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A. Jafari Koshkouei; Alan S. I. Zinober

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SLIDING MODE CONTROLLER–OBSERVER DESIGN FOR MULTIVARIABLE LINEAR SYSTEMS WITH UNMATCHED UNCERTAINTY

A. JAFARI KOSHKOUEI AND ALAN S. I. ZINOBER

This paper presents sufficient conditions for the sliding mode control of a system with disturbance input. The behaviour of the sliding dynamics in the presence of unmatched uncertainty is also studied. When a certain sufficient condition on the gain feedback matrix of the discontinuous controller and the disturbance bound holds, then the disturbance does not affect the sliding system. The design of asymptotically stable sliding observers for linear multivariable systems is presented. A sliding observer design ensures that in the presence of unmatched uncertainty, the estimated state nearly approaches the actual state. The error of the approximation depends upon the distance and bound of the unmatched uncertainty. However, certain sufficient conditions should be satisfied for the asymptotic stability of the error system.

1. INTRODUCTION

Sliding mode controller-observers have been widely studied in recent years [1], [4], [7] – [11], [13], [14]. Their robustness and insensitivity with respect to unknown parameter variations [1] and simplicity of design, make sliding mode a powerful approach. Analysis and comparison of several types of observer [12] show that the sliding mode observer is good from the point of view of robustness, implementation, numerical stability, applicability, ease of tuning and overall evaluation.

Edwards and Spurgeon [4] modified the Utkin observer [13] and extended the discontinuous observer to nonlinear systems. They developed a robust discontinuous observer. Sira-Ramírez and Spurgeon [11] discussed the matching conditions of the sliding mode observer for linear systems, and also studied the generalized observer canonical form. Koshkouei and Zinober [6] have described methods for designing an asymptotically stable observer, the existence of the sliding mode and the stability of state reconstruction systems of MIMO linear systems with disturbance input. They also studied an observer for a SISO linear system with unmatched uncertainty and presented certain conditions for the stability of the system [7].

Dorling and Zinober [1] compared full and reduced order Luenberger observers with the Utkin observer. They reported some difficulty in the selection of an appropriate constant switched gain to ensure that the sliding mode occurs, and discussed the elimination of chattering. However, the unmatched uncertainty was shown to affect the ideal dynamics prescribed by the chosen sliding surface.

There are two approaches for designing a sliding mode observer. The first approach is based upon the equivalent control technique. By using an appropriate state transformation, the system equation can be converted to two suitable subsystems. An attractive sliding hyperplane is selected and then a full state observer for the reduced order sliding system is designed. In the second approach a full state observer is designed. Then sliding mode techniques are applied to stabilize the resulting error system. A sliding mode observer, like sliding control, is usually a discontinuous observer.

In this paper the second approach is applied and we extend the work of Koshkouei and Zinober [7] to multivariable systems. First some results relating to sliding dynamics are presented, and then an observer for a system which may *not* satisfy the *matching condition*, is developed.

The main problems in sliding mode design are the selection of an attractive sliding hyperplane, sliding control, particularly the gain feedback matrix for the discontinuous controller-observer and the reduction of the influence of disturbances as much as possible. Sometimes a certain condition can be applied to eliminate the effect of the disturbance. The matching condition is a sufficient condition that rejects the disturbance in the sliding mode. However, a control-observer satisfying a certain condition may be designed for a system with unmatched uncertainty, so that the sliding system is free of the influence of the disturbance.

In this paper a sliding observer for full order systems with disturbance input is designed. This system may not be ideally in the sliding mode and the uncertainty may not satisfy the matching condition. Similarly to discontinuous controllers, there exist many methods to eliminate observer chattering, including a continuous approximation for the discontinuous feedforward compensation signals [2], if chatter is undesirable.

To establish the stability of the error system, suitable conditions on the disturbance input are needed; (i) the matching condition, (ii) the convergency of the norm of the disturbance input signal to zero, (iii) the norm of the disturbance signal to be bounded by the output error, i. e. there exists a real function M (or a real number M) such that $\|\xi\| \leq M\|C(x-\hat{x})\|$ where ξ is the disturbance input and $\hat{x}$ is the estimate of the state x. Otherwise, the asymptotic stability of the error system may not exist in the presence of the disturbance input. Note that, since the output is accessible, so is the estimated output.

In Section 2 the sufficient conditions for the existence of the sliding mode for a linear system with disturbance input, are studied. In Section 3 an approach to observer design and the stability of the state reconstruction error system for time-invariant multivariable systems using the Lyapunov method, is developed. We establish methods to find the feedforward injection map and the external feedforward compensation signal, which correspond respectively to the control input distribution map and the input of the reconstruction error system. Sufficient conditions for the existence of the sliding mode in the reconstruction error system are proposed to ensure ultimate boundedness or asymptotic stability of the error system. When there is unmatched uncertainty, system stability may not be achieved. However, a region may exist in which the state error trajectory converges to the sliding surface within a finite time and thereafter remains on this surface to the origin. In this case the disturbance rejection problem for the sliding system may not be completely satisfied but, when the sliding mode occurs, the state trajectory moves within a neighbourhood of the sliding surface to the origin. An example illustrating the results is presented in Section 4.

