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GLOBAL ASYMPTOTIC STABILISATION OF AN ACTIVE MASS DAMPER FOR A FLEXIBLE BEAM[1]

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In this paper, a finite dimensional approximated model of a mechanical system constituted by a vertical heavy flexible beam with lumped masses placed along the beam and a mobile mass located at the tip, is proposed; such a model is parametric in the approximation order, so that a prescribed accuracy in the representation of the actual system can be easily obtained with the proposed model. The system itself can be understood as a simple representation of a building subject to transverse vibrations, whose vibrating modes are damped by a control action performed at the top by means of a mobile mass. A simple PD control law, which requires only the measurement of the position and velocity of the mobile mass with respect to the end-point of the beam, is shown to globally asymptotically stabilise all the flexible modes considered in the approximated model, regardless of the chosen approximation order, under a technical assumption that is satisfied in many cases of practical interest. Simulation runs confirm the effectiveness of the proposed control law in achieving both position regulation of the mobile mass and vibration control.

1. INTRODUCTION

In the last decade, great deal of attention has been paid to the problem of modelling and controlling flexible structures [2]-[4], [7]-[12], [14]. The classic modelling approach, which is used to represent mechanical systems, is usually based on the rigidity assumption; however, such an assumption, in most cases, leads to heavy limitations on the maximum speeds and accelerations supported by the systems themselves. A solution to this problem may be to take into account the deformations of the structures in the modelling process, and to analyse the behaviour of the deformed bodies, possibly under the action of certain control laws. In such a way, the deformations may be inquired to verify, for instance, certain security bounds related to the structure elasticity, or, simply, to monitor such deformations occurring in the flexible components of the system.

The planar mechanical structure analysed in this paper is constituted by a heavy flexible beam clamped on an inertial base, at one of its extremities, and rigidly

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connected, at the other extremity, to a platform where a mass is free to move perpendicularly to the direction assumed by the undeformed beam (see Figure 1). The mass is assumed to be subject to a force (which constitutes the input of the system) exerted by an actuator placed at the end-point of the beam. Moreover, it is assumed that H lumped masses are located at fixed points along the beam, with H being an arbitrary non-negative integer. The motivation for studying such a system arises in civil engineering, when studying the transverse vibrations of a building of H storeys.

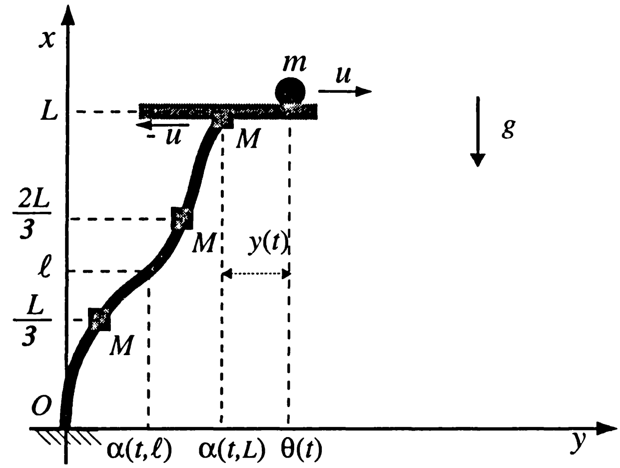

Fig. 1. The mechanical system (in the case of $H = 3$ storeys).

In Section 2, an infinite dimensional model of the mechanical system, taking into account the distributed properties of the beam, is derived, first. Hence, explicit expressions are derived for the equations of an approximated dynamic model of order N of the mechanical structure. Such a model is parametric with respect to the approximation order N, i.e., the same equations can be used to obtain a representation as accurate as necessary by choosing suitable values for N. The dynamic equations of both models are derived by neglecting any kind of friction or damping in the system; in real applications, the presence of damping in the mechanical structures will possibly increase the robustness of the closed-loop system, with respect to the stability requirement.

In Section 3, it is shown that, under certain assumptions, which are not too restrictive, the same PD controller asymptotically stabilises the approximated equations of motion, for any number H of storeys and for any approximation order N of the model. The feedback PD control law requires the measurements of the only relative position and velocity of the mobile mass, with respect to the end-point of the beam; whence, no deformation measurements are needed for the implementation of the control scheme on the real system. The result obtained is global, so that the prescribed control law can be actually applied in the domain of validity of the Bernoulli approximation of elastic deformations.

Finally, in Section 4, some simulation runs are presented, which test the performance of the PD controller on a finite dimensional model, in the simplest case of absence of lumped masses along the beam. The results of the simulations confirm the effectiveness of the control law, reporting satisfactory behaviour of the time responses, and asymptotic stability of the closed-loop system.

2. MOTION EQUATIONS

The mechanical system under consideration is constituted by a heavy flexible beam, whose mass per unit length and elastic constant are denoted by ρ and k, respectively, by a mass m, and by $H \in \mathbb{N}$ masses M. The system under consideration is seen as a model for an H-storey building, having an active mass damper actuator on its top floor. Hence, the mass m (which is free to move on a platform perpendicular to the direction assumed by the undeformed beam, under the action of an external force $u(t)$, exerted between the mass and the end point of the beam) is the representation of the mobile mass, whereas the H masses (which are placed along the beam, at equally spaced positions) are lumped representations of the H storeys. The beam and the masses are located on a plane, where an inertial reference frame (x, y) is defined (see Figure 1), whose origin O coincides with one of the extremities of the beam, which is clamped to an inertial base. The beam, having length L, lies on the x axis of the reference frame when undeformed.

