_{yy}] v_{1,h,k} \\ &\quad - \frac{k^2}{3} \int_{\Omega} [\alpha_{22}(\sigma_2)_{yy} + \alpha_{21}(\sigma_1)_{yy}] v_{2,h,k} \\ &\quad + O(H^4) \|\boldsymbol{\sigma}\|_{4,p} \|\mathbf{v}_{h,k}\|_{0,q}, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}, \end{aligned}$$

where σ_1, σ_2 and $v_{1,h,k}, v_{2,h,k}$ are the first components and the second components of the vector-valued functions $\boldsymbol{\sigma}$ and $\mathbf{v}_{h,k}$, respectively, and $q = p/(p-1)$ is the conjugate of p and $\alpha = (\alpha_{ij})_{2 \times 2}$.

Theorem 2.2. Assume that $u \in W^{3,p}(\Omega)$ for some $p \in [1, \infty]$. Then, for a sufficiently smooth function $c(x, y)$ we have the following asymptotic expansion:

$$(c(u - P_{h,k}^0 u), w_{h,k}) = \frac{h^2}{3} \int_{\Omega} c_x u_x w_{h,k} + \frac{k^2}{3} \int_{\Omega} c_y u_y w_{h,k} + O(H^4) \|u\|_{3,p} \|w_{h,k}\|_{0,q},$$

$$w_{h,k} \in W_{h,k},$$

where $q = p/(p-1)$ is the conjugate of p .

From Theorems 2.1 and 2.2 we immediately obtain

Corollary 2.1. If $\boldsymbol{\sigma} \in \mathbf{V} \cap (W^{2,p}(\Omega))^2$ for some $p \in [1, \infty]$, then, we have for sufficiently smooth functions α_{ij} ($1 \leq i, j \leq 2$) that

$$|(\alpha \cdot (\boldsymbol{\sigma} - \boldsymbol{\Pi}_{h,k}^0 \boldsymbol{\sigma}), \mathbf{v}_{h,k})| \leq CH^2 \|\boldsymbol{\sigma}\|_{2,p} \|\mathbf{v}_{h,k}\|_{0,q}, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k},$$

where $q = p/(p-1)$.

Corollary 2.2. If $u \in W^{1,p}(\Omega)$, we have for a sufficiently smooth function $c(x, y)$ that

$$|(c(u - P_{h,k}^0 u), w_{h,k})| \leq CH^2 \|u\|_{1,p} \|w_{h,k}\|_{0,q}, \quad w_{h,k} \in W_{h,k},$$

where $q = p/(p-1)$.

3. THE RICHARDSON EXTRAPOLATION

On the basis of Theorems 2.1 and 2.2, we discuss in this section the asymptotic expansion of the error between the mixed finite element solution and the Raviart-Thomas projection of the exact solution of (1.1) in the L^∞ -norm in order to establish the asymptotic error expansion of the mixed finite element approximation in maximum norm. First, let us define the following two linear operators M^* and M^{**} for any smooth function $f(t)$ defined on $(0, T)$ by

$$(M^* f)(t) := \int_0^t M(t, s) f(s) ds \quad \text{and} \quad (M^{**} f)(t) := \int_t^T M(s, t) f(s) ds.$$

Then, we have the following three lemmas according to [13].

Lemma 3.1. *We have*

$$\int_0^T M * f(t)g(t) dt = \int_0^T f(t)M ** g(t) dt.$$

Lemma 3.2. *Assume that $f(t), g(t) \in L^1(0, T^*)$ and there exists $C > 0$ such that for an arbitrary non-negative $\varphi(t) \in C^\infty[0, T]$,*

$$\left| \int_0^T f(t)\varphi(t) dt \right| \leq C \left| \int_0^T g(t)(1 + \varphi(t)) dt \right|, \quad 0 \leq T \leq T^*.$$

Then, we have

$$|f(t)| \leq C \left| g(t) + \int_0^t g(s) ds \right|, \quad \forall t \in (0, T), \quad \text{a.e.}$$

Especially, if

$$\left| \int_0^T f(t)\varphi(t) dt \right| \leq C \left| \int_0^T g(t)\varphi(t) dt \right|,$$

then,

$$|f(t)| \leq C|g(t)|, \quad \forall t \in (0, T), \quad \text{a.e.}$$

Lemma 3.3. *Suppose that the matrix $A(t)$ is positive definite. Then, the norms $\|\boldsymbol{\sigma}\|_0 := (\boldsymbol{\sigma}, \boldsymbol{\sigma})^{1/2}$ and $\|\boldsymbol{\sigma}\|_{A^{-1}} := (A^{-1}\boldsymbol{\sigma}, \boldsymbol{\sigma})^{1/2}$ are equivalent.*

Next we shall define two regularized Green's functions with memory terms for problem (1.1) in mixed form in the fashion analogous to that employed earlier for finite elements methods of elliptic and integro-differential equations ([39] and [28]) as well as mixed finite element methods of elliptic equations [35], which will play an important role in the analysis of L^∞ -norm estimates.

For an arbitrary point $z_0 \in \bar{\Omega}$ the first pair regularized Green's function $(\mathbf{G}_1, \lambda_1) = (\mathbf{G}_1(z, z_0), \lambda_1(z, z_0))$ with memory is defined by

$$(3.1) \quad \begin{aligned} \alpha \mathbf{G}_1 + M ** \mathbf{G}_1 - \nabla \lambda_1 &= 0 && \text{in } \Omega \times (0, T), \\ \operatorname{div} \mathbf{G}_1 &= \delta_1^h \varphi_1(t) && \text{in } \Omega \times (0, T), \\ \lambda_1 &= 01 && \text{on } \partial\Omega \times (0, T), \end{aligned}$$

where $\varphi_1(t) \in C^\infty(0, T)$, and $\delta_1^{h,k} = \delta_1^{h,k}(z, z_0) \in W_h$ is the regularized Dirac δ -function at any fixed point $z_0 \in \bar{\Omega}$. Also, the second pair regularized Green's

function $(\mathbf{G}_2, \lambda_2) = (\mathbf{G}_2(z, z_0), \lambda_2(z, z_0))$ is defined such that

$$(3.2) \quad \begin{aligned} \alpha \mathbf{G}_2 + M ** \mathbf{G}_2 - \nabla \lambda_2 &= \delta_2^h \varphi_2(t) && \text{in } \Omega \times (0, T), \\ \operatorname{div} \mathbf{G}_2 &= 0 && \text{in } \Omega \times (0, T), \\ \lambda_2 &= 0 && \text{on } \partial\Omega \times (0, T), \end{aligned}$$

where $\varphi_2(t) \in C^\infty(0, T)$ and $\delta_2^{h,k}$ is either $(\delta_2^{h,k}, 0)$ or $(0, \delta_2^{h,k})$ with $\delta_2^{h,k}$ being a regularized Dirac δ -function at z_0 . From [13] we know that

$$(3.3) \quad \begin{aligned} \|w_{h,k}\|_\infty &\leq C|(w_{h,k}, \delta_1^{h,k})|, && w_h \in W_h, \\ \|\mathbf{v}_{h,k}\|_\infty &\leq C|(\mathbf{v}_{h,k}, \delta_2^{h,k})|, && \mathbf{v}_{h,k} \in \mathbf{V}_{h,k}. \end{aligned}$$

In addition, we have the following estimates for the regularized Green's functions [13].

Lemma 3.4. *Assume that $(\mathbf{G}_1, \lambda_1)$, $(\mathbf{G}_2, \lambda_2)$ and $(\mathbf{G}_1^{h,k}, \lambda_1^{h,k})$, $(\mathbf{G}_2^{h,k}, \lambda_2^{h,k})$ are the true solutions and the mixed finite element solutions of (3.1) and (3.2), respectively. Then, we have*

$$\begin{aligned} \|\mathbf{G}_1^{h,k}\|_0 &\leq C|\log H|^{1/2}(1 + \varphi_1(t)), \\ \|\nabla \lambda_1^{h,k}\|_0 &\leq C|\log H|^{1/2}(1 + \varphi_1(t)), \\ \|\mathbf{G}_2^{h,k}\|_1 &\leq C|\log H|(1 + \varphi_2(t)), \\ \|\lambda_2^{h,k}\|_0 &\leq C|\log H|^{1/2}(1 + \varphi_2(t)), \\ \|\nabla \lambda_2^{h,k}\|_0 &\leq CH^{-1}(1 + \varphi_2(t)). \end{aligned}$$

3.1. The global Richardson extrapolation in two directions

We first discuss the extrapolation method of mixed finite element approximation for (1.1) in both x and y directions as follows.

Theorem 3.1. *Suppose that $(u, \boldsymbol{\sigma})$ and $(u_{h,k}, \boldsymbol{\sigma}_{h,k})$ are the exact solution of (2.1) and its mixed finite element solution, respectively, with the chosen initial value $u_{h,k}(0) = P_{h,k}^0 u_0$. Then, in the sense of L^∞ -norm we have the following asymptotic expansions under the conditions that $(u, \boldsymbol{\sigma})$, c , α and M are sufficiently smooth:*

$$\begin{aligned} u_{h,k} - P_{h,k}^0 u &= H^2 \xi_{h,k} + O(H^4 |\log H|^{1/2}), \\ \boldsymbol{\sigma}_{h,k} - \boldsymbol{\Pi}_{h,k}^0 \boldsymbol{\sigma} &= H^2 \boldsymbol{\eta}_{h,k} + O(H^3), \end{aligned}$$

where $(\xi_{h,k}, \boldsymbol{\eta}_{h,k}) \in W_{h,k} \times \mathbf{V}_{0,h,k}$ and $P_{h,k}^0 \times \boldsymbol{\Pi}_{h,k}^0: W \times \mathbf{V}_0 \rightarrow W_{h,k} \times \mathbf{V}_{0,h,k}$ is the Raviart-Thomas projection operator.