In this paper $\sigma_M(\cdot)$, $\sigma_m(\cdot)$ $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ refer to the largest singular value, the smallest singular value, the largest and the smallest eigenvalue of $(\cdot)$, respectively. We also use p.d., p.d.s. and u.p.d.s. for positive definite, positive definite symmetric and unique positive definite symmetric.

2. SUFFICIENT SLIDING MODE CONDITIONS FOR SYSTEMS WITH UNMATCHED UNCERTAINTY

Consider the time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) + \Gamma\xi(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where $x \in \mathbb{R}^n$ is the state variable, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ is full rank, $u \in \mathbb{R}^m$ is the control, $C \in \mathbb{R}^{m\times n}$ such that CB is a nonsingular matrix, $y \in \mathbb{R}^m$ is the "output" required to implement sliding mode control, $\Gamma \in \mathbb{R}^{n\times m}$ is the perturbation input map and $\xi \in \mathbb{R}^m$ is the bounded disturbance input, i. e. there exists a positive real number M such that $\|\xi\| \leq M$. The real number M is chosen as small as possible and if $\sup\|\xi(t)\|$ is known, $M = \sup\|\xi(t)\|$. We assume that (A, B) is completely controllable and (A, C) completely observable.

The sliding surface is $y = Cx = 0$. The ideal sliding mode occurs if there exists a finite time t_s such that

$$y = Cx = 0, \quad t \geq t_s \tag{3}$$

where the time t_s is the time when the sliding mode is reached. The equivalent control is the theoretical effective linear control of the system during the sliding mode and is given by

$$u_{eq} = -(CB)^{-1}(CAx + C\Gamma\xi). \tag{4}$$

Substituting the equivalent control (4) in (1) gives the reduced order system

$$\dot{x} = (I - B(CB)^{-1}C)\,Ax + (I - B(CB)^{-1}C)\,\Gamma\xi \tag{5}$$

during the sliding mode. Let $y = \begin{bmatrix} y_1 & y_2 & \dots & y_m \end{bmatrix}^T$. Choose the control

$$u = -(CB)^{-1}(CAx + C\Gamma\xi + K_1 \operatorname{sgn} y) \tag{6}$$

where $\operatorname{sgn} y = \begin{bmatrix} \operatorname{sgn} y_1 & \operatorname{sgn} y_2 & \dots & \operatorname{sgn} y_m \end{bmatrix}^T$, sgn being the signum function, and the design gain matrix K_1 is a diagonal matrix with positive elements

$$K_1 = \operatorname{diag}(k_{11}, k_{12}, \dots, k_{1m}).$$

Then

$$\begin{aligned} y^T \dot{y} &= y^T(CAx + CBu + C\Gamma\xi) \\ &= -y^T K_1 \operatorname{sgn} y \\ &= -(k_{11}|y_1| + k_{12}|y_2| + \ldots + k_{1m}|y_m|) \\ &< 0. \end{aligned} \tag{7}$$

Differentiating (2) and using (6) yields the output signal dynamics

$$\dot{y} = -K_1 \operatorname{sgn} y. \tag{8}$$

Hence, for any i, $1 \le i \le m$,

$$\dot{y}_i = -k_{1i} \operatorname{sgn} y_i \tag{9}$$

and then

$$y_i = -k_{1i}(t - t_{i_s}) \operatorname{sgn} y_i \tag{10}$$

where t_{i_s} is the time to reach the surface $y_i = 0$. Therefore, the output behaviour is governed by

$$y = -K_1(t - t_s) \operatorname{sgn} y. \tag{11}$$

From (10)

$$t_s = \max_{1 \le i \le m} \frac{|y_i(0)|}{k_{1i}}. \tag{12}$$

Since for all i, $1 \le i \le m$, $|y_i(0)| \le \|y(0)\|$ and $\sigma_m(K_1) \le k_{1i}$, (12) yields

$$t_s \le \frac{\|y(0)\|}{\sigma_m(K_1)}.$$

Note that, for $t \in [0, t_s]$, the state variable x moves to the the sliding surface $y = 0$. The output dynamics (8) shows that y asymptotically converges to $y = 0$ and for any i, $1 \le i \le m$, the rate of change of y_i is guaranteed to be $-k_{1i}$ (k_{1i}) for y_i positive (negative), i.e. the velocity of y to the sliding surface $y = 0$ is $[k_{11}\ k_{12}\ \ldots\ k_{1m}]$.