Under the assumption of small deformations, the Cartesian coordinates of an infinitesimal element of the beam at time $t \in \mathbb{R}$, $t \geq 0$, expressed in the reference frame (x, y), are $(\ell, \alpha(t, \ell))$, with $\ell \in [0, L]$, whereas the Cartesian coordinates of the mass m at time t, expressed in the same reference frame, are $(L, \theta(t))$. The Cartesian coordinates of the masses M at time t are given by $(\ell_i, \alpha(t, \ell_i))$, with $\ell_i := i\,L/H$, $i = 1, 2, \ldots, H$. Hence, variables $\alpha(t, \ell)$ and $\theta(t)$ can be taken as the generalised coordinates, which uniquely describe the configuration of the mechanical system.

In the following, in order to simplify the notation, the derivative with respect to t will be denoted by $\dot{}$, and the derivative with respect to ℓ will be denoted by superscript $'$.

Due to the assumption of small deformations, the effects of the gravity force can be neglected. In addition, any dissipative force, such as viscous friction or internal damping due to deformation, will not be considered in this model, since its presence will possibly increase the stability properties of the overall system.

The kinetic energy T_b of the beam and of the masses M, which constitute a single body representing the overall building, can be expressed by the following functional depending on α:

$$T_b := \frac{\rho}{2} \int_0^L \dot{\alpha}^2(t, \ell)\, \mathrm{d}\ell + \sum_{i=1}^{H} \frac{M}{2} \dot{\alpha}^2(t, \ell_i). \tag{1}$$

The potential energy U_b (due only to flexure, by assumption) of the single body can be expressed by the following functional depending on α:

$$U_b := \frac{k}{2} \int_0^L (\alpha''(t,\, \ell))^2 \, \mathrm{d}\ell; \tag{2}$$

notice that the potential energy of the lumped masses M is constant and can be taken to be equal to zero, whence no terms due to the masses M appear in U_b.

The kinetic energy T_m of the mass m is:

$$T_m := \frac{m}{2} \dot{\theta}^2(t); \tag{3}$$

the potential energy of the mass m is constant, similarly to that of the masses M, and is taken to be equal to zero as well.

Now, assume that an external force $u(t)$ acts on the mass m and on the end-point of the beam, and that the associated generalised potential (see [5]) is given by

$$U_u = -u(t)\,(\theta(t) - \alpha(t,\, L)). \tag{4}$$

Moreover, assume that, at each time $t \geq 0$, the function $\alpha(t,\, \ell)$ can be expressed by the following series expansion:

$$\alpha(t,\, \ell) = \sum_{h=0}^{+\infty} \gamma_h(t)\, \sigma_h(\ell), \quad \ell \in [0,\, L]. \tag{5}$$

The functions $\sigma_h(\ell)$, $h \in \mathbb{Z}$, $h \geq 0$, which constitute a complete set, are given by

$$\sigma_h(\ell) := a_h \left(\sinh(\omega_h\, \ell) - \sin(\omega_h\, \ell) - \frac{\sinh(\omega_h\, L) + \sin(\omega_h\, L)}{\cosh(\omega_h\, L) + \cos(\omega_h\, L)} (\cosh(\omega_h\, \ell) - \cos(\omega_h\, \ell)) \right), \tag{6}$$

with the reals ω_h, $h \in \mathbb{Z}$, $h \geq 0$, being the countable solutions of the following equation $\left(\text{i.e., } \omega_h \approx \frac{2\,h+1}{2} \frac{\pi}{L}\right)$:

$$1 + \cos(\omega_h\, L)\, \cosh(\omega_h\, L) = 0. \tag{7}$$

The normalisation constants a_h, $h \in \mathbb{Z}$, $h \geq 0$, are chosen so that the following relation is fulfilled:

$$\int_0^L \sigma_h(\ell)\, \sigma_k(\ell)\, \mathrm{d}\ell = \begin{cases} 1, & h = k, \\ 0, & h \neq k, \end{cases} \tag{8}$$

which implies

$$\int_0^L \sigma_h''(\ell)\, \sigma_k''(\ell)\, \mathrm{d}\ell = \begin{cases} \omega_h^4, & h = k, \\ 0, & h \neq k. \end{cases} \tag{9}$$

Remark 1. The functions $\sigma_h(\cdot)$ given in (6) are the eigenfunctions of the eigenvalue problem resulting from an infinite dimensional model of the considered system, when the mass m and the masses M are absent. The boundary conditions at the extremities of the beam, in such a case, guarantee the possibility of satisfying equations (8), so that the choice of the set of functions $\sigma_i(\cdot)$, given in equation (6), presents remarkable advantages in the subsequent computations, with respect to other complete sets of functions which could be chosen in order to perform a series expansion such as (5).