Proof. Set

$$\varrho_{h,k} := u_{h,k} - P_{h,k}^0 u \quad \text{and} \quad \boldsymbol{\theta}_{h,k} := \boldsymbol{\sigma}_{h,k} - \boldsymbol{\Pi}_{h,k}^0 \boldsymbol{\sigma}.$$

Then, it follows from (2.5) and (2.7) that

$$\begin{aligned} (3.4) \quad & (\alpha \boldsymbol{\theta}_{h,k}, \mathbf{v}_{h,k}) + \int_0^t (M(t,s) \boldsymbol{\theta}_{h,k}(s), \mathbf{v}_{h,k}) \, ds + (\varrho_{h,k}, \nabla \cdot \mathbf{v}_{h,k}) \\ &= (\alpha(\boldsymbol{\sigma} - \boldsymbol{\Pi}_{h,k}^0 \boldsymbol{\sigma}), \mathbf{v}_{h,k}) + \int_0^t (M(t,s)(\boldsymbol{\sigma} - \boldsymbol{\Pi}_{h,k}^0 \boldsymbol{\sigma})(s), \mathbf{v}_{h,k}) \, ds, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}, \\ & \quad (\varrho_{h,k,t}, w_{h,k}) - (\nabla \cdot \boldsymbol{\theta}_{h,k}, w_{h,k}) - (c \varrho_{h,k}, w_{h,k}) \\ & \quad = -(c(u - P_{h,k}^0 u), w_{h,k}), \quad w_{h,k} \in W_{h,k}. \end{aligned}$$

Thus, from Theorems 2.1 and 2.2 we know that for $\mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}$ and $w_{h,k} \in W_{h,k}$

$$(3.5) \quad \begin{aligned} & (\alpha(\boldsymbol{\sigma} - \boldsymbol{\Pi}_{h,k}^0 \boldsymbol{\sigma}) + M * (\boldsymbol{\sigma} - \boldsymbol{\Pi}_{h,k}^0 \boldsymbol{\sigma}), \mathbf{v}_{h,k}) = H^2 \mathbf{L}_{h,k}(\mathbf{v}_{h,k}) + O(H^4) \|\mathbf{v}_{h,k}\|_{0,q}, \\ & -(c(u - P_{h,k}^0 u), w_{h,k}) = H^2 G_{h,k}(w_{h,k}) + O(H^4) \|w_{h,k}\|_{0,q}, \end{aligned}$$

where $\alpha = (\alpha_{ij})_{2 \times 2}$, $M = (m_{ij})_{2 \times 2}$, and

$$\begin{aligned} G_{h,k}(\varphi) &= -\frac{1}{3} \left(\frac{h}{H} \right)^2 \int_{\Omega} c_x u_x \varphi - \frac{1}{3} \left(\frac{k}{H} \right)^2 \int_{\Omega} c_y u_y \varphi, \\ \mathbf{L}_{h,k}(\boldsymbol{\psi}) &= \frac{1}{3} \left(\frac{h}{H} \right)^2 \int_{\Omega} \{ -[\alpha_{11}(\boldsymbol{\sigma}_1)_{xx} + \alpha_{12}(\boldsymbol{\sigma}_2)_{xx}] \boldsymbol{\psi}_1 + [(\alpha_{22})_x(\boldsymbol{\sigma}_2)_x - \alpha_{21}(\boldsymbol{\sigma}_1)_{xx}] \boldsymbol{\psi}_2 \} \\ & \quad + \frac{1}{3} \left(\frac{h}{H} \right)^2 \int_{\Omega} \int_0^t \{ -[m_{11}(\boldsymbol{\sigma}_1)_{xx} + m_{12}(\boldsymbol{\sigma}_2)_{xx}] \boldsymbol{\psi}_1 \\ & \quad \quad \quad + [(m_{22})_x(\boldsymbol{\sigma}_2)_x - m_{21}(\boldsymbol{\sigma}_1)_{xx}] \boldsymbol{\psi}_2 \} \, ds \\ & \quad + \frac{1}{3} \left(\frac{k}{H} \right)^2 \int_{\Omega} \{ [(\alpha_{11})_y(\boldsymbol{\sigma}_1)_y - \alpha_{12}(\boldsymbol{\sigma}_2)_{yy}] \boldsymbol{\psi}_1 - [\alpha_{22}(\boldsymbol{\sigma}_2)_{yy} + \alpha_{21}(\boldsymbol{\sigma}_1)_{yy}] \boldsymbol{\psi}_2 \} \\ & \quad + \frac{1}{3} \left(\frac{k}{H} \right)^2 \int_{\Omega} \int_0^t \{ [(m_{11})_y(\boldsymbol{\sigma}_1)_y - m_{12}(\boldsymbol{\sigma}_2)_{yy}] \boldsymbol{\psi}_1 \\ & \quad \quad \quad - [m_{22}(\boldsymbol{\sigma}_2)_{yy} + m_{21}(\boldsymbol{\sigma}_1)_{yy}] \boldsymbol{\psi}_2 \} \, ds \end{aligned}$$

with $\boldsymbol{\psi} = (\boldsymbol{\psi}_1, \boldsymbol{\psi}_2)$ being a vector-valued function. Evidently,

$$(3.6) \quad \mathbf{L}_{h/2,k/2}(\boldsymbol{\psi}) = \mathbf{L}_{h,k}(\boldsymbol{\psi}) \quad \text{and} \quad G_{h/2,k/2}(\varphi) = G_{h,k}(\varphi).$$

Let $(\xi, \boldsymbol{\eta}) \in W \times \mathbf{V}_0$ and $(\xi_{h,k}, \boldsymbol{\eta}_{h,k}) \in W_{h,k} \times \mathbf{V}_{0,h,k}$ be the variational solution and the mixed finite element solution of the auxiliary problem, respectively,

$$(3.7) \quad \begin{aligned} (\alpha \boldsymbol{\eta}, \mathbf{v}) + (M * \boldsymbol{\eta}, \mathbf{v}) \, ds + (\xi, \nabla \cdot \mathbf{v}) &= \mathbf{L}_{h,k}(\mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_0, \\ (\xi_t, w) - (\nabla \cdot \boldsymbol{\eta}, w) - (c\xi, w) &= G_{h,k}(w), \quad w \in W, \\ \xi(0) &= 0. \end{aligned}$$

Then, from (3.4), (3.5), and (3.7) one finds that

$$\begin{aligned} (\alpha(\boldsymbol{\theta}_{h,k} - H^2 \boldsymbol{\eta}_{h,k}) + M * \boldsymbol{\theta}_{h,k}(s) - H^2 \boldsymbol{\eta}_{h,k}(s), \mathbf{v}_{h,k}) + (\varrho_{h,k} - H^2 \xi_{h,k}, \nabla \cdot \mathbf{v}_{h,k}) \\ = O(H^4) \|\mathbf{v}_{h,k}\|_{0,q}, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}, \\ (\varrho_{h,k,t} - H^2 \xi_{h,k,t}, w_{h,k}) - (\nabla \cdot (\boldsymbol{\theta}_{h,k} - H^2 \boldsymbol{\eta}_{h,k}), w_{h,k}) - (c(\varrho_{h,k} - H^2 \xi_{h,k}), w_{h,k}) \\ = O(H^4) \|w_{h,k}\|_0, \quad w_{h,k} \in W_{h,k}. \end{aligned}$$

Let

$$\boldsymbol{\theta}_{h,k}^* := \boldsymbol{\theta}_{h,k} - H^2 \boldsymbol{\eta}_{h,k} \quad \text{and} \quad \varrho_{h,k}^* := \varrho_{h,k} - H^2 \xi_{h,k}.$$

Thus, we have

$$(3.8) \quad (\alpha \boldsymbol{\theta}_{h,k}^* + M * \boldsymbol{\theta}_{h,k}^*, \mathbf{v}_{h,k}) + (\varrho_{h,k}^*, \nabla \cdot \mathbf{v}_{h,k}) = O(H^4) \|\mathbf{v}_{h,k}\|_{0,q}, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}.$$

Moreover, in [16] we have proved that

$$(3.9) \quad \|\boldsymbol{\theta}_{h,k}^*\|_0 + \|\varrho_{h,k}^*\|_0 \leq CH^4.$$

Set $\mathbf{v}_{h,k} = \mathbf{G}_1^{h,k}$ in (3.8). Then, it follows from Lemma 3.4 for $q = 2$ that

$$(\alpha \boldsymbol{\theta}_{h,k}^* + M * \boldsymbol{\theta}_{h,k}^*, \mathbf{G}_1^{h,k}) + (\varrho_{h,k}^*, \nabla \cdot \mathbf{G}_1^{h,k}) = O(H^4 |\log H|^{1/2}) (1 + \varphi_1(t)),$$

and from Lemma 3.1 and (3.1) that

$$\begin{aligned} \left| \int_0^T (\varrho_{h,k}^*, \delta_1^{h,k}) \varphi_1(t) \, dt \right| &\leq \left| \int_0^T (\alpha \mathbf{G}_1^{h,k} + M * \mathbf{G}_1^{h,k}, \boldsymbol{\theta}_{h,k}^*) \, dt \right| \\ &\quad + CH^4 |\log H|^{1/2} \int_0^T (1 + \varphi_1(t)) \, dt \\ &= \left| \int_0^T (\nabla \lambda_1^{h,k}, \boldsymbol{\theta}_{h,k}^*) \, dt \right| + CH^4 |\log H|^{1/2} \int_0^T (1 + \varphi_1(t)) \, dt. \end{aligned}$$

Hence, from Lemma 3.4 and (3.9) we get

$$\left| \int_0^T (\varrho_{h,k}^*, \delta_1^{h,k}) \varphi_1(t) \, dt \right| \leq C |\log H|^{1/2} H^4 \int_0^T (1 + \varphi_1(t)) \, dt$$

which, together with Lemma 3.2 and (3.3), leads to

$$(3.10) \quad \|\varrho_{h,k}^*\|_\infty \leq CH^4 |\log H|^{1/2}.$$

Letting $\mathbf{v}_{h,k} = \mathbf{G}_2^{h,k}$ in (3.8), we have according to Lemma 3.4 that

$$(\alpha \boldsymbol{\theta}_{h,k}^* + M * \boldsymbol{\theta}_{h,k}^*, \mathbf{G}_2^{h,k}) + (\varrho_{h,k}^*, \nabla \cdot \mathbf{G}_2^{h,k}) = O(H^4 |\log H|)(1 + \varphi_2(t)).$$

Since $\nabla \cdot \mathbf{G}_2^{h,k} = 0$ by the mixed finite element approximation of (3.2), we further obtain by means of Lemmas 3.1, 3.4, and (3.9) that

$$\begin{aligned} \left| \int_0^T (\boldsymbol{\theta}_{h,k}^*, \delta_2^{h,k}) \varphi_2(t) dt \right| &\leq \left| \int_0^T (\nabla \lambda_2^{h,k}, \boldsymbol{\theta}_{h,k}^*) dt \right| + CH^4 |\log H| \int_0^T (1 + \varphi_2(t)) dt \\ &\leq CH^3 \int_0^T (1 + \varphi_2(t)) dt. \end{aligned}$$

Finally, we get via Lemma 3.2 and (3.3) that

$$\|\boldsymbol{\theta}_{h,k}^*\|_\infty \leq CH^3$$

which, together with (3.10), completes the proof of Theorem 3.1. \square

For the future need we shall next present a superconvergent estimate for the mixed finite element solution of (3.7). For this purpose we first introduce the so-called mixed Ritz-Volterra projection.