If ξ is unknown, the control law (6) cannot be implemented. So an estimate $\hat{\xi}$ of ξ is required. Let $\hat{\xi} = M \operatorname{sgn}(Cx)$. The control law (6) becomes

$$\begin{aligned} u &= -(CB)^{-1}\left(CAx + C\Gamma\hat{\xi} + K_1 \operatorname{sgn} y\right) \\ &= -(CB)^{-1}\left(CAx + (C\Gamma M + K_1) \operatorname{sgn} y\right) \\ &= -(CB)^{-1}\left(CAx + K \operatorname{sgn} y\right). \end{aligned} \tag{13}$$

Therefore the control is given by

$$u = -(CB)^{-1}(CAx + K \operatorname{sgn} y) \tag{14}$$

where the matrix $K - C\Gamma M$ is a p.d. matrix. For simplicity, select the matrix K to be a diagonal matrix with positive real entries such that

$$M\sigma_M(C\Gamma) < \sigma_m(K).$$

Then the output signal is given by

$$\dot{y} = C\Gamma\xi - K\operatorname{sgn} y \tag{15}$$

for

$$C\Gamma\xi = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{bmatrix}.$$

The reaching condition of the sliding mode is

$$\dot{y}_i y_i < 0, \qquad \forall\, i,\ 1 \le i \le m. \tag{16}$$

and

$$y^T \dot{y} = y^T (C\Gamma\xi - K\operatorname{sgn} y)$$

Let $K = \operatorname{diag}(k_1, k_2, \ldots, k_m)$. Hence

$$y^T \dot{y} = y^T \begin{bmatrix} \gamma_1 - k_1 \operatorname{sgn} y_1 \\ \gamma_2 - k_2 \operatorname{sgn} y_2 \\ \vdots \\ \gamma_m - k_m \operatorname{sgn} y_m \end{bmatrix}$$

Therefore, if for all i, $1 \le i \le m$

$$y_i(\gamma_i - k_i \operatorname{sgn} y_i) < 0$$

then $y^T \dot{y} < 0$. Thus, a sufficient condition for the existence of the sliding mode is that

$$|\gamma_i| < k_i, \qquad \forall\, i, \quad 1 \le i \le m \tag{17}$$

Hence, if

$$\gamma_1^2 + \gamma_2^2 + \ldots + \gamma_m^2 < \min_{1 \le i \le m} k_i^2 = \sigma_m^2(K) \tag{18}$$

then (17) is true. From (18)

$$\|C\Gamma\xi\| < \sigma_m(K) \tag{19}$$

but

$$\|C\Gamma\xi\| \le M\sigma_M(C\Gamma)$$

so, if

$$M\sigma_M(C\Gamma) < \sigma_m(K) \tag{20}$$

then (17) is true, i.e. (20) is a sufficient condition for (17) and for the existence of the sliding mode control.

If (15) is satisfied, then

$$\dot{y}_i = \gamma_i - k_i \operatorname{sgn} y_i, \qquad \forall\, i,\ 1 \le i \le m$$

and

$$y_i = \left(\int_{t_{is}}^{t} \gamma_i \, \mathrm{d}t \right) - k_i(t - t_{is}) \operatorname{sgn} y_i$$

where t_{is} is the time to reach the sliding surface $y_i = 0$. Therefore, (16) implies

$$\dot{y}_i \operatorname{sgn} y_i = \gamma_i \operatorname{sgn} y_i - k_i < 0.$$

Hence, a sufficient condition for the state trajectories to converge to the surface $y_i = 0$ is

$$\gamma_i \operatorname{sgn} y_i < k_i$$

which holds if (17) is satisfied. For any i, $1 \leq i \leq m$, there is a real number η_i such that $|\gamma_i| \leq \eta_i < k_i$ and $\eta_i \leq ||C\Gamma||M$. Hence, the ith output y_i satisfies

$$|y_i| \begin{cases} \geq (\eta_i - k_i)(t - t_{is}) & \text{if} \quad t < t_{is} \\ = 0 & \text{if} \quad t \geq t_{is} \end{cases}$$

and

$$t_{is} \leq \frac{|y_i(0)|}{k_i - \eta_i}, \qquad \forall\, i, \ 1 \leq i \leq m.$$

Assume that the condition (20) is true. Then

$$\begin{aligned} t_s &= \max_{1 \leq i \leq m} t_{is} \\ &\leq \max_{1 \leq i \leq m} \frac{y_i(0)}{k_i - \eta_i} \\ &\leq \frac{||y(0)||}{\sigma_m(K) - M\sigma_M(C\Gamma)}. \end{aligned}$$

Now, we study the effect of the uncertainty on the sliding system. The system in the sliding mode is independent of the disturbance input ξ if $\operatorname{rank}(B, \Gamma) = \operatorname{rank}(B)$. This condition was first studied by Draženović [3] and was proved by El-Ghezawi et al [5] using the properties of projectors. Here we present a new proof for an equivalent condition. Our proof gives more information about the influence of the disturbance on the system and clarifies which parts of the system (5) have no effect on the system in the sliding mode.

Assume T is an orthogonal matrix such that

$$TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{21}$$

where B_2 is a non-singular matrix. Let $Tx = z$, then

$$\dot{z} = TAT^T z + TBu + T\Gamma\xi \tag{22}$$

and

$$z^T = [\, {z_1}^T \ {z_2}^T \,], \quad z_1 \in I\!R^{n-m}, \ z_2 \in I\!R^m. \tag{23}$$

Therefore

$$\dot{z}_1 = A_{11}z_1 + A_{12}z_2 + \Gamma_1\xi \tag{24}$$
$$\dot{z}_2 = A_{21}z_1 + A_{22}z_2 + B_2u + \Gamma_2\xi \tag{25}$$

where

$$TAT^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad T\Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}.$$