With these positions, the energies given in (1), (2) and (4), become:

$$T_b = \frac{\rho}{2}\sum_{h=0}^{+\infty}\dot\gamma_h^2(t) + \frac{M}{2}\sum_{i=1}^{H}\left(\sum_{h=0}^{+\infty}\dot\gamma_h(t)\sigma_h(\ell_i)\right)^2, \tag{10a}$$

$$U_b = \frac{k}{2}\sum_{h=0}^{+\infty}\gamma_h^2(t)\omega_h^4, \tag{10b}$$

$$U_u = -u(t)\left(\theta(t) - \sum_{h=0}^{+\infty}\gamma_h(t)\,\sigma_h(L)\right). \tag{10c}$$

The variables $\theta(t)$ and $\gamma_h(t)$, $h \in \mathbb{Z}$, $h \geq 0$, can be taken as the generalised coordinates describing the configuration of the mechanical system under consideration, and the related Euler–Lagrange equations are given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\theta} - \frac{\partial\mathcal{L}}{\partial\theta} = 0, \tag{11a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\gamma_h} - \frac{\partial\mathcal{L}}{\partial\gamma_h} = 0, \quad h \in \mathbb{Z},\ h \geq 0, \tag{11b}$$

where the Lagrangian function $\mathcal{L}$ is given by $\mathcal{L} := T_b + T_m - U_b - U_u$. By (3) and (10), the Euler–Lagrange equations (11) can be recast as follows:

$$m\,\ddot\theta(t) = u(t), \tag{12a}$$

$$\rho\,\ddot\gamma_h(t) + M\sum_{i=1}^{H}\sigma_h(\ell_i)\sum_{j=0}^{+\infty}\ddot\gamma_j(t)\sigma_j(\ell_i) + \rho\,\Omega_h^2\,\gamma_h(t) = -\sigma_h(L)\,u(t),\ h\in\mathbb{Z},\ h\geq 0, \tag{12b}$$

where $\Omega_h := \omega_h^2\sqrt{\frac{k}{\rho}}$. The countable set of equations (12) constitutes the infinite dimensional model of the mechanical system under consideration, which will be taken here as an "exact" model.

In order to obtain a finite-dimensional approximated model of the system, the sum in equation (5) is now truncated to the first N terms, with N being an arbitrary positive integer. Therefore, the function $\alpha(t,\ell)$ will be represented by its N-order approximation:

$$\alpha(t,\ell) \approx \alpha_N(t,\ell) := \sum_{h=0}^{N-1}\gamma_h(t)\,\sigma_h(\ell). \tag{13}$$

In the following, only the approximated model obtained by (13) will be considered.

To this end, taking into account relations (8) and (9), the substitution of (13) into (1), (2) and (4) allows the N-order approximation of the kinetic and potential energies to be obtained as follows:

$$T_{b,N} = \frac{\rho}{2}\sum_{h=0}^{N-1}\dot{\gamma}_h^2(t) + \frac{M}{2}\sum_{i=1}^{H}\left(\sum_{h=0}^{N-1}\dot{\gamma}_h^2(t)\sigma_h(\ell_i)\right)^2, \tag{14a}$$

$$U_{b,N} = \frac{k}{2}\sum_{h=0}^{N-1}\gamma_h^2(t)\omega_h^4, \tag{14b}$$

$$U_{u,N} = -u(t)\left(\theta(t) - \sum_{h=0}^{N-1}\gamma_h(t)\,\sigma_h(L)\right). \tag{14c}$$

Since $\theta(t)$ and $\gamma_h(t)$, $h = 0, \ldots, N-1$, can be taken as the generalised coordinates describing the configuration of the approximated mechanical system, the Euler–Lagrange equations for such a N-order approximated system can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}_N}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}_N}{\partial \theta} = 0, \tag{15a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}_N}{\partial \dot{\gamma}_h} - \frac{\partial \mathcal{L}_N}{\partial \gamma_h} = 0, \quad h = 0, \ldots, N-1, \tag{15b}$$

where the Lagrangian function $\mathcal{L}_N$ of the approximated system is given by $\mathcal{L}_N := T_{b,N} + T_m - U_{b,N} - U_{u,N}$.

By (14) and (3), equations (15) become:

$$m\,\ddot{\theta}(t) = u(t), \tag{16a}$$

$$\rho\,\ddot{\gamma}_h(t) + M\sum_{i=1}^{H}\sigma_h(\ell_i)\sum_{j=0}^{N-1}\ddot{\gamma}_j(t)\sigma_j(\ell_i) + \rho\,\Omega_h^2\,\gamma_h(t) = -\sigma_h(L)\,u(t),\ h=0,\ldots,N-1. \tag{16b}$$

Equations (16) are the N-order approximated equations of motion. Note that the accuracy of such an approximated model can be chosen as high as necessary, by a proper choice of the integer N.