Definition 3.1. For $(u, \boldsymbol{\sigma}) \in W \times \mathbf{V}$ its mixed Ritz-Volterra projection

$$(\bar{u}_{h,k}, \bar{\boldsymbol{\sigma}}_{h,k}): [0, T] \rightarrow W_{h,k} \times \mathbf{V}_{0,h,k}$$

is defined by

$$\begin{aligned} (\alpha(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}_{h,k}) + M * (\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}_{h,k}), \mathbf{v}_{h,k}) + (u - \bar{u}_{h,k}, \nabla \cdot \mathbf{v}_{h,k}) &= 0, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{h,k}, \\ (\nabla \cdot (\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}_{h,k}), w_{h,k}) + (c(u - \bar{u}_{h,k}), w_{h,k}) &= 0, \quad w_{h,k} \in W_{h,k}. \end{aligned}$$

In [10] it has been shown that the mixed Ritz-Volterra projection $(\bar{u}_{h,k}, \bar{\boldsymbol{\sigma}}_{h,k}) \in W_{h,k} \times \mathbf{V}_{h,k}$ for an arbitrary pair $(u, \boldsymbol{\sigma}) \in W \times \mathbf{V}$ exists uniquely. In addition, we have [10]

$$(3.11) \quad \|P_{h,k}^0 u - \bar{u}_{h,k}\|_W + \|\mathbf{\Pi}_{h,k}^0 \boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}_{h,k}\|_{\mathbf{V}} = O(H^2),$$

where

$$\|u\|_W := \|u\|_0 \quad \text{and} \quad \|\boldsymbol{\sigma}\|_{\mathbf{V}} := (\|\boldsymbol{\sigma}\|_0^2 + \|\nabla \cdot \boldsymbol{\sigma}\|_0^2)^{1/2}.$$

In order to obtain superconvergence of the mixed finite element approximation for the variational equation (3.7), we choose the initial data approximation as the mixed elliptic projection; i.e.,

$$(3.12) \quad (\alpha(0)(\boldsymbol{\eta}_{h,k} - \boldsymbol{\eta}(0)), \mathbf{v}_{h,k}) + (\xi_{h,k}(0) - \xi(0), \nabla \cdot \mathbf{v}_{h,k}) = 0, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k},$$

$$(\nabla \cdot (\boldsymbol{\eta}_{h,k}(0) - \boldsymbol{\eta}(0)), w_{h,k}) + (c(0)(\xi_{h,k}(0) - \xi(0)), w_{h,k}) = 0, \quad w_{h,k} \in W_{h,k}.$$

Theorem 3.2. *Assume that $(\xi, \boldsymbol{\eta})$ and $(\xi_{h,k}, \boldsymbol{\eta}_{h,k})$ are the exact solution of (3.7) and its mixed finite element solution, respectively, and $(\xi_{h,k}(0), \boldsymbol{\eta}_{h,k}(0))$ is chosen to satisfy (3.12). Then, we have*

$$\|\xi_{h,k} - P_{h,k}^0 \xi\|_0 + \|(\xi_{h,k} - P_{h,k}^0 \xi)_t\|_0 + \|\boldsymbol{\eta}_{h,k} - \boldsymbol{\Pi}_{h,k}^0 \boldsymbol{\eta}\|_0 \leqslant CH^2.$$

Proof. Let $\tau_{h,k} := \xi_{h,k} - P_{h,k}^0 \xi$ and $\gamma_{h,k} := \boldsymbol{\eta}_{h,k} - \boldsymbol{\Pi}_{h,k}^0 \boldsymbol{\eta}$. Then, from (2.7) and the mixed finite element error equation of (3.7) one derives that

$$(3.13) \quad (\alpha \gamma_{h,k} + M * \gamma_{h,k}, \mathbf{v}_{h,k}) + (\tau_{h,k}, \nabla \cdot \mathbf{v}_{h,k})$$

$$= (\alpha(\boldsymbol{\eta} - \boldsymbol{\Pi}_{h,k}^0 \boldsymbol{\eta}) + M * (\boldsymbol{\eta} - \boldsymbol{\Pi}_{h,k}^0 \boldsymbol{\eta}), \mathbf{v}_{h,k}), \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k},$$

$$(\tau_{h,k,t}, w_{h,k}) - (\nabla \cdot \gamma_{h,k}, w_{h,k}) - (c\tau_{h,k}, w_{h,k})$$

$$= -(c(\xi - P_{h,k}^0 \xi), w_{h,k}), \quad w_{h,k} \in W_{h,k},$$

and by Corollaries 2.1 and 2.2 that

$$(\alpha \gamma_{h,k} + M * \gamma_{h,k}, \mathbf{v}_{h,k}) + (\tau_{h,k}, \nabla \cdot \mathbf{v}_{h,k}) = O(H^2) \|\mathbf{v}_{h,k}\|_{0,q}, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k},$$

$$(\tau_{h,k,t}, w_{h,k}) - (\nabla \cdot \gamma_{h,k}, w_{h,k}) - (c\tau_{h,k}, w_{h,k}) = O(H^2) \|w_{h,k}\|_{0,q}, \quad w_{h,k} \in W_{h,k}.$$

Therefore, setting $w_{h,k} = \tau_{h,k}$ and $\mathbf{v}_{h,k} = \gamma_{h,k}$ in (3.13), we obtain from their sum that

$$\frac{1}{2} \frac{d}{dt} \|\tau_{h,k}\|_0^2 - (c\tau_{h,k}, \tau_{h,k}) + \|\gamma_{h,k}\|_{A^{-1}}^2$$

$$\leqslant - \int_0^t (M(t,s)\gamma_{h,k}(s), \gamma_{h,k}) ds + CH^2 (\|\gamma_{h,k}\|_0 + \|\tau_{h,k}\|_0),$$

and by means of Lemma 3.3 and the ε -type inequality that

$$\frac{d}{dt} \|\tau_{h,k}\|_0^2 + \|\gamma_{h,k}\|_0^2 \leqslant C \left\{ \int_0^t \|\gamma_{h,k}(s)\|_0^2 ds + \|\tau_{h,k}\|_0^2 + H^4 \right\}.$$

Integrating the above inequality from 0 to t and noticing $\tau_{h,k}(0) = 0$ leads to

$$\|\tau_{h,k}\|_0^2 + \int_0^t \|\gamma_{h,k}(s)\|_0^2 ds \leq C \int_0^t \left\{ \|\tau_{h,k}(s)\|_0^2 + \int_0^s \|\gamma_{h,k}(\tau)\|_0^2 d\tau \right\} ds + CH^4$$

which, together with Gronwall's lemma, implies

$$\|\tau_{h,k}\|_0^2 + \int_0^t \|\gamma_{h,k}(s)\|_0^2 ds \leq CH^4.$$

Thus,

$$(3.14) \quad \|\tau_{h,k}\|_0 \leq CH^2.$$

In order to get the estimate for $\gamma_{h,k}$ we first differentiate (3.13) to obtain

$$\begin{aligned} (\alpha_t \gamma_{h,k} + \alpha \gamma_{h,k,t} + M \gamma_{h,k} + M_t * \gamma_{h,k}, \mathbf{v}_{h,k}) + (\tau_{h,k,t}, \nabla \cdot \mathbf{v}_{h,k}) &= O(H^2) \|\mathbf{v}_{h,k}\|_0, \\ \mathbf{v}_{h,k} &\in \mathbf{V}_{0,h,k}, \end{aligned}$$

and then by setting $\mathbf{v}_{h,k} = \gamma_{h,k}$ in the above equation and $w_{h,k} = \tau_{h,k,t}$ in (3.13), we have from their sum that

$$(3.15) \quad \begin{aligned} \|\tau_{h,k,t}\|_0^2 + (\alpha \gamma_{h,k,t}, \gamma_{h,k}) + (\alpha_t \gamma_{h,k}, \gamma_{h,k}) \\ = -(M \gamma_{h,k} + M_t * \gamma_{h,k}, \gamma_{h,k}) + (c \tau_{h,k}, \tau_{h,k,t}). \end{aligned}$$

It follows from

$$\alpha(\gamma_{h,k}^2)_t = (\alpha \gamma_{h,k}^2)_t - \alpha_t \gamma_{h,k}^2$$

that

$$\begin{aligned} (\alpha \gamma_{h,k,t}, \gamma_{h,k}) &= \int_{\Omega} \alpha \gamma_{h,k,t} \gamma_{h,k} = \frac{1}{2} \int_{\Omega} \alpha \frac{d}{dt} (\gamma_{h,k}^2) \\ &= \frac{1}{2} \int_{\Omega} \frac{d}{dt} (\alpha \gamma_{h,k}^2) - \frac{1}{2} \int_{\Omega} \alpha_t \gamma_{h,k}^2 \\ &= \frac{1}{2} \frac{d}{dt} \|\gamma_{h,k}\|_{A^{-1}}^2 - \frac{1}{2} (\alpha_t \gamma_{h,k}, \gamma_{h,k}) \end{aligned}$$

which, together with (3.15) and the ε -type inequality, produces that

$$\frac{1}{2} \frac{d}{dt} \|\gamma_{h,k}\|_{A^{-1}}^2 + \frac{1}{2} (\alpha_t \gamma_{h,k}, \gamma_{h,k}) \leq C \left\{ \|\gamma_{h,k}\|_0^2 + \int_0^t \|\gamma_{h,k}\|_0^2 ds \right\} + C \|\tau_{h,k}\|_0^2,$$

and then via (3.14), integrating from 0 to t and using Lemma 3.3,

$$\|\gamma_{h,k}\|_0^2 \leq \|\gamma_{h,k}(0)\|_0^2 + C \left\{ H^4 + \int_0^t \|\gamma_{h,k}(s)\|_0^2 ds \right\}.$$