The system in the sliding mode is independent of ξ if $\Gamma_1 = 0$. We now prove that a sufficient condition for the reduced order system (24) to be independent of ξ, is that there exists an $m \times m$ matrix D such that

$$\Gamma = BD. \tag{26}$$

In fact, $\Gamma_1 = 0$ if and only if $\Gamma = BD$. Suppose $\Gamma = BD$, then

$$\begin{aligned} T\Gamma &= TBD \\ &= \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} D \\ &= \begin{bmatrix} 0_{(n-m)\times m} \\ B_2D \end{bmatrix}. \end{aligned} \tag{27}$$

Therefore, $\Gamma_1 = 0$ and $\Gamma_2 = B_2D$. Conversely, assume $\Gamma_1 = 0$. Since B_2 is full rank

$$\begin{aligned} \Gamma_2 &= B_2(B_2^{-1}\Gamma_2) \\ &= B_2D \end{aligned}$$

where $D = B_2^{-1}\Gamma_2$. Then

$$\begin{aligned} T\Gamma &= \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \\ &= \begin{bmatrix} 0_{(n-m)\times m} \\ B_2D \end{bmatrix} \\ &= TBD. \end{aligned}$$

Since T is an invertible matrix, $\Gamma = BD$.

However, the system (24) may be independent of ξ but $\Gamma_1 \neq 0$. Therefore for $m \neq 1$, the condition (26) is not a necessary condition for the independence of the reduced order system (24). In the general case, the necessary and sufficient condition for independence of the system (24) from the perturbation signal ξ is that $\xi \in N(\Gamma_1)$, where $N(\Gamma_1)$ is the null space of Γ_1. Note that $\Gamma = BD$ is equivalent to rank(B, Γ) =rank(B) [5]. If $m = 1$, Γ_1 is a real number and $\Gamma_1\xi = 0$ if and only if $\Gamma_1 = 0$. In this case, the system in the sliding mode is independent of ξ if and only if there exists a real number ρ such that

$$\Gamma = \rho B$$

which is equivalent to rank$[B, \Gamma] = 1$ [7].

3. SLIDING MODE OBSERVER DESIGN

Sliding observers potentially offer advantages similar to those of sliding controllers, in particular, inherent robustness to parametric uncertainty and straightforward application to important classes of systems.

Here, suitable state estimation of the system (1) – (2) is considered so that the estimate of the state is close to the actual state as time increases. This yields a reconstruction error system which is asymptotically stable or ultimately bounded. A method for sliding observer design and sufficient conditions for the existence of the sliding mode and the sliding region, are presented below.

A robust observer for the system (1) – (2) with an estimate of the disturbance input $\xi(t)$ is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + H(y(t) - \hat{y}(t)) + \Gamma\hat{\xi}(t) \tag{28}$$

where $\hat{x}$ is the state estimate, $\hat{y} = C\hat{x}$, $\hat{\xi}$ is an estimate of the disturbance ξ, and $H \in I\!R^{n\times m}$ is the observer gain matrix. Since the behaviour of the observer (28) is based upon an estimate of the disturbance, a suitable choice of $\hat{\xi}$ is required. An estimate $\hat{\xi}$ of ξ is $\hat{\xi} = L\,\mathrm{sgn}\,(Cx)$ with a suitable gain matrix L. Clearly if ξ is known, we set $\hat{\xi} = \xi$. In the absence of uncertainty the observer will be asymptotically stable if we select H such that $A - HC$ is a stable matrix.

The general form of the sliding observer (28) for the system (1) – (2) may be selected as

$$\dot{\hat{x}} = A\hat{x} + Bu + H(y - \hat{y}) + \Lambda v \tag{29}$$
$$\hat{y} = C\hat{x} \tag{30}$$

where $v \in I\!R^m$ is an external discontinuous feedforward compensation signal and $\Lambda \in I\!R^{n\times m}$ is the feedforward injection map such that $C\Lambda$ is a nonsingular matrix. The state reconstruction error is defined as $e = x - \hat{x}$. Subtracting (29) from (1) gives the dynamical reconstruction error system

$$\dot{e} = (A - HC)\,e + \Gamma\xi - \Lambda v \tag{31}$$
$$e_y = Ce \tag{32}$$

where $e_y = y - \hat{y}$ is the output reconstruction error. The initial state $x_0 = x(t_0)$ is unknown and $\hat{x}_0 = \hat{x}(t_0)$ can be arbitrarily assigned. A suitable value for $\hat{x}_0$ is a point on the sliding surface $y = 0$, i. e. $C\hat{x}_0 = 0$.