3. STABILISATION OF THE MECHANICAL SYSTEM

The purpose of this section is to achieve global asymptotic stability of the mechanical system under the action of a suitable control law; due to practical problems, only the relative position and velocity of the mobile mass m with respect to the end-point of the beam can be assumed to be measurable. Therefore, since only the approximated model is considered, it is assumed that the following variables are available for feedback:

$$y(t) = \theta(t) - \sum_{h=0}^{N-1}\gamma_h(t)\,\sigma_h(L), \tag{17a}$$

$$\dot{y}(t) \;=\; \dot{\theta}(t) - \sum_{h=0}^{N-1} \dot{\gamma}_h(t)\,\sigma_h(L). \tag{17b}$$

Under such an assumption, the regulation problem under consideration can be stated as follows.

Problem 1. Find (if any) a static feedback control law

$$u(t) = f\left(y(t),\, \dot{y}(t)\right), \tag{18}$$

with $f(\cdot,\,\cdot)$ being a suitable function, from $y(t)$ and $\dot{y}(t)$ such that the closed-loop system (16), (17), (18) has $[\theta\;\gamma_0\;\ldots\;\gamma_{N-1}\;\dot{\theta}\;\dot{\gamma}_0\;\ldots\;\dot{\gamma}_{N-1}]^T = [0\;0\;\ldots\;0\;0\;0\;\ldots\;0]^T$ as globally asymptotically stable equilibrium point.

The control law considered as possible solution to Problem 1 is a classical PD from $y(t)$ and $\dot{y}(t)$:

$$u(t) = -k_p\, y(t) - k_v\, \dot{y}(t), \tag{19}$$

where k_p, k_v are two suitable real constants.

The closed-loop system obtained by applying the control law (19) to system (16), (17) can be recast as follows:

$$m\,\ddot{\theta}(t) + k_p\, y(t) + k_v\, \dot{y}(t) \;=\; 0, \tag{20a}$$

$$\mathcal{B}\,\ddot{\gamma}(t) + \mathcal{H}\,\gamma(t) - k_p\,\sigma\, y(t) - k_v\,\sigma\,\dot{y}(t) \;=\; 0, \tag{20b}$$

where $\gamma(t) \in I\!R^N$ is given by $\gamma(t) := [\gamma_0(t)\;\;\gamma_1(t)\;\;\ldots\;\;\gamma_{N-1}(t)]^T$, the square N-dimensional matrix $\mathcal{B}$ is given by:

$$\mathcal{B} := \begin{bmatrix} \rho + M\sum_{i=1}^{H}(\sigma_0(\ell_i))^2 & M\sum_{i=1}^{H}\sigma_0(\ell_i)\sigma_1(\ell_i) & \ldots & M\sum_{i=1}^{H}\sigma_0(\ell_i)\sigma_{N-1}(\ell_i) \\ M\sum_{i=1}^{H}\sigma_0(\ell_i)\sigma_1(\ell_i) & \rho + M\sum_{i=1}^{H}(\sigma_1(\ell_i))^2 & \ldots & M\sum_{i=1}^{H}\sigma_1(\ell_i)\sigma_{N-1}(\ell_i) \\ \vdots & \vdots & \vdots & \vdots \\ M\sum_{i=1}^{H}\sigma_0(\ell_i)\sigma_{N-1}(\ell_i) & M\sum_{i=1}^{H}\sigma_1(\ell_i)\sigma_{N-1}(\ell_i) & \ldots & \rho + M\sum_{i=1}^{H}(\sigma_{N-1}(\ell_i))^2 \end{bmatrix},$$

$\mathcal{H}$ is the N-dimensional, diagonal, square matrix given by $\mathcal{H} := \mathrm{diag}\left(\rho\,\Omega_0^2,\, \rho\,\Omega_1^2,\, \ldots,\, \rho\,\Omega_{N-1}^2\right)$, and the vector $\sigma \in I\!R^N$ is given by $\sigma := [\sigma_0(L)\;\;\sigma_1(L)\;\;\ldots\;\;\sigma_{N-1}(L)]^T$. Notice that the dependence of matrices $\mathcal{B}$ and $\mathcal{H}$ and vector σ on the physical parameters L, ρ, k, and M of the system has been omitted for the sake of simplicity.

The following assumption is needed in order to prove the main result of this section; such an assumption is not too restrictive, as shown in the subsequent Lemma 1 and Remark 2.

Assumption 1. Matrix $\mathcal{B}^{-1}\mathcal{H}$ has N distinct eigenvalues, and, in addition, for each eigenvector $v \in \mathbb{R}^N$ of $\mathcal{B}^{-1}\mathcal{H}$, one has $\sigma^T v \neq 0$.

Assumption 1 is "robust" with respect to small variations of the physical parameters L, ρ, k, and M, of the system, as stated in the following lemma (notice that matrix $\mathcal{B}^{-1}\mathcal{H}$ is independent of mass m).

Lemma 1. If Assumption 1 is satisfied for $L = \overline{L}$, $\rho = \overline{\rho}$, $k = \overline{k}$, and $M = \overline{M}$, for some $\overline{L}$, $\overline{\rho}$, $\overline{k}$, $\overline{M} \in \mathbb{R}^+$, then there exists a neighbourhood $\mathcal{U} \subset \mathbb{R}^4$ of $[\overline{L} \;\; \overline{\rho} \;\; \overline{k} \;\; \overline{M}]^T$ such that such an assumption is satisfied for all $[L \;\; \rho \;\; k \;\; M]^T \in \mathcal{U}$.