To estimate $\|\gamma_{h,k}(0)\|_0$ we set $t = 0$ and $\mathbf{v}_{h,k} = \gamma_{h,k}(0)$ in (3.13) to gain that

$$(\alpha(0)\gamma_{h,k}(0), \gamma_{h,k}(0)) \leq CH^2 \|\gamma_{h,k}(0)\|_0, \quad \text{where } \tau_{h,k}(0) = 0 \text{ is used,}$$

and by Lemma 3.3 that

$$\|\gamma_{h,k}(0)\|_0 \leq CH^2.$$

Thus,

$$\|\gamma_{h,k}\|_0^2 \leq C \left\{ H^4 + \int_0^t \|\gamma_{h,k}\|_0^2 ds \right\}$$

which, together with Gronwall's lemma, demonstrates

$$(3.16) \quad \|\gamma_{h,k}\|_0 \leq CH^2.$$

Hence, only $\|\tau_{h,k,t}\|_0$ remains to be estimated to finish the proof of the theorem.

We differentiate (3.13) to obtain

$$\begin{aligned} (\alpha_t \gamma_{h,k} + \alpha \gamma_{h,k,t} + M \gamma_{h,k} + M_t * \gamma_{h,k}, \mathbf{v}_{h,k}) + (\tau_{h,k,t}, \nabla \cdot \mathbf{v}_{h,k}) &= O(H^2) \|\mathbf{v}_{h,k}\|_0, \\ (\tau_{h,k,t}, w_{h,k}) - (\nabla \cdot \gamma_{h,k,t}, w_{h,k}) - (c_t \tau_{h,k} + c \tau_{h,k,t}, w_{h,k}) &= O(H^2) \|w_{h,k}\|_0, \end{aligned}$$

and thus, letting $\mathbf{v}_{h,k} = \gamma_{h,k,t}$ and $w_{h,k} = \tau_{h,k,t}$ yields from their sum, Lemma 3.3, the ε -inequality, (3.14) and (3.16) that

$$\frac{d}{dt} \|\tau_{h,k,t}\|_0^2 \leq C \{ H^4 + \|\tau_{h,k,t}\|_0^2 \},$$

or

$$(3.17) \quad \|\tau_{h,k,t}\|_0^2 \leq \|\tau_{h,k,t}(0)\|_0^2 + C \left\{ H^4 + \int_0^t \|\tau_{h,k,t}(s)\|_0^2 ds \right\}.$$

From Definition 3.1 and (3.12) we derive that

$$\begin{aligned} (\alpha(0)(\bar{\boldsymbol{\eta}}_{h,k} - \boldsymbol{\eta}_{h,k})(0), \mathbf{v}_{h,k}) + (\bar{\xi}_{h,k}(0) - \xi_{h,k}(0), \nabla \cdot \mathbf{v}_{h,k}) &= 0, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}, \\ (\nabla \cdot (\bar{\boldsymbol{\eta}}_{h,k} - \boldsymbol{\eta}_{h,k})(0), w_{h,k}) + (c(0)(\bar{\xi}_{h,k} - \xi_{h,k})(0), w_{h,k}) &= 0, \quad w_{h,k} \in W_{h,k}, \end{aligned}$$

where $(\bar{\xi}_{h,k}, \bar{\boldsymbol{\eta}}_{h,k}) \in W_{h,k} \times \mathbf{V}_{0,h,k}$ is the Ritz-Volterra projection of $(\xi, \boldsymbol{\eta})$. Therefore, the uniqueness of the solution to (3.12) implies

$$\bar{\boldsymbol{\eta}}_{h,k}(0) - \boldsymbol{\eta}_{h,k}(0) = \bar{\xi}_{h,k}(0) - \xi_{h,k}(0) = 0$$

which, together with (3.11), indicates that

$$\|\nabla \cdot \gamma_{h,k}(0)\|_0 = \|\nabla \cdot (\bar{\boldsymbol{\eta}}_{h,k} - \mathbf{\Pi}_{h,k}^0 \boldsymbol{\eta})(0)\|_0 \leq CH^2.$$

Thus, it follows from (3.14), setting $t = 0$ and $w_{h,k} = \tau_{h,k,t}(0)$ in (3.13) that

$$\|\tau_{h,k,t}(0)\|_0^2 \leq CH^2 \|\tau_{h,k,t}(0)\|_0$$

or

$$\|\tau_{h,k,t}(0)\|_0 \leq CH^2.$$

This, together with (3.17) and Gronwall's lemma, claims that

$$\|\tau_{h,k,t}\|_0 \leq CH^2.$$

□

Theorem 3.3. *We have under the conditions of Theorem 3.2 that*

$$|\log H|^{1/2} \|\xi_{h,k} - P_{h,k}^0 \xi\|_\infty + \|\boldsymbol{\eta}_{h,k} - \mathbf{\Pi}_{h,k}^0 \boldsymbol{\eta}\|_\infty \leq CH^2 |\log H|.$$

Proof. Set $\mathbf{v}_{h,k} = \mathbf{G}_1^{h,k}$ in (3.13) and follow the steps for (3.10) to get

$$(3.18) \quad \|\tau_{h,k}\|_\infty \leq CH^2 |\log H|^{1/2}.$$

It follows from Theorem 3.2 and (3.13) that

$$\begin{aligned} |(\nabla \cdot \gamma_{h,k}, w_{h,k})| &\leq \|\tau_{h,k,t}\|_0 \|w_{h,k}\|_0 + C \|\tau_{h,k}\|_0 \|w_{h,k}\|_0 + CH^2 \|w_{h,k}\|_0 \\ &\leq CH^2 \|w_{h,k}\|_0, \end{aligned}$$

which implies

$$(3.19) \quad \|\nabla \cdot \gamma_{h,k}\|_0 \leq CH^2.$$

Let $\mathbf{v}_{h,k} = \mathbf{G}_2^{h,k}$ in (3.13) to obtain according to Lemmas 3.4 and 3.1, Green's formula and (3.19) that

$$\begin{aligned} & \left| \int_0^T (\gamma_{h,k}, \delta_2^{h,k}) \varphi_2(t) dt \right| \\ & \leq \left| \int_0^T (\lambda_2^{h,k}, \nabla \cdot \gamma_{h,k}) dt \right| + CH^2 |\log H| \int_0^T (1 + \varphi_2(t)) dt \\ & \leq CH^2 |\log H|^{1/2} \int_0^T (1 + \varphi_2(t)) dt + CH^2 |\log H| \int_0^T (1 + \varphi_2(t)) dt. \end{aligned}$$

Thus, Lemma 3.2 and (3.3) lead to

$$\|\gamma_{h,k}\|_\infty \leq CH^2 |\log H|, \quad \text{where } \nabla \cdot \mathbf{G}_2^{h,k} = 0 \text{ has been used.}$$

This, together with (3.18), completes the proof of the theorem. \square

As we have done in our previous work [10], we now utilize an interpolation postprocessing method to gain the global extrapolation approximations of higher accuracy for the pressure field and the velocity field. For this end, we need to define two postprocessing interpolation operator $\Pi_{3h,3k}^2$ and $P_{4h,4k}^3$ to satisfy

$$\begin{aligned} (3.20) \quad & \Pi_{3h,3k}^2 \Pi_{h,k}^0 = \Pi_{3h,3k}^3, \\ & \|\Pi_{3h,3k}^2 \mathbf{v}_{h,k}\|_{0,p} \leq C \|\mathbf{v}_{h,k}\|_{0,p} \quad \forall \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}, \quad 1 \leq p \leq \infty, \\ & \|\Pi_{3h,3k}^2 \boldsymbol{\sigma} - \boldsymbol{\sigma}\|_{0,p} \leq CH^3 \|\boldsymbol{\sigma}\|_{3,p} \quad \forall \boldsymbol{\sigma} \in (W^{3,p}(\Omega))^2, \quad 1 \leq p \leq \infty, \\ & P_{4h,4k}^3 P_{h,k}^0 = P_{4h,4k}^3, \\ & \|P_{4h,4k}^3 w_{h,k}\|_{0,p} \leq C \|w_{h,k}\|_{0,p} \quad \forall w_{h,k} \in W_{h,k}, \quad 1 \leq p \leq \infty, \\ & \|P_{4h,4k}^3 u - u\|_{0,p} \leq CH^4 \|u\|_{4,p} \quad \forall u \in W^{4,p}(\Omega), \quad 1 \leq p \leq \infty. \end{aligned}$$

We assume that the rectangular partition $T_{h,k}$ has been obtained from $T_{3h,3k}$ with mesh size $3H$ and $T_{4h,4k}$ with mesh size $4H$ by subdividing each element of $T_{3h,3k}$ and $T_{4h,4k}$ into nine and sixteen small congruent rectangles, respectively. Let $\tau := \bigcup_{i=1}^9 e_i$

and $\hat{\tau} := \bigcup_{i=1}^{16} \hat{e}_i$ with $e_i, \hat{e}_i \in T_{h,k}$. Then, we can define two projection interpolation operators $\Pi_{3h,3k}^2$ and $P_{4h,4k}^3$ associated with $T_{3h,3k}$ and $T_{4h,4k}$ of degree at most 2 and 3 in x and y on τ and $\hat{\tau}$, respectively, according to the following conditions:

$$\begin{aligned} (3.21) \quad & \Pi_{3h,3k}^2 \boldsymbol{\sigma}|_\tau \in Q_{3,2}(\tau) \times Q_{2,3}(\tau), \quad P_{4h,4k}^3 u|_{\hat{\tau}} \in Q_{3,3}(\hat{\tau}), \\ & \int_{s_i} (\boldsymbol{\sigma} - \Pi_{3h,3k}^2 \boldsymbol{\sigma}) \cdot \mathbf{n} ds = 0, \quad i = 1, 2, \dots, 24, \quad \text{and} \\ & \int_{\hat{e}_i} (u - P_{4h,4k}^3 u) = 0, \quad i = 1, 2, \dots, 16, \quad \text{respectively,} \end{aligned}$$

where s_i ($i = 1, 2, \dots, 24$) is one of the twenty-four sides of the nine small elements e_i ($i = 1, 2, \dots, 9$). It is easy to check that the two operators $\Pi_{3h,3k}^2$ and $P_{4h,4k}^3$ defined by (3.21) possess the properties described in (3.20).