If the control u is not directly accessible, the conventional estimate is in the form (14), i. e.

$$u = -(CB)^{-1}\,(CA\hat{x} + K\mathrm{sgn}\,(C\hat{x})) \tag{33}$$

where the matrix K is a diagonal matrix with positive entries such that $M\sigma_M(C\Gamma) < \sigma_m(K)$. The ideal sliding mode for the system (31)-(32) satisfies $e_y = 0$, $\dot{e}_y = 0$ [2]. The virtual equivalent feedforward input is given by

$$v_{eq} = (C\Lambda)^{-1}(CAe + C\Gamma\xi). \tag{34}$$

Substituting (34) in the state reconstruction error system (31) gives the reduced order system

$$\dot{e} = (I - \Lambda(C\Lambda)^{-1}C)\,Ae + (I - \Lambda(C\Lambda)^{-1}C)\,\Gamma\xi \tag{35}$$

with m of the eigenvalues (35) zero and the $n-m$ remaining eigenvalues to be assigned [2]. The reduced order system is independent of the disturbance input signal if there exists an $m \times m$ matrix D such that

$$\Gamma = \Lambda D. \tag{36}$$

The error system in the sliding mode is now studied. Assume T is an orthogonal matrix (21) and $CT^T = [C_1 \;\; C_2]$. Consider a second transformation

$$T_s = \begin{bmatrix} I_{n-m} & 0 \\ C_1 & C_2 \end{bmatrix}. \tag{37}$$

Then

$$T_s T x = \begin{bmatrix} y_1 \\ y \end{bmatrix} \tag{38}$$

and the system (1) – (2) is converted to

$$\begin{aligned} \dot{y}_1(t) &= \hat{A}_{11}y_1(t) + \hat{A}_{12}y(t) + \hat{\Gamma}_1\xi & (39)\\ \dot{y}(t) &= \hat{A}_{21}y_1(t) + \hat{A}_{22}y(t) + CBu(t) + \hat{\Gamma}_2\xi & (40)\end{aligned}$$

where

$$\begin{aligned} & T_s T A T^T T_s^{-1} \\ = & \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \\ = & \begin{bmatrix} A_{11} - A_{12}C_2^{-1}C_1 & A_{12}C_2^{-1} \\ C_1A_{11} + C_2A_{21} - (C_1A_{12} + C_2A_{22})C_2^{-1}C_1 & (C_1A_{12} + C_2A_{22})C_2^{-1} \end{bmatrix}\end{aligned}$$

and

$$T_s T\Gamma = \begin{bmatrix} \hat{\Gamma}_1 \\ \hat{\Gamma}_2 \end{bmatrix} = \begin{bmatrix} \Gamma_1 \\ C_1\Gamma_1 + C_2\Gamma_2 \end{bmatrix}.$$

Using the transformation T_sT the observer (29) is given by

$$\begin{aligned} \dot{\hat{y}}_1(t) &= \hat{A}_{11}\hat{y}_1(t) + \hat{A}_{12}\hat{y}(t) + H_1e_y + \Lambda_1 v & (41)\\ \dot{\hat{y}}(t) &= \hat{A}_{21}\hat{y}_1(t) + \hat{A}_{22}\hat{y}(t) + CBu(t) + H_2e_y + \Lambda_2 v & (42)\end{aligned}$$

where

$$T_s T\hat{x} = \begin{bmatrix} \hat{y}_1 \\ \hat{y} \end{bmatrix}, \quad T_s T H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}. \tag{43}$$

Subtracting (41) – (42) from (39) – (40), the error system is given by

$$\begin{aligned} \dot{e}_1(t) &= \hat{A}_{11}e_1(t) + \hat{A}_{12}e_y(t) + \hat{\Gamma}_1\xi - \Lambda_1 v - H_1e_y & (44)\\ \dot{e}_y(t) &= \hat{A}_{21}e_1(t) + \hat{A}_{22}e_y(t) + \hat{\Gamma}_2\xi - \Lambda_2 v - H_2e_y & (45)\end{aligned}$$

where

$$e_1 = y_1 - \hat{y}_1.$$

The sliding mode occurs if $e_y = 0$ and $\dot{e}_y = 0$. Assume that Λ_2 is a nonsingular matrix. The equivalent feedforward input (34) is obtained from the subsystem (45)

$$v_{eq} = \Lambda_2^{-1}\left(\hat{A}_{21}e_1 + \hat{\Gamma}_2\xi\right). \tag{46}$$

The subsystem (44) yields the error system in the sliding mode

$$\dot{e}_1(t) \quad = \quad \hat{A}_{11}e_1(t) + \hat{\Gamma}_1\xi - \Lambda_1 v_{eq}. \tag{47}$$

Substituting (46) in (47), the reduced order error system is

$$\dot{e}_1(t) \quad = \quad (\hat{A}_{11} - \Lambda_1\Lambda_2^{-1}\hat{A}_{21})e_1(t) + (\hat{\Gamma}_1 - \Lambda_1\Lambda_2^{-1}\hat{\Gamma}_2)\xi. \tag{48}$$

Since (A, C) is observable, the pair $(\hat{A}_{11}, \hat{A}_{21})$ is also observable and $\hat{A}_{11} - \Lambda_1\Lambda_2^{-1}\hat{A}_{21}$ can be assigned arbitrary eigenvalues with negative real parts by a suitable choice of Λ. The bounded inputs $\Gamma\xi$ guarantee bounded error e, but the asymptotic stability of (48) is not guaranteed in general. However, some sufficient conditions ensure the stability of the reduced order error system. A sufficient condition for the reduced order system to be free of the influence of the disturbance ξ is that

$$\hat{\Gamma}_1 - \Lambda_1\Lambda_2^{-1}\hat{\Gamma}_2 = 0. \tag{49}$$

If condition (36) holds, then (49) is also satisfied.