P r o o f. All the entries of matrices $\mathcal{B}$ and $\mathcal{H}$ are continuous functions of the parameters L, ρ, k, M, and matrix $\mathcal{B}$ is non-singular; hence, also the entries of matrix $\mathcal{B}^{-1}\mathcal{H}$ are continuous functions of such parameters. By taking into account that the eigenvalues of $\mathcal{B}^{-1}\mathcal{H}$ are all distinct for $L = \overline{L}$, $\rho = \overline{\rho}$, $k = \overline{k}$, and $M = \overline{M}$, and that the eigenvalues of a square matrix are continuous functions of its entries, one has that the N eigenvalues of $\mathcal{B}^{-1}\mathcal{H}$ are all distinct in a suitable neighbourhood Θ of $[\overline{L} \;\; \overline{\rho} \;\; \overline{k} \;\; \overline{M}]^T$, and this proves that the first statement of Assumption 1 holds in Θ. Now, let $\lambda_i(\cdot, \cdot, \cdot, \cdot)$, $i = 1, 2, \ldots, N$, be N scalar continuous functions such that, for each $[L \;\; \rho \;\; k \;\; M]^T \in \Theta$, $\{\lambda_1(L, \rho, k, M), \lambda_2(L, \rho, k, M), \ldots, \lambda_N(L, \rho, k, M)\}$ is the set of the distinct eigenvalues of $\mathcal{B}^{-1}\mathcal{H}$. Due to the fact that the N eigenvalues of $\mathcal{B}^{-1}\mathcal{H}$ are distinct in Θ, it is possible to define N continuous vector functions $v_1(\cdot, \cdot, \cdot, \cdot)$, $v_2(\cdot, \cdot, \cdot, \cdot)$, $\ldots$, $v_n(\cdot, \cdot, \cdot, \cdot) \in \mathbb{R}^N$, such that, in a suitable neighbourhood $\overline{\Theta}$ of $[\overline{L} \;\; \overline{\rho} \;\; \overline{k} \;\; \overline{M}]^T$, $\overline{\Theta} \subset \Theta$, for each $i = 1, 2, \ldots, N$, $v_i(L, \rho, k, M)$ is an eigenvector of $\mathcal{B}^{-1}\mathcal{H}$ relative to the eigenvalue $\lambda_i(L, \rho, k, M)$. Since the vector σ is a continuous function of L, then, for each $i = 1, 2, \ldots, N$, the product $\sigma^T v_i$ is a continuous function of the parameters, which is non-null at $[\overline{L} \;\; \overline{\rho} \;\; \overline{k} \;\; \overline{M}]^T$, whence it is non null in a suitable neighbourhood $\Theta_i \subset \overline{\Theta}$. The proof of the lemma is completed, by letting $\mathcal{U} = \bigcap_{i=1}^N \Theta_i$. □

Remark 2. It is easy to see that, if $M = 0$, then Assumption 1 holds for any L, ρ, $k \in \mathbb{R}^+$: in this case one has $\mathcal{B} = \rho I$, with I being the N-dimensional identity matrix, and matrix $\mathcal{H}$ is diagonal by definition, with its diagonal elements being all distinct; hence, the eigenvalues of $\mathcal{B}^{-1}\mathcal{H}$ are all distinct, and the eigenvectors of $\mathcal{B}^{-1}\mathcal{H}$ are the N vectors of the canonical basis of $\mathbb{R}^N$. The required property $\sigma^T v \neq 0$ for $M = 0$ and arbitrary L, ρ, $k \in \mathbb{R}^+$, easily follows from the consideration that all the components of vector σ are different from zero; this can be proven by direct computation, by taking into account equations (6) and (7).

Hence, by Lemma 1, Assumption 1 holds for a significant class of systems, namely those representing buildings in which the mass M of the storeys is not too big as compared with the mass of the whole structure.

In the following theorem, the global asymptotic stability of the closed-loop system (20) is stated and proven.

Theorem 1. For each $N \in \mathbb{Z}$, $N \geq 1$, under Assumption 1, if $k_p > 0$ and $k_v > 0$, then system (20) has $[\theta\ \gamma_0 \ldots \gamma_{N-1}\ \dot{\theta}\ \dot{\gamma}_0 \ldots \dot{\gamma}_{N-1}]^T = [0\ 0 \ldots 0\ 0\ 0 \ldots 0]^T$ as globally asymptotically stable equilibrium point.

Remark 3. Theorem 1 proves the effectiveness of a simple PD control law in stabilising the N-order approximated model (with N arbitrarily high) of the mechanical system under consideration; it is stressed that the structure of the proposed control algorithm is independent of the chosen approximation order.