We are now in a position to assert our main result in this section. In fact, using Theorems 3.1, 3.3, (3.20) and following the procedure for Theorem 4.2 in [16] we can establish:

Theorem 3.4. *Under the conditions of Theorem 3.1 we have in the sense of L^∞ -norm that*

$$\begin{aligned} P_{4h,4k}^3 u_{h,k} - u &= H^2 \xi + O(H^4 |\log H|^{1/2}), \\ \Pi_{3h,3k}^2 \sigma_{h,k} - \sigma &= H^2 \eta + O(H^3), \end{aligned}$$

where $(\xi, \eta) \in W \times \mathbf{V}_0$ is the variational solution of (3.7).

On the basis of Theorem 3.4 we can generate the approximations of higher precision for problem (1.1) by Richardson extrapolation. For this end, in addition to $W_{h,k} \times \mathbf{V}_{0,h,k}$, we employ another Raviart-Thomas mixed finite element space $W_{h/2,k/2} \times \mathbf{V}_{0,h/2,k/2}$ of the lowest order gained by subdividing each element $e_i \in T_{h,k}$ into four small congruent elements $\hat{e}_{i,j} \in T_{h/2,k/2}$ ($j = 1, 2, 3, 4$). Let $(u_{h/2,k/2}, \sigma_{h/2,k/2}) \in W_{h/2,k/2} \times \mathbf{V}_{0,h/2,k/2}$ and $P_{2h,2k}^3 \times \Pi_{3h/2,3k/2}^2$ stand for the mixed finite element solution and the Raviart-Thomas projection operator with respect to this new partition, respectively. Then, we know from Theorem 3.4 that

$$P_{2h,2k}^3 u_{h/2,k/2} - u = \left(\frac{H}{2}\right)^2 \xi + O(H^4 |\log H|^{1/2}),$$

which gives by applying once the Richardson extrapolation that

$$(3.22) \quad \frac{4P_{2h,2k}^3 u_{h/2,k/2} - P_{4h,4k}^3 u_{h,k}}{3} = u + O(H^4 |\log H|^{1/2}).$$

Similarly, we have

$$(3.23) \quad \frac{4\Pi_{3h/2,3k/2}^2 \sigma_{h/2,k/2} - \Pi_{3h,3k}^2 \sigma_{h,k}}{3} = \sigma + O(H^3).$$

An important application of the approximations of higher accuracy given by (3.22) and (3.23) is that they can provide efficient a posteriori error estimators to assess the accuracy of the approximate solutions in applications. In fact, following the same way as that for Theorem 5.3 in [16] we are led to

Theorem 3.5. *Under the assumptions of Theorem 3.1, we have*

$$\begin{aligned}\|u - P_{2h,2k}^3 u_{h/2,k/2}\|_\infty &= \frac{1}{3} \|P_{2h,2k}^3 u_{h/2,k/2} - P_{4h,4k}^3 u_{h,k}\|_\infty + O(H^4 |\log H|^{1/2}), \\ \|\sigma - \Pi_{3h/2,3k/2}^2 \sigma_{h/2,k/2}\|_\infty &= \frac{1}{3} \|\Pi_{3h/2,3k/2}^2 \sigma_{h/2,k/2} - \Pi_{3h,3k}^2 \sigma_{h,k}\|_\infty + O(H^3).\end{aligned}$$

Moreover, if there exist positive constants C_1, C_2 and sufficiently small $\varepsilon_1, \varepsilon_2 \in (0, 1)$ such that

$$(3.24) \quad \|u - P_{2h,2k}^3 u_{h/2,k/2}\|_\infty \geq C_1 H^{4-\varepsilon_1} |\log H|^{1/2},$$

$$(3.25) \quad \|\sigma - \Pi_{3h/2,3k/2}^2 \sigma_{h/2,k/2}\|_\infty \geq C_2 H^{3-\varepsilon_2},$$

then we have

$$\begin{aligned}\lim_{H \rightarrow 0} \frac{3 \|u - P_{2h,2k}^3 u_{h/2,k/2}\|_\infty}{\|P_{2h,2k}^3 u_{h/2,k/2} - P_{4h,4k}^3 u_{h,k}\|_\infty} &= 1, \\ \lim_{H \rightarrow 0} \frac{3 \|\sigma - \Pi_{3h/2,3k/2}^2 \sigma_{h/2,k/2}\|_\infty}{\|\Pi_{3h/2,3k/2}^2 \sigma_{h/2,k/2} - \Pi_{3h,3k}^2 \sigma_{h,k}\|_0} &= 1.\end{aligned}$$

From Theorem 3.4 we know that the optimal convergence rates of

$$\|u - P_{2h,2k}^3 u_{h/2,k/2}\|_\infty \quad \text{and} \quad \|\sigma - \Pi_{3h/2,3k/2}^2 \sigma_{h/2,k/2}\|_\infty$$

are $O(H^2)$. Therefore, the assumed conditions (3.24) and (3.25) seem to be reasonable.

3.2. The global Richardson extrapolation in one direction

It is clear according to (3.22) and (3.23) that this extrapolation approach requires a global refinement to generate an approximation of higher accuracy, which wastes computing time and memory. Therefore, it is natural for us to develop a new type of extrapolation to overcome this shortcoming. In fact, here we shall construct a new extrapolation formula of a partial refinement, in which the meshes are refined just in one direction, x - or y -direction, such that it is more efficient and more suitable for parallel computations.

Theorem 3.6. Under the conditions of Theorem 3.1 we have in L^∞ -norm that

$$\begin{aligned} u_{h,k} - P_{h,k}^0 u &= h^2 \xi_{h,k}^1 + k^2 \xi_{h,k}^2 + O(H^4 |\log H|^{1/2}), \\ \boldsymbol{\sigma}_{h,k} - \boldsymbol{\Pi}_{h,k}^0 \boldsymbol{\sigma} &= h^2 \boldsymbol{\eta}_{h,k}^1 + k^2 \boldsymbol{\eta}_{h,k}^2 + O(H^3), \end{aligned}$$

where $(\xi_{h,k}^1, \boldsymbol{\eta}_{h,k}^1), (\xi_{h,k}^2, \boldsymbol{\eta}_{h,k}^2) \in W_{h,k} \times \mathbf{V}_{0,h,k}$.

Proof. Let $(\xi^1, \boldsymbol{\eta}^1), (\xi^2, \boldsymbol{\eta}^2) \in W \times \mathbf{V}_0$ and $(\xi_{h,k}^1, \boldsymbol{\eta}_{h,k}^1), (\xi_{h,k}^2, \boldsymbol{\eta}_{h,k}^2) \in W_{h,k} \times \mathbf{V}_{0,h,k}$ be the exact solutions and the mixed finite element solutions of the following two auxiliary variational problems, respectively:

$$(3.26) \quad \begin{aligned} (\alpha \boldsymbol{\eta}^1 + M * \boldsymbol{\eta}^1, \mathbf{v}) + (\xi^1, \nabla \cdot \mathbf{v}) &= \mathbf{L}_1(\mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_0, \\ (\xi_t^1, w) - (\nabla \cdot \boldsymbol{\eta}^1, w) - (c \xi^1, w) &= L_3(w), \quad w \in W, \\ \xi^1(0) &= 0, \end{aligned}$$

and

$$(3.27) \quad \begin{aligned} (\alpha \boldsymbol{\eta}^2 + M * \boldsymbol{\eta}^2, \mathbf{v}) + (\xi^2, \nabla \cdot \mathbf{v}) &= \mathbf{L}_2(\mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_0, \\ (\xi_t^2, w) - (\nabla \cdot \boldsymbol{\eta}^2, w) - (c \xi^2, w) &= L_4(w), \quad w \in W, \\ \xi^2(0) &= 0, \end{aligned}$$

where

$$\begin{aligned} \mathbf{L}_1(\mathbf{v}) &= \frac{1}{3} \int_{\Omega} \{ -[\alpha_{11}(\sigma_1)_{xx} + \alpha_{12}(\sigma_2)_{xx}]v_1 + [(\alpha_{22})_x(\sigma_2)_x - \alpha_{21}(\sigma_1)_{xx}]v_2 \} \\ &\quad + \frac{1}{3} \int_{\Omega} \int_0^t \{ -[m_{11}(\sigma_1)_{xx} + m_{12}(\sigma_2)_{xx}](s)v_1 \\ &\quad \quad + [(m_{22})_x(\sigma_2)_x - m_{21}(\sigma_1)_{xx}](s)v_2 \} ds, \\ \mathbf{L}_2(\mathbf{v}) &= \frac{1}{3} \int_{\Omega} \{ [(\alpha_{11})_y(\sigma_1)_y - \alpha_{12}(\sigma_2)_{yy}]v_1 - [\alpha_{22}(\sigma_2)_{yy} + \alpha_{21}(\sigma_1)_{yy}]v_2 \} \\ &\quad + \frac{1}{3} \int_{\Omega} \int_0^t \{ [(m_{11})_y(\sigma_1)_y - m_{12}(\sigma_2)_{yy}](s)v_1 \\ &\quad \quad - [m_{22}(\sigma_2)_{yy} + m_{21}(\sigma_1)_{yy}](s)v_2 \} ds, \\ L_3(w) &= -\frac{1}{3} \int_{\Omega} c_x u_x w, \\ L_4(w) &= -\frac{1}{3} \int_{\Omega} c_y u_y w. \end{aligned}$$