Remark 3.1. It is possible to find the reduced order system (48) directly. Consider the transformation

$$T_\Lambda = \begin{bmatrix} I_{n-m} & -\Lambda_1\Lambda_2^{-1} \\ 0 & I_m \end{bmatrix}. \tag{50}$$

Then the error system (44)–(45) is converted to

$$\begin{aligned} \dot{\tilde{e}}_1(t) &= \tilde{A}_{11}e_1(t) + \tilde{A}_{12}e_y - (H_1 - \Lambda_1\Lambda_2^{-1}H_2)\,e_y(t) + (\hat{\Gamma}_1 - \Lambda_1\Lambda_2^{-1}\hat{\Gamma}_2)\,\xi & (51)\\ \dot{e}_y(t) &= \hat{A}_{21}\tilde{e}_1(t) + \tilde{A}_{22}e_y(t) + \hat{\Gamma}_2\xi - H_2e_y - \Lambda_2 v & (52) \end{aligned}$$

where $\tilde{A}_{11} = \hat{A}_{11} - \Lambda_1\Lambda_2^{-1}\hat{A}_{21}$, $\tilde{A}_{12} = \hat{A}_{12} - \Lambda_1\Lambda_2^{-1}\hat{A}_{22} + \tilde{A}_{11}\Lambda_1\Lambda_2^{-1}$ and $\tilde{A}_{22} = \hat{A}_{22} + \hat{A}_{21}\Lambda_1\Lambda_2^{-1}$, $\tilde{e}_1 = e_1 - \Lambda_1\Lambda_2^{-1}e_y$. In the sliding mode

$$\dot{\tilde{e}}_1 = (\hat{A}_{11} - \Lambda_1\Lambda_2^{-1}\hat{A}_{21})\tilde{e}_1 + (\hat{\Gamma}_1 - \Lambda_1\Lambda_2^{-1}\hat{\Gamma}_2)\,\xi. \tag{53}$$

Since in the sliding mode $\tilde{e}_1 = e_1$, the reduced order system (53) coincides with (48).

One needs to obtain H, Λ and v so that the stability of the observer system is preserved. The observer gain H can be found in two ways; the pole assignment method, i. e. assigning n prespecified eigenvalues to the matrix $A - HC$; and the LQ method.

In the LQ method, the algebraic Riccati equation (ARE)

$$AP + PA^T - PC^T R^{-1} CP = -Q \tag{54}$$

with Q, R arbitrary semi-p.d.s. and p.d.s. matrices respectively, has a u.p.d.s. matrix solution P. Then $A^T - C^T H^T$ is stable with

$$H = PC^T R^{-1} \tag{55}$$

which is equivalent to the stability of $A - HC$. Similarly to [7], the matrix Λ can be found in several ways.

Let P_f be the u.p.d.s. solution of the Lyapunov equation

$$(A - HC)\, P_f + P_f (A - HC)^T = -Q_f \tag{56}$$

where Q_f is an arbitrary p.d.s. matrix. Consider the discontinuous feedforward input

$$v = W \frac{Ce}{\|Ce\|} \tag{57}$$

where W is an $m \times m$ diagonal p.d. matrix with

$$\lambda_{\min}(W) \geq M \|D\| \frac{\lambda_{\max}(CP_f^{-1}C^T)}{\lambda_{\min}(CP_f^{-1}C^T)}. \tag{58}$$

Assume the condition (36) is satisfied. We now find conditions such that

$$\lim_{t \to \infty} e(t) = 0.$$

Let

$$\Lambda = P_f^{-1} C^T W^{-1}. \tag{59}$$

Since $C\Lambda$ is a nonsingular matrix and W is a p.d. matrix, $C\Lambda W$ is nonsingular and

$$\lambda_{\min}(C\Lambda W) = \lambda_{\min}(CP_f^{-1}C^T) \neq 0.$$

The quadratic stability of the reconstruction error system is guaranteed by (58) and (59). A Lyapunov function candidate is

$$V(e) = e^T P_f e. \tag{60}$$

If $Ce \neq 0$, then

$$\begin{aligned} \dot{V} &= e^T((A - HC)\, P_f + P_f (A - HC)^T)\, e + 2e^T C^T W^{-1} D\xi - 2e^T C^T \frac{Ce}{\|Ce\|} \\ &\leq -e^T Q_f e + 2\, \|e^T C^T\| \, (\|W^{-1} D\| M - 1) \\ &\leq -e^T Q_f e + 2\, \|e^T C^T\| \left(\frac{1}{\lambda_{\min}(W)} \|D\| M - 1 \right) \\ &< 0 \end{aligned} \tag{61}$$

since

$$\lambda_{\min}(W) \geq M\|D\| \frac{\lambda_{\max}(CP_f^{-1}C^T)}{\lambda_{\min}(CP_f^{-1}C^T)} \geq M\|D\|$$

If $Ce = 0$, $v = v_{eq}$ and

$$\begin{aligned} \dot{V} &= -e^T Q_f e + 2e^T P_f P_f^{-1} C^T W^{-1} D\xi - 2e^T P_f P_f^{-1} C^T W^{-1} v_{eq} \\ &= -e^T Q_f e + 2e^T C^T W^{-1} D\xi - 2e^T C^T W^{-1} v_{eq} \\ &= -e^T Q_f e \\ &< 0. \end{aligned} \tag{62}$$