Proof of Theorem 1. In the following, the dependence on variable t is sometimes omitted for the sake of brevity. For any $N \in \mathbb{Z}$, $N \geq 1$, consider the following positive definite and radially unbounded function of θ, $\dot{\theta}$, γ_h, $\dot{\gamma}_h$, $h = 0, \ldots, N-1$,

$$V_N := \frac{m}{2}\dot{\theta}^2 + \frac{1}{2}\dot{\gamma}^T \mathcal{B}\dot{\gamma} + \frac{1}{2}\gamma^T \mathcal{H}\gamma + \frac{k_p}{2} y^2, \tag{21}$$

to be used as a candidate Lyapunov function. It is easy to compute the total time derivative of V_N along the dynamics of system (20) (see also Theorem 12.28 of [13]):

$$\begin{aligned} \dot{V}_N &= m\dot{\theta}\ddot{\theta} + \dot{\gamma}^T \mathcal{B}\ddot{\gamma} + \gamma^T \mathcal{H}\dot{\gamma} + k_p\, y\, \dot{y} \\ &= m\dot{\theta}\ddot{\theta} + \dot{\gamma}^T \mathcal{B}\left(-\mathcal{B}^{-1}\mathcal{H}\gamma + \mathcal{B}^{-1} k_p\, \sigma\, y + k_v\, \mathcal{B}^{-1}\sigma\, \dot{y}\right) + \gamma^T \mathcal{H}\dot{\gamma} + k_p\, y\, \dot{y} \\ &= -k_p\, \dot{\theta}\, y - k_v\, \dot{\theta}\, \dot{y} + k_p\, \dot{\gamma}^T \sigma\, y + k_v\, \dot{\gamma}^T \sigma\, \dot{y} + k_p\, y\, \dot{y} \\ &= -k_v\, \dot{y}^2. \end{aligned} \tag{22}$$

Since the function V_N defined in (21) is globally positive definite and its total time derivative (22) is globally negative semi-definite, then Theorem 25.1 of [6] proves that the equilibrium point $[\theta\ \gamma_0 \ldots \gamma_{N-1}\ \dot{\theta}\ \dot{\gamma}_0 \ldots \dot{\gamma}_{N-1}]^T = [0\ 0 \ldots 0\ 0\ 0 \ldots 0]^T$ is stable. Let $\mathcal{E}$ be the set of points $[\theta\ \gamma_0 \ldots \gamma_{N-1}\ \dot{\theta}\ \dot{\gamma}_0 \ldots \dot{\gamma}_{N-1}]^T$ such that $\dot{V}_N = 0$. Since $k_v > 0$, if $\theta(t)$, $\dot{\theta}(t)$, $\gamma_h(t)$, $\dot{\gamma}_h(t)$, $h = 0, \ldots, N-1$, $t \geq 0$, is an half trajectory of (20) entirely contained in $\mathcal{E}$, then it satisfies the following relations:

$$\dot{y}(t) = 0, \quad \forall t \geq 0, \tag{23a}$$

$$m\ddot{\theta}(t) + k_p\, y(t) = 0, \quad \forall t \geq 0, \tag{23b}$$

$$\mathcal{B}\ddot{\gamma}(t) + \mathcal{H}\gamma(t) - k_p\, \sigma\, y(t) = 0, \quad \forall t \geq 0. \tag{23c}$$

Equation (23a) implies $\ddot{y}(t) = \ddot{\theta}(t) - \sum_{h=0}^{N-1}\ddot{\gamma}_h(t)\,\sigma_h(L) = 0$ for all $t \geq 0$ and $y(t) = y$ for all $t \geq 0$, for some $y \in \mathbb{R}$; hence, one obtains:

$$\begin{aligned} \ddot{\theta}(t) &= \sum_{h=0}^{N-1} \ddot{\gamma}_h(t)\,\sigma_h(L) \\ &= \sigma^T \ddot{\gamma}(t), \quad \forall t \geq 0. \end{aligned} \tag{24}$$

Equation (23c) can be rewritten as:

$$\mathcal{B}\ddot{\gamma}(t) + \mathcal{H}\gamma(t) = k_p\, \sigma\, y, \quad \forall t \geq 0,$$

and its solutions are of the form

$$\gamma(t) = \sum_{i=1}^{N-1} (\zeta_i \sin(\mu_i t) + \eta_i \cos(\mu_i t)) + \mathcal{H}^{-1} k_p \sigma y, \tag{25}$$

where $\mu_i \in I\!R$, $\mu_i > 0$, are N real numbers such that μ_i^2, $i = 0, 1, \ldots, N-1$, are the (distinct) eigenvalues of $\mathcal{B}^{-1}\mathcal{H}$ (in addition, they are all real and positive, as it is proven in Section 10−2 of [5]) and, for every $i = 0, 1, \ldots, N-1$, $\zeta_i = c_{i,\zeta} v_i$, $\eta_i = c_{i,\eta} v_i$ with v_i being eigenvector of $\mathcal{B}^{-1}\mathcal{H}$, relative to the eigenvalue μ_i^2, and $c_{i,\zeta}$, $c_{i,\eta}$ being suitable reals.