From (3.4), Theorems 2.1 and 2.2 one finds that

$$\begin{aligned}
(3.28) \quad & (\alpha \boldsymbol{\theta}_{h,k} + M * \boldsymbol{\theta}_{h,k}, \mathbf{v}_{h,k}) + (\varrho_{h,k}, \nabla \cdot \mathbf{v}_{h,k}) \\
& = h^2 \mathbf{L}_1(\mathbf{v}_{h,k}) + k^2 \mathbf{L}_2(\mathbf{v}_{h,k}) + O(H^4) \|\mathbf{v}_{h,k}\|_{0,q}, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}, \\
& \quad (\varrho_{h,k,t}, w_{h,k}) - (\nabla \cdot \boldsymbol{\theta}_{h,k}, w_{h,k}) - (c \varrho_{h,k}, w_{h,k}) \\
& = h^2 L_3(w_{h,k}) + k^2 L_4(w_{h,k}) + O(H^4) \|w_{h,k}\|_{0,q}, \quad w_{h,k} \in W_{h,k}.
\end{aligned}$$

Let

$$\hat{\boldsymbol{\theta}}_{h,k} := \boldsymbol{\theta}_{h,k} - h^2 \boldsymbol{\eta}_{h,k}^1 - k^2 \boldsymbol{\eta}_{h,k}^2, \quad \hat{\varrho}_{h,k} := \varrho_{h,k} - h^2 \xi_{h,k}^1 - k^2 \xi_{h,k}^2.$$

Then, it follows from (3.26)–(3.28) that

$$\begin{aligned}
(3.29) \quad & (\alpha \hat{\boldsymbol{\theta}}_{h,k} + M * \hat{\boldsymbol{\theta}}_{h,k}, \mathbf{v}_{h,k}) + (\hat{\varrho}_{h,k}, \nabla \cdot \mathbf{v}_{h,k}) = O(H^4) \|\mathbf{v}_{h,k}\|_{0,q}, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}, \\
& (\hat{\varrho}_{h,k,t}, w_{h,k}) - (\nabla \cdot \hat{\boldsymbol{\theta}}_{h,k}, w_{h,k}) - (c \hat{\varrho}_{h,k}, w_{h,k}) = O(H^4) \|w_{h,k}\|_{0,q}, \quad w_{h,k} \in W_{h,k}.
\end{aligned}$$

It has been shown in [16] that

$$(3.30) \quad \|\hat{\varrho}_{h,k}\|_0 + \|\hat{\boldsymbol{\theta}}_{h,k}\|_0 \leq CH^4.$$

Thus, we can also obtain by using (3.29)–(3.30) and following the procedure for the estimates of $\|\varrho_{h,k}^*\|_\infty$ and $\|\boldsymbol{\theta}_{h,k}^*\|_\infty$ in Theorem 3.1 that

$$\|\hat{\varrho}_{h,k}\|_\infty \leq CH^4 |\log H|^{1/2} \quad \text{and} \quad \|\hat{\boldsymbol{\theta}}_{h,k}\|_\infty \leq CH^3.$$

□

Also, we can get the analog of Theorem 3.4 by means of the analog of Theorem 3.2.

Theorem 3.7. *Under the conditions of Theorem 3.1 we have in the L^∞ -norm that*

$$\begin{aligned}
P_{4h,4k}^3 u_{h,k} - u & = h^2 \xi^1 + k^2 \xi^2 + O(H^4 |\log H|^{1/2}), \\
\Pi_{3h,3k}^2 \boldsymbol{\sigma}_{h,k} - \boldsymbol{\sigma} & = h^2 \boldsymbol{\eta}^1 + k^2 \boldsymbol{\eta}^2 + O(H^3),
\end{aligned}$$

where $(\xi^1, \boldsymbol{\eta}^1), (\xi^2, \boldsymbol{\eta}^2) \in W \times \mathbf{V}_0$.

Like (3.22) and (3.23), from Theorem 3.7 we can obtain the following unidirectional Richardson extrapolation in the L^∞ norm:

$$\begin{aligned}
(3.31) \quad & \frac{4(\Pi_{3h/2,3k}^2 \boldsymbol{\sigma}_{h/2,k} + \Pi_{3h,3k/2}^2 \boldsymbol{\sigma}_{h,k/2}) - 5\Pi_{3h,3k}^2 \boldsymbol{\sigma}_{h,k}}{3} = \boldsymbol{\sigma} + O(H^3), \\
& \frac{4(P_{2h,4k}^3 u_{h/2,k} + P_{4h,2k}^3 u_{h,k/2}) - 5P_{4h,4k}^3 u_{h,k}}{3} = u + O(H^4 |\log H|^{1/2}),
\end{aligned}$$

where $(u_{h/2,k}, \boldsymbol{\sigma}_{h/2,k})$, $(u_{h,k/2}, \boldsymbol{\sigma}_{h,k/2})$ and $(u_{h,k}, \boldsymbol{\sigma}_{h,k})$ are the mixed finite element solutions corresponding to the meshes $T_{h/2,k}$, $T_{h,k/2}$ and $T_{h,k}$, respectively. Here, $T_{h/2,k}$ and $T_{h,k/2}$ are the meshes gained by subdividing each element of $T_{h,k}$ into two small congruent rectangles in x - and y -direction, respectively.

Similar to Theorem 3.5, a posteriori error indicators can also be constructed by virtue of (3.31). In fact, we have:

Theorem 3.8. *Under the conditions of Theorem 3.1 we have*

$$\begin{aligned}
& \|u - P_{2h,4k}^3 u_{h/2,k}\|_\infty \\
&= \frac{1}{3} \|P_{2h,4k}^3 u_{h/2,k} + 4P_{4h,2k}^3 u_{h,k/2} - 5P_{4h,4k}^3 u_{h,k}\|_\infty + O(H^4 |\log H|^{1/2}), \\
& \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_{3h/2,3k}^2 \boldsymbol{\sigma}_{h/2,k}\|_\infty \\
&= \frac{1}{3} \|\boldsymbol{\Pi}_{3h/2,3k}^2 \boldsymbol{\sigma}_{h/2,k} + 4\boldsymbol{\Pi}_{3h,3k/2}^2 \boldsymbol{\sigma}_{h,k/2} - 5\boldsymbol{\Pi}_{3h,3k}^2 \boldsymbol{\sigma}_{h,k}\|_\infty + O(H^3), \\
& \|u - P_{4h,2k}^3 u_{h,k/2}\|_\infty \\
&= \frac{1}{3} \|4P_{2h,4k}^3 u_{h/2,k} + P_{4h,2k}^3 u_{h,k/2} - 5P_{4h,4k}^3 u_{h,k}\|_\infty + O(H^4 |\log H|^{1/2}), \\
& \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_{3h,3k/2}^2 \boldsymbol{\sigma}_{h,k/2}\|_\infty \\
&= \frac{1}{3} \|4\boldsymbol{\Pi}_{3h/2,3k}^2 \boldsymbol{\sigma}_{h/2,k} + \boldsymbol{\Pi}_{3h,3k/2}^2 \boldsymbol{\sigma}_{h,k/2} - 5\boldsymbol{\Pi}_{3h,3k}^2 \boldsymbol{\sigma}_{h,k}\|_\infty + O(H^3).
\end{aligned}$$

In addition, if there exist positive constants C_1, C_2, C_3, C_4 and sufficiently small $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in (0, 1)$ such that

$$\begin{aligned}
& \|u - P_{2h,4k}^3 u_{h/2,k}\|_\infty \geq C_1 H^{4-\varepsilon_1} |\log H|^{1/2}, \\
& \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_{3h/2,3k}^2 \boldsymbol{\sigma}_{h/2,k}\|_\infty \geq C_2 H^{3-\varepsilon_2}, \\
& \|u - P_{4h,2k}^3 u_{h,k/2}\|_\infty \geq C_3 H^{4-\varepsilon_3} |\log H|^{1/2}, \\
& \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_{3h,3k/2}^2 \boldsymbol{\sigma}_{h,k/2}\|_\infty \geq C_4 H^{3-\varepsilon_4},
\end{aligned}$$

then

$$\begin{aligned}
& \lim_{H \rightarrow 0} \frac{3 \|u - P_{2h,4k}^3 u_{h/2,k}\|_\infty}{\|P_{2h,4k}^3 u_{h/2,k} + 4P_{4h,2k}^3 u_{h,k/2} - 5P_{4h,4k}^3 u_{h,k}\|_\infty} = 1, \\
& \lim_{H \rightarrow 0} \frac{3 \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_{3h/2,3k}^2 \boldsymbol{\sigma}_{h/2,k}\|_\infty}{\|\boldsymbol{\Pi}_{3h/2,3k}^2 \boldsymbol{\sigma}_{h/2,k} + 4\boldsymbol{\Pi}_{3h,3k/2}^2 \boldsymbol{\sigma}_{h,k/2} - 5\boldsymbol{\Pi}_{3h,3k}^2 \boldsymbol{\sigma}_{h,k}\|_\infty} = 1, \\
& \lim_{H \rightarrow 0} \frac{3 \|u - P_{4h,2k}^3 u_{h,k/2}\|_\infty}{\|4P_{2h,4k}^3 u_{h/2,k} + P_{4h,2k}^3 u_{h,k/2} - 5P_{4h,4k}^3 u_{h,k}\|_\infty} = 1, \\
& \lim_{H \rightarrow 0} \frac{3 \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_{3h,3k/2}^2 \boldsymbol{\sigma}_{h,k/2}\|_\infty}{\|4\boldsymbol{\Pi}_{3h/2,3k}^2 \boldsymbol{\sigma}_{h/2,k} + \boldsymbol{\Pi}_{3h,3k/2}^2 \boldsymbol{\sigma}_{h,k/2} - 5\boldsymbol{\Pi}_{3h,3k}^2 \boldsymbol{\sigma}_{h,k}\|_\infty} = 1.
\end{aligned}$$

4. INTERPOLATION DEFECT CORRECTION

In this section we propose and investigate an interpolation defect correction scheme (see, for example, [19], [21], [22]) applied to the mixed finite element solution $(u_{h,k}, \boldsymbol{\sigma}_{h,k}) \in W_{h,k} \times \mathbf{V}_{0,h,k}$ to obtain approximations with higher convergence rate. Also, these new approximations are naturally used to form a posteriori error estimators in order to estimate the actual accuracy of the mixed finite element solutions.