Therefore

$$\lim_{t\to\infty} e(t) = 0.$$

A condition for the existence of the sliding mode is now established. Since the system is stable, the convergent sliding mode exists, i. e. $\lim_{t\to\infty} e_y(t) = 0$. We now find a region, the so-called sliding region, such that after a finite time, the state error trajectory lies in the vicinity of the sliding surface and inside the region. Consider

$$\begin{aligned} e_y^T \cdot e_y &\leq \|Ce\|.\|C(A-HC)e\| + e^T C^T C\left(P_f^{-1} C^T W^{-1}\right) D\xi \\ &\quad -e^T C^T C P_f^{-1} C^T W^{-1} W \frac{Ce}{\|Ce\|} \\ &\leq \|Ce\|.\|C(A-HC)e\| + \|e^T C^T\| \lambda_{\max}(CP_f^{-1}C^T) \|W^{-1}D\| M \\ &\quad -e^T C^T \lambda_{\min}\left(CP_f^{-1}C^T\right) \frac{Ce}{\|Ce\|} \\ &\leq \|Ce\| \Big[\|C(A-HC)\|.\|e\| + \lambda_{\max}(CP_f^{-1}C^T)\|W^{-1}D\|M \\ &\quad - \lambda_{\min}(CP_f^{-1}C^T)\Big]. \end{aligned} \tag{63}$$

A sufficient condition for the sliding mode is that the right-hand side of (63) be nonpositive. Thus

$$\|e\| \leq \frac{\lambda_{\min}(CP_f^{-1}C^T) - \lambda_{\max}\left(CP_f^{-1}C^T\right)\|W^{-1}D\|M}{\sigma_M(C(A-HC))} = r_s \tag{64}$$

or

$$\|e\| \leq \frac{\lambda_{\min}(CP_f^{-1}C^T) - \lambda_{\max}(CP_f^{-1}C^T)\|W^{-1}D\|M}{\sigma_M(C)\,\sigma_M(A-HC)} = \hat{r}_s. \tag{65}$$

Remark 3.2. One may choose $\Lambda = P_f^{-1}C^T$, then all the conditions for the stability of the system and sliding mode remain intact, and only the velocity of the state approaching the origin and the state trajectory dynamics on the sliding surface may differ from that of (59).

Remark 3.3. Since $V(e) = e^T P_f e$ and $\dot{V}(e) \leq -e^T Q_f e$, then

$$\frac{\dot{V}}{V} \leq -\frac{e^T Q_f e}{e^T P_f e} \tag{66}$$

The real function $g(e(t)) = e^T Q_f e / \left(e^T P_f e\right)$ takes its minimum at the point $e_{\min}$ satisfying $\dot{g}(e_{\min}) = 0$, which yields $P_f^{-1} Q_f e_{\min} - g(e_{\min})\, e_{\min} = 0$. So $\mu = g(e_{\min})$ is the minimum eigenvalue of the matrix $P_f^{-1} Q_f$. Therefore

$$V(e(t)) \leq V(e(t_0))\; e^{-\mu(t-t_0)}.$$

Then

$$\begin{aligned} \|e\|^2 &\leq \frac{e^T P_f e}{\lambda_{\min}(P_f)} \\ &\leq \frac{V(e(t_0))}{\lambda_{\min}(P_f)}\; e^{-\mu(t-t_0)} \end{aligned} \tag{67}$$

Hence $e(t)$ is bounded and approaches zero at least as fast as $e^{-\mu t/2}$.

3.1. Sliding error system with unmatched uncertainty

When the matching condition does not hold, the disturbance may affect the sliding system. However, if the norm of the disturbance input signal converges to zero, or the norm of the disturbance signal is bounded by the norm of the output error, the error system is asymptotically stable [7]. Otherwise, only ultimate boundedness may be deduced, i. e. the state error trajectory enters a region centred on the origin and thereafter remains within this region.

The behaviour of the system depends upon the norm of a matrix which we named the 'unmatched uncertainty matrix' [7]. The norm of this matrix, ϵ, is called the 'unmatched uncertainty distance'.

Let $D \in \mathbb{R}^{m\times m}$ be a matrix such that

$$\left[\begin{array}{cc} 0_{m\times(n-m)} & I_m \end{array}\right] \Gamma = \left[\begin{array}{cc} 0_{m\times(n-m)} & I_m \end{array}\right] \Lambda D. \tag{68}$$

Set $E = \Gamma - \Lambda D$ and $\|E\| = \epsilon$.

Definition 3.1. For the system (1), the uncertainty $\Gamma\xi$ is said to satisfy a matching condition with ϵ-approximation (or ϵ-matching condition) if there exists a matrix D such that

(i) condition (68) holds;

(ii) $\|E\| = \|\Gamma - \Lambda D\| = \epsilon$.

If the 'unmatched uncertainty distance' is zero, the matching condition is completely satisfied. ΛD is the matched uncertainty matrix and E is the unmatched uncertainty matrix of the matrix Γ. So the unmatched distance is $||E|| = \epsilon$. In the SISO case, the unmatched disturbance distance is the length of the disturbance vector. So, if ϵ is very small, the stability of the system is practically achieved. Therefore, when $\epsilon > 0$, the error system (31) may no longer be generally asymptotically stable.