By substituting the expression of $\ddot{\theta}(t)$ given by (24) into (23b) and taking into account (25), one has:

$$m \sum_{i=0}^{N-1} \mu_i^2 \left(\sigma^T \zeta_i \sin(\mu_i t) + \sigma^T \eta_i \cos(\mu_i t)\right) = k_p y, \quad \forall t \geq 0. \tag{26}$$

Since the functions $\sin(\mu_0 t)$, $\cos(\mu_0 t)$, $\ldots$, $\sin(\mu_{N-1} t)$, $\cos(\mu_{N-1} t)$, 1, are linearly independent over $[0, +\infty)$, by taking into account that $\mu_h \neq 0$ for all $h = 0, \ldots, N-1$, that $k_p > 0$ and $m > 0$, and Assumption 1, relation (26) implies that $\zeta_h = 0$, $\eta_h = 0$, $h = 0, \ldots, N-1$, and $y = 0$. Hence, from (25), $\gamma_h(t) = 0$, $h = 0, \ldots, N-1$, for all $t \geq 0$, and, by recalling equation (17a), it follows that $\theta(t) = 0$ for all $t \geq 0$.

The above discussion shows that the largest invariant subset contained in $\mathcal{E}$ is constituted by the only equilibrium point $[\theta\ \gamma_0 \ldots \gamma_{N-1}\ \dot{\theta}\ \dot{\gamma}_0 \ldots \dot{\gamma}_{N-1}]^T = [0\ 0 \ldots 0\ 0\ 0 \ldots 0]^T$; whence, Theorem 26.1 of [6] proves the attractivity of such an equilibrium point. Since stability has already been proven, the global asymptotic stability of the mentioned equilibrium point follows, by taking into account that V_N is globally positive definite and radially unbounded. □

4. SIMULATIONS

In this section, the results of some significant simulations of the behaviour of the closed-loop system (20), are reported. Two case studies have been considered, in the case of $M = 0$, characterised by different choices of the feedback constants k_p and k_v of the controller. In both cases, the order of the approximated model has been chosen as $N = 10$. It has been verified that a higher order approximation does not affect significantly the time responses obtained from the simulations; whence, the accuracy of the model obtained by the chosen approximation order is sufficiently high to guarantee a good fitness with the real case.

The physical parameters of the system have been chosen as:

$$L = 1\,[\mathrm{m}], \quad m = 7.86 \cdot 10^{-2}\,[\mathrm{kg}], \quad \rho = 7.86 \cdot 10^{-1} \left[\frac{\mathrm{kg}}{\mathrm{m}}\right], \quad k = 2.05 \cdot 10^{2}\,\left[\mathrm{N\,m}^2\right],$$

which correspond to the parameters of a hardened steel beam having a square cross section with edges 1 cm long, and to a mobile mass, whose mass m has a ratio

1 : 10 with the total mass of the beam itself (see Figure 2). With these parameters, the mass m is sufficiently heavy to perform a satisfactory control action on the system, without being too heavy compared to the weight of the whole structure. For such a choice of the system parameters and of the approximation order, the reals ω_h, a_h, $h = 0, 1, \ldots, 9$, have been calculated as reported in Table 1.

Table 1. Values of the real constants ω_h, a_h, $h = 0, 1, \ldots, 9$.

	$h=0$	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=6$	$h=7$	$h=8$	$h=9$
ω_h	1.88	4.69	7.85	11.00	14.14	17.28	20.42	23.56	26.70	29.85
a_h	0.73	1.02	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00

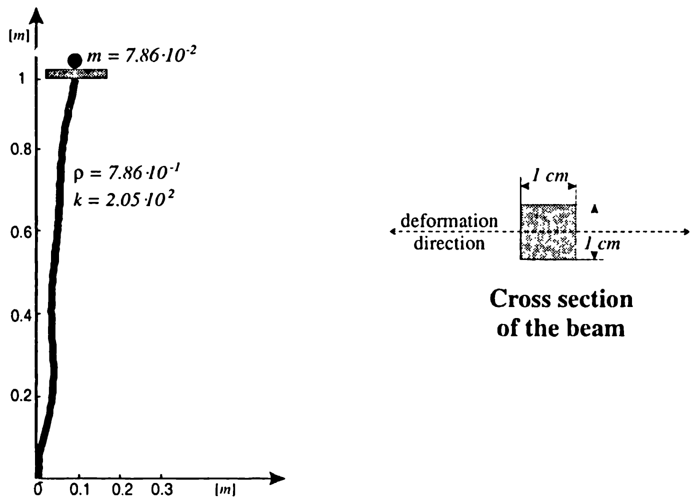

Fig. 2. Physical parameters and initial conditions of the system.

As for the initial conditions at the initial time $t = 0$, the velocities have been assumed to be all equal to zero (i.e., $[\dot{\theta}(0)\ \dot{\gamma}_0(0) \ldots \dot{\gamma}_9(0)]^T = [0\ 0 \ldots 0]^T$), and the initial configuration of the beam has been chosen as a deformed configuration characterised by the following coordinates:

$$\begin{aligned} [\gamma_0(0) \ldots \gamma_9(0)]^T &= [0.05\ 0.01\ 0.01\ 0.005\ 0.005\ 0\ 0\ 0\ 0\ 0]^T, \\ \theta(0) &= \sum_{h=0}^{9} \sigma_h(L)\, \gamma_h(0), \end{aligned}$$

namely, the higher order modes have been assumed to be zero at the initial time, and the variable $y(0)$ has been set to zero as well (see Figure 2).