First of all, for the future need we construct two projection interpolation operators $P_{2h,2k}^1$ and $\mathbf{\Pi}_{2h,2k}^1$ associated with the mesh $T_{2h,2k}$ to satisfy

$$(4.1) \quad \begin{aligned} \mathbf{\Pi}_{2h,2k}^1 \mathbf{\Pi}_{h,k}^0 &= \mathbf{\Pi}_{2h,2k}^1, \\ \|\mathbf{\Pi}_{2h,2k}^1 \mathbf{v}_{h,k}\|_{0,p} &\leq C \|\mathbf{v}_{h,k}\|_{0,p} \quad \forall \mathbf{v}_{h,k} \in \mathbf{V}_{h,k}, \quad 1 \leq p \leq \infty, \\ \|\mathbf{\Pi}_{2h,2k}^1 \boldsymbol{\sigma} - \boldsymbol{\sigma}\|_{0,p} &\leq CH^2 \|\boldsymbol{\sigma}\|_{2,p} \quad \forall \boldsymbol{\sigma} \in (W^{2,p}(\Omega))^2, \quad 1 \leq p \leq \infty, \\ P_{2h,2k}^1 P_{h,k}^0 &= P_{2h,2k}^1, \\ \|P_{2h,2k}^1 w_{h,k}\|_{0,p} &\leq C \|w_{h,k}\|_{0,p} \quad \forall w_{h,k} \in W_{h,k}, \quad 1 \leq p \leq \infty, \\ \|P_{2h,2k}^1 u - u\|_{0,p} &\leq CH^2 \|u\|_{2,p} \quad \forall u \in W^{2,p}(\Omega), \quad 1 \leq p \leq \infty. \end{aligned}$$

Then, like that seen in the last section, it is assumed that the rectangular partition $T_{h,k}$ has been obtained from $T_{2h,2k}$ with mesh size $2H$ by subdividing each element of $T_{2h,2k}$ into four small congruent rectangles. Set $\hat{e} := \bigcup_{i=1}^4 e_i$ with $e_i \in T_{h,k}$. And thus, the two interpolation operators $\mathbf{\Pi}_{2h,2k}^1$ and $P_{2h,2k}^1$ of degree at most 1 in x and y on \hat{e} , respectively, are defined as follows:

$$\begin{aligned} \mathbf{\Pi}_{2h,2k}^1 \boldsymbol{\sigma}|_{\hat{e}} &\in Q_{2,1}(\hat{e}) \times Q_{1,2}(\hat{e}), \quad P_{2h,2k}^1 u|_{\hat{e}} \in Q_{1,1}(\hat{e}), \\ \int_{s_i} (\boldsymbol{\sigma} - \mathbf{\Pi}_{2h,2k}^1 \boldsymbol{\sigma}) \cdot \mathbf{n} \, ds &= 0, \quad i = 1, 2, \dots, 12, \\ \int_{e_i} (u - P_{2h,2k}^1 u) &= 0, \quad i = 1, 2, 3, 4, \end{aligned}$$

where s_i ($i = 1, 2, \dots, 12$) are the twelve edges of the four small elements e_i ($i = 1, 2, 3, 4$). We can also check that $\mathbf{\Pi}_{2h,2k}^1$ and $P_{2h,2k}^1$ defined above possess the properties indicated in (4.1).

In addition, we also need a pair of mixed finite element projection operators $R_{h,k} \times \mathbf{S}_{h,k}: W \times \mathbf{V}_0 \rightarrow W_{h,k} \times \mathbf{V}_{0,h,k}$ defined by

$$\begin{aligned} ((R_{h,k}u)_t - u_t, w_{h,k}) - (\nabla \cdot (\mathbf{S}_{h,k}\boldsymbol{\sigma} - \boldsymbol{\sigma}), w_{h,k}) - (c(R_{h,k}u - u), w_{h,k}) &= 0, \\ w_{h,k} &\in W_{h,k}, \\ (\alpha(\mathbf{S}_{h,k}\boldsymbol{\sigma} - \boldsymbol{\sigma}) + M * (\mathbf{S}_{h,k}\boldsymbol{\sigma} - \boldsymbol{\sigma}), \mathbf{v}_{h,k}) + (\nabla \cdot \mathbf{v}_{h,k}, R_{h,k}u - u) &= 0, \\ \mathbf{v}_{h,k} &\in \mathbf{V}_{0,h,k}, \\ R_{h,k}u(0) &= P_{h,k}^0 u(0). \end{aligned}$$

Then, $(R_{h,k}u, \mathbf{S}_{h,k}\boldsymbol{\sigma})$ is the solution of (2.3) if $(u, \boldsymbol{\sigma})$ is the solution of (2.1).

First of all, we discuss the interpolation defect correction method in the L^2 -norm. To this purpose, let us recall the following lemma from [16].

Lemma 4.1. *Assume that $(u, \boldsymbol{\sigma})$ and $(u_{h,k}, \boldsymbol{\sigma}_{h,k})$ are the exact solution of (2.1) and its mixed finite element solution, respectively, with the chosen initial value $u_{h,k}(0) = P_{h,k}^0 u_0$. Then, in the sense of L^2 -norm we have under the conditions that $(u, \boldsymbol{\sigma})$, c , α and M are sufficiently smooth that*

$$(4.2) \quad P_{4h,4k}^3 u_{h,k} - u = H^2 \xi + O(H^4),$$

$$(4.3) \quad \Pi_{4h,4k}^3 \boldsymbol{\sigma}_{h,k} - \boldsymbol{\sigma} = H^2 \boldsymbol{\eta} + O(H^4),$$

where $(\xi, \boldsymbol{\eta}) \in W \times \mathbf{V}_0$ is the variational solution of (3.7).

Theorem 4.1. *Suppose that the conditions of Lemma 4.1 are fulfilled. Then, we have*

$$\|u_{h,k}^* - u\|_0 + \|\boldsymbol{\sigma}_{h,k}^* - \boldsymbol{\sigma}\|_0 \leq CH^4,$$

where

$$\begin{aligned} u_{h,k}^* &:= P_{4h,4k}^3 u_{h,k} + P_{2h,2k}^1 u_{h,k} - P_{2h,2k}^1 R_{h,k} P_{4h,4k}^3 u_{h,k}, \\ \boldsymbol{\sigma}_{h,k}^* &:= \Pi_{4h,4k}^3 \boldsymbol{\sigma}_{h,k} + \Pi_{2h,2k}^1 \boldsymbol{\sigma}_{h,k} - \Pi_{2h,2k}^1 \mathbf{S}_{h,k} \Pi_{4h,4k}^3 \boldsymbol{\sigma}_{h,k}. \end{aligned}$$

Proof. Multiplying (4.2) by the operator $(I - P_{2h,2k}^1 R_{h,k})$, where I is the identity operator, results in

$$\begin{aligned} (I - P_{2h,2k}^1 R_{h,k})(P_{4h,4k}^3 u_{h,k} - u) & \\ &= H^2(I - P_{2h,2k}^1 R_{h,k})\xi + O(H^4) \\ &= H^2(\xi - P_{2h,2k}^1 \xi) + H^2(P_{2h,2k}^1 \xi - P_{2h,2k}^1 \xi_{h,k}) + O(H^4) \\ &= H^2 P_{2h,2k}^1 (P_{h,k}^0 \xi - \xi_{h,k}) + O(H^4), \end{aligned}$$

since

$$\|\xi - P_{2h,2k}^1 \xi\|_0 \leq CH^2 \|\xi\|_2 \quad \text{and} \quad P_{2h,2k}^1 P_{h,k}^0 = P_{2h,2k}^1$$

according to the properties of the operator $P_{2h,2k}^1$ described in (4.1). Furthermore, it follows from Theorem 3.2 and the inequality

$$\|P_{2h,2k}^1 (P_{h,k}^0 \xi - \xi_{h,k})\|_0 \leq C \|P_{h,k}^0 \xi - \xi_{h,k}\|_0$$

that

$$(I - P_{2h,2k}^1 R_{h,k})(P_{4h,4k}^3 u_{h,k} - u) = O(H^4),$$

and the left-hand side is nothing but

$$(I - P_{2h,2k}^1 R_{h,k})(P_{4h,4k}^3 u_{h,k} - u) = u_{h,k}^* - u.$$

Similarly, we can gain in terms of (4.3) that

$$\|\sigma_{h,k}^* - \sigma\|_0 = O(H^4).$$

□

In the same way we can also obtain from Theorems 3.3 and 3.4 the following result.

Theorem 4.2. *We have under the conditions of Theorem 3.1 that*

$$\begin{aligned} \|\hat{u}_{h,k} - u\|_\infty &\leq CH^4 |\log H|^{1/2}, \\ \|\hat{\sigma}_{h,k} - \sigma\|_\infty &\leq CH^3, \end{aligned}$$

where

$$\begin{aligned} \hat{u}_{h,k} &:= P_{4h,4k}^3 u_{h,k} + P_{2h,2k}^1 u_{h,k} - P_{2h,2k}^1 R_{h,k} P_{4h,4k}^3 u_{h,k}, \\ \hat{\sigma}_{h,k} &:= \Pi_{3h,3k}^2 \sigma_{h,k} + \sigma_{h,k} - \mathbf{S}_{h,k} \Pi_{3h,3k}^2 \sigma_{h,k}. \end{aligned}$$

Analogously to Section 3 we can utilize the superconvergent approximation provided in Theorems 4.1 and 4.2 to establish a posteriori error estimators for the mixed finite element solution of problem (2.1). In fact, we have

Theorem 4.3. *If the conditions of Theorem 4.1 are satisfied, then*

$$\begin{aligned}\|u - u_{h,k}\|_0 &= \|u_{h,k}^* - u_{h,k}\|_0 + O(H^4), \\ \|\sigma - \sigma_{h,k}\|_0 &= \|\sigma_{h,k}^* - \sigma_{h,k}\|_0 + O(H^4).\end{aligned}$$

Furthermore, if there exist positive constants C_1, C_2 and sufficiently small $\varepsilon_1, \varepsilon_2 \in (0, 1)$ such that

$$\begin{aligned}\|u - u_{h,k}\|_0 &\geq C_1 H^{4-\varepsilon_1}, \\ \|\sigma - \sigma_{h,k}\|_0 &\geq C_2 H^{4-\varepsilon_2},\end{aligned}$$

we then have

$$\begin{aligned}\lim_{H \rightarrow 0} \frac{\|u - u_{h,k}\|_0}{\|u_{h,k}^* - u_{h,k}\|_0} &= 1, \\ \lim_{H \rightarrow 0} \frac{\|\sigma - \sigma_{h,k}\|_0}{\|\sigma_{h,k}^* - \sigma_{h,k}\|_0} &= 1.\end{aligned}$$

Theorem 4.4. *We have under the conditions of Theorem 4.2 that*

$$\begin{aligned}\|u - u_{h,k}\|_\infty &= \|\hat{u}_{h,k} - u_{h,k}\|_\infty + O(H^4 |\log H|^{1/2}), \\ \|\sigma - \sigma_{h,k}\|_\infty &= \|\hat{\sigma}_{h,k} - \sigma_{h,k}\|_\infty + O(H^3).\end{aligned}$$

In addition, if there exist positive constants C_1, C_2 and sufficiently small $\varepsilon_1, \varepsilon_2 \in (0, 1)$ such that

$$\begin{aligned}\|u - u_{h,k}\|_\infty &\geq C_1 H^{4-\varepsilon_1} |\log H|^{1/2}, \\ \|\sigma - \sigma_{h,k}\|_\infty &\geq C_2 H^{3-\varepsilon_2},\end{aligned}$$

then

$$\begin{aligned}\lim_{H \rightarrow 0} \frac{\|u - u_{h,k}\|_\infty}{\|\hat{u}_{h,k} - u_{h,k}\|_\infty} &= 1, \\ \lim_{H \rightarrow 0} \frac{\|\sigma - \sigma_{h,k}\|_\infty}{\|\hat{\sigma}_{h,k} - \sigma_{h,k}\|_\infty} &= 1.\end{aligned}$$

References

- [1] *I. Babuška*: The finite element method with Lagrangian multipliers. *Numer. Math.* 20 (1973), 179–192. zbl
- [2] *H. Blum*: Asymptotic Error Expansion and Defect in the Finite Element Method. University of Heidelberg, Institut für Angewandte Mathematik, Heidelberg.
- [3] *H. Blum, Q. Lin, R. Rannacher*: Asymptotic error expansion and Richardson extrapolation for linear finite elements. *Numer. Math.* 49 (1986), 11–38. zbl
- [4] *F. Brezzi*: On the existence, uniqueness, and approximation of saddle point problems arising from Lagrangian multipliers. *RAIRO, Anal. Numér.* 2 (1974), 129–151. zbl

- [5] *H. Brunner, Y. Lin, S. Zhang*: Higher accuracy methods for second-kind Volterra integral equations based on asymptotic expansions of iterated Galerkin methods. *J. Integral Equations Appl.* *10* (1998), 375–396. zbl
- [6] *J. R. Cannon, Y. Lin*: Non-classical H^1 projection and Galerkin methods for nonlinear parabolic integro-differential equations. *Calcolo* *25* (1988), 187–201. zbl
- [7] *J. R. Cannon, Y. Lin*: A priori L^2 error estimates for finite element methods for nonlinear diffusion equations with memory. *SIAM J. Numer. Anal.* *27* (1990), 595–607. zbl
- [8] *C. Chen, Y. Huang*: Higher Accuracy Theory of FEM. Hunan Science Press, Changsha, 1995.
- [9] *J. Douglas, Jr., J. E. Roberts*: Global estimates for mixed methods for second order elliptic equations. *Math. Comput.* *44* (1985), 39–52. zbl
- [10] *R. E. Ewing, Y. Lin, T. Sun, J. Wang, S. Zhang*: Sharp L^2 error estimates and superconvergence of mixed finite element methods for nonFickian flows in porous media. *SIAM J. Numer. Anal.* *40* (2002), 1538–1560. zbl
- [11] *R. E. Ewing, Y. Lin, J. Wang*: A numerical approximation of nonFickian flows with mixing length growth in porous media. *Acta Math. Univ. Comenian. (N. S.)* *70* (2001), 75–84. zbl
- [12] *R. E. Ewing, Y. Lin, J. Wang*: A backward Euler method for mixed finite element approximations of nonFickian flows with non-smooth data in porous media. Preprint.
- [13] *R. E. Ewing, Y. Lin, J. Wang, S. Zhang*: L^∞ -error estimates and superconvergence in maximum norm of mixed finite element methods for nonFickian flows in porous media. *Int. J. Numer. Anal. Model.* *2* (2005), 301–328. zbl
- [14] *G. Fairweather, Q. Lin, Y. Lin, J. Wang, S. Zhang*: Asymptotic expansions and Richardson extrapolation of approximate solutions for second order elliptic problems on rectangular domains by mixed finite element methods. *SIAM J. Numer. Anal.* *44* (2006), 1122–1149.
- [15] *P. Helfrich*: Asymptotic expansion for the finite element approximations of parabolic problems. *Bonn. Math. Schr.* *158* (1984), 11–30. zbl
- [16] *S. Jia, D. Li, S. Zhang*: Asymptotic expansions and Richardson extrapolation of approximate solutions for integro-differential equations by mixed finite element methods. *Adv. Comput. Math.* To appear.
- [17] *M. N. LeRoux, V. Thomée*: Numerical solutions of semilinear integro-differential equations of parabolic type with nonsmooth data. *SIAM J. Numer. Anal.* *26* (1989), 1291–1309.
- [18] *Q. Lin, I. H. Sloan, R. Xie*: Extrapolation of the iterated-collocation method for integral equations of the second kind. *SIAM J. Numer. Anal.* *27* (1990), 1535–1541. zbl
- [19] *Q. Lin, N. Yan*: The Construction and Analysis of High Efficiency Finite Element Methods. Hebei University Publishers, 1996.
- [20] *Q. Lin, S. Zhang*: An immediate analysis for global superconvergence for integrodifferential equations. *Appl. Math.* *42* (1997), 1–21. zbl
- [21] *Q. Lin, S. Zhang, N. Yan*: Asymptotic error expansion and defect correction for Sobolev and viscoelasticity type equations. *J. Comput. Math.* *16* (1998), 57–62. zbl
- [22] *Q. Lin, S. Zhang, N. Yan*: High accuracy analysis for integrodifferential equations. *Acta Math. Appl. Sin.* *14* (1998), 202–211. zbl
- [23] *Q. Lin, S. Zhang, N. Yan*: Methods for improving approximate accuracy for hyperbolic integro-differential equations. *Syst. Sci. Math. Sci.* *10* (1997), 282–288. zbl
- [24] *Q. Lin, S. Zhang, N. Yan*: Extrapolation and defect correction for diffusion equations with boundary integral conditions. *Acta Math. Sci.* *17* (1997), 409–412. zbl
- [25] *Q. Lin, S. Zhang, N. Yan*: An acceleration method for integral equations by using interpolation post-processing. *Adv. Comput. Math.* *9* (1998), 117–128. zbl

- [26] *T. Lin, Y. Lin, M. Rao, S. Zhang*: Petrov-Galerkin methods for linear Volterra integro-differential equations. *SIAM J. Numer. Anal.* *38* (2000), 937–963. zbl
- [27] *Y. Lin*: On maximum norm estimates for Ritz-Volterra projections and applications to some time-dependent problems. *J. Comput. Math.* *15* (1997), 159–178. zbl
- [28] *Y. Lin, V. Thomée, L. Wahlbin*: Ritz-Volterra projection onto finite element spaces and applications to integrodifferential and related equations. *SIAM J. Numer. Anal.* *28* (1991), 1047–1070.
- [29] *G. Marchuk, V. Shaidurov*: *Difference Methods and Their Extrapolation*. Springer, New York, 1983.
- [30] *B. Neta, J. Igwe*: Finite difference versus finite elements for solving nonlinear integro-differential equations. *J. Math. Anal. Appl.* *112* (1985), 607–618.
- [31] *A. K. Pani, V. Thomée, L. Wahlbin*: Numerical methods for hyperbolic and parabolic integro-differential equations. *J. Integral Equations Appl.* *4* (1992), 533–584. zbl
- [32] *I. H. Sloan, V. Thomée*: Time discretization of an integro-differential equation of parabolic type. *SIAM J. Numer. Anal.* *23* (1986), 1052–1061. zbl
- [33] *V. Thomée, N. Zhang*: Error estimates for semidiscrete finite element methods for parabolic integro-differential equations. *Math. Comput.* *53* (1989), 121–139. zbl
- [34] *J. Wang*: Superconvergence and extrapolation for mixed finite element methods on rectangular domains. *Math. Comput.* *56* (1991), 477–503. zbl
- [35] *J. Wang*: Asymptotic expansions and L^∞ -error estimates for mixed finite element methods for second order elliptic problems. *Numer. Math.* *55* (1989), 401–430. zbl
- [36] *N. Yan, K. Li*: An extrapolation method for BEM. *J. Comput. Math.* *2* (1989), 217–224. zbl
- [37] *S. Zhang, T. Lin, Y. Lin, M. Rao*: Extrapolation and a-posteriori error estimators of Petrov-Galerkin methods for non-linear Volterra integro-differential equations. *J. Comp. Math.* *19* (2001), 407–422. zbl
- [38] *A. Zhou, C. B. Liem, T. M. Shih, T. Lü*: A multi-parameter splitting extrapolation and a parallel algorithm. *Syst. Sci. Math. Sci.* *10* (1997), 253–260. zbl
- [39] *Q. Zhu, Q. Lin*: *Superconvergence Theory of the Finite Element Methods*. Hunan Science Press, 1989.

Authors' addresses: *Shanghai Jia*, School of Applied Mathematics, Central University of Finance and Economics, Beijing 100081, P. R. China, e-mail: shjia@lsec.cc.ac.cn; *Deli Li*, Department of Mathematical Sciences, Lakehead University, Thunder Bay, Ontario P7B 5E1, Canada, e-mail: deli.li@lakeheadu.ca; *Tang Liu, Shuhua Zhang*, Department of Mathematics, Tianjin University of Finance and Economics, Tianjin 300222, P. R. China, e-mail: szhang@tjufe.edu.cn.