As regards to the first choice of the control parameters, the time responses $\theta(t)$, $\alpha_N(t, L)$, $y(t)$ and $u(t)$, corresponding to the following values of k_p and k_v:

$$k_p = 1\,,\ k_v = 0.5\,,$$

are shown in Figure 3. In particular, the first plot represents the absolute position $\theta(t)$ of the mobile mass, the second plot represents the position $\alpha_N(t, L)$ of the end-point of the beam, the third plot represents the relative position $y(t)$ of the mobile mass with respect to the end-point of the beam, and the fourth plot represents the control action $u(t)$.

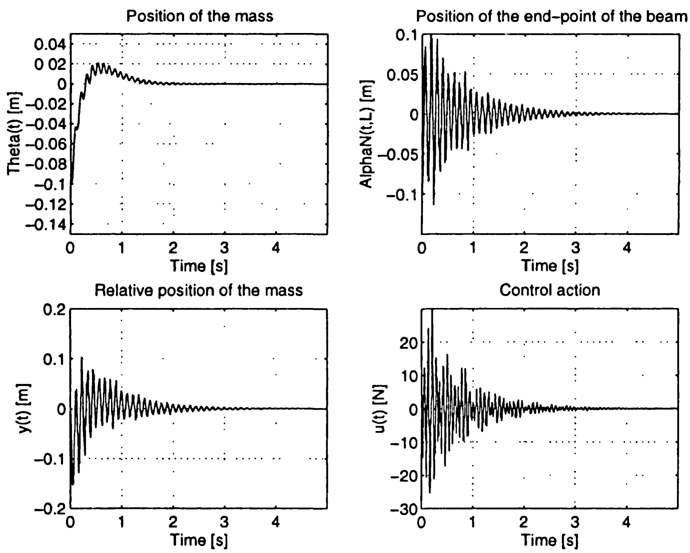

Fig. 3. Time responses of the closed loop system corresponding to the controller gains $k_p = 1$, $k_v = 0.5$.

It can be observed, from all the time responses, that the dissipative action of the derivative control efficiently reduces the intrinsic oscillations of the mechanism. The power exerted by the controller (i.e., the modulus of the control signal) is high because of the high potential energy of the system at the initial time; as a matter of fact, the hardened steel bar is widely deformed at time $t = 0$ with respect to its stiffness.

As regards to the second choice of the control parameters, the time responses $\theta(t)$, $\alpha_N(t, L)$, $y(t)$ and $u(t)$, respectively, corresponding to the following values of k_p and k_v:

$$k_p = 100\,, \; k_v = 3\,,$$

are shown in Figure 4.

In this second case, the controller gains have been incremented to improve the performance of the control action. As a consequence, the dynamics of the closed loop system are faster (the time responses are damped to their steady-state values in 0.5 s instead of 3 s) but the power needed by the control action is higher (the force peaks are more than double in this case with respect to the previous one). It should be noticed that a stronger control action highly improves the performance of the controller in damping the intrinsic oscillations of the system.

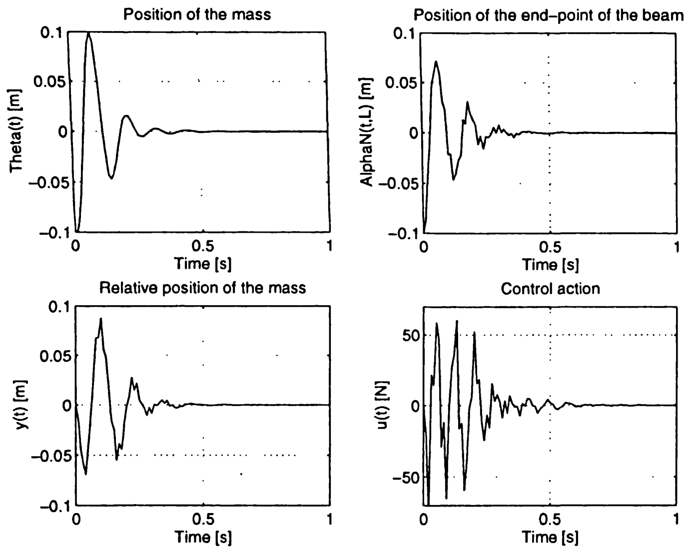

Fig. 4. Time responses of the closed loop system corresponding to the controller gains $k_p = 100$, $k_v = 3$.

5. CONCLUSIONS

A finite-dimensional approximated model, parametric in the order N of approximation, of a flexible beam, with a mobile mass located at its end-point and H lumped masses placed along its length, has been obtained by considering the natural vibration modes of a clamped beam. Such a mechanical system is to be understood as a simple representation of an H-storey building subject to intrinsic vibration and controlled by means of an actuator exerting a relative force between the upper storey and a mass located on the roof. This modelling approach allowed to prove that, under an assumption, which is fulfilled in many cases of interest, the same feedback PD control law from the relative position and velocity of the mobile mass with respect to the position of the end-point of the beam (namely, a local measurement), asymptotically stabilises the vibration modes considered in the approximated model, for any choice of the order N of approximation. The proposed control law has been tested by means of simulations, which confirmed the effectiveness in achieving position regulation and stabilisation of the elastic modes of the beam, in the simplified case in which the mass of the storeys is negligible with respect to the total mass of the building.

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