Definition 3.2. The solution $e(t; e_0, t_0)$ of (31) is

(i) *ultimately bounded*, with respect to a compact set $S \subset I\!R^n$, if there exists a nonnegative time $T(t_0, e_0, S)$ such that for all $t \geq t_0 + T(t_0, e_0, S)$, $e(t) \in S$.

(ii) *uniformly ultimately bounded* with respect to a compact set $S \subset I\!R^n$, if $T(e_0, S)$, possibly dependent on e_0 but not on t_0, defined as (i), is independent of t_0, i.e. there exists a nonnegative time $T(e_0, S)$ such that for all $t \geq t_0 + T(e_0, S)$, $e(t) \in S$.

Consider the Lyapunov function (60). Then similarly to (61) one obtains

$$\begin{aligned} \dot{V} &= e^T \left((A - HC) P_f + P_f (A - HC)^T\right) e + 2e^T P_f (\Lambda D + E) \xi - 2e^T P_f \Lambda v \\ &\leq -||e||^2 \lambda_{\min}(Q_f) + 2||e|| \cdot ||P_f E|| M \\ &= -||e|| \left\{(1-\theta)\lambda_{\min}(Q_f)||e|| + \theta\lambda_{\min}(Q_f)||e|| - 2||P_f E|| M\right\} \end{aligned} \tag{69}$$

where $0 < \theta \leq 1$. So, if

$$e \notin \Omega_E = \left\{ e \in I\!R^n \mid \ ||e|| \leq \frac{2||P_f E|| M}{\theta \lambda_{\min}(Q_f)} = r_\Omega \right\} \tag{70}$$

$\dot{V} < 0$. Since V is a monotonically decreasing function on the outside of the set, the maximum value of V on the compact set Ω_E is attained for $||e|| = \dfrac{2||P_f E|| M}{\theta \lambda_{\min}(Q_f)}$, and the state error trajectory enters the closed ellipsoid

$$\mathcal{E}_E = \left\{ e \in I\!R^n \mid V(e(t)) \leq r^2 \right\} \tag{71}$$

where

$$r = \frac{2M||P_f E|| \sqrt{\lambda_{\max}(P)}}{\theta \lambda_{\min}(Q_f)}.$$

The solution of the error system is uniformly ultimately bounded with ultimate boundedness ratio

$$b_u = \frac{2||P_f E|| M}{\theta \lambda_{\min}(Q_f)} \sqrt{\frac{\lambda_{\max}(P_f)}{\lambda_{\min}(P_f)}} \tag{72}$$

and ultimate boundedness radius

$$r_u = \inf \left\{ r \in I\!R \mid \mathcal{E}_E \subset B_r \right\}$$

where B_r is a ball with radius r centred on the origin. Note that all trajectories starting inside $\mathcal{E}_E$, remain within this set for all future time, and all trajectories starting outside $\mathcal{E}_E$ enter this compact set within a finite time and remain inside thereafter.

Remark 3.4. Since $||P_f E|| \leq ||P_f||.||E|| = \epsilon||P_f||$, one can conclude that

$$\dot{V} \ \leq \ -||e||\left\{(1-\theta)\lambda_{\min}(Q_f)||e|| + \theta\lambda_{\min}(Q_f)||e|| - 2\epsilon||P_f||M\right\}. \tag{73}$$

So V is a monotonically decreasing function on the outside of the set

$$\Omega_\epsilon = \left\{ e \in I\!R^n \mid \ ||e|| \leq \frac{2\epsilon M \lambda_{\max}(P_f)}{\theta\lambda_{\min}(Q_f)} \right\}. \tag{74}$$

The ratio $\lambda_{\max}(P_f)/\lambda_{\min}(Q_f)$ is minimized by the choice $Q = I$. Consider

$$\mathcal{E}_\epsilon = \left\{ e \in I\!R^n \mid V(e(t)) \leq {r_1}^2 \right\} \tag{75}$$

where

$$r_1 = \frac{2\epsilon M \lambda_{\max}^{3/2}(P_f)}{\theta\lambda_{\min}(Q_f)}.$$

The state error trajectory enters the ellipsoid $\mathcal{E}_\epsilon$ in finite time and remains inside thereafter.

We now show that the state trajectory enters the set $\mathcal{E}_E$ in finite time and remains inside thereafter. Consider

$$\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}\left(e(t)^T P_f\, e(t)\, e^{\mu t}\right) \\
= \ & \frac{\mathrm{d}}{\mathrm{d}t}\left(e^T(t)\, P_f e(t)\right)\, e^{\mu t} + \mu e^T(t)\, P_f e(t)\, e^{\mu t} \\
\leq \ & \left(-||e(t)||^2\lambda_{\min}(Q_f) + 2||e(t)||.||P_f E||M\right) e^{\mu t} + \lambda_{\min}(Q_f)||e(t)||^2 e^{\mu t} \\
= \ & 2||e(t)||.||P_f E|| M e^{\mu t}
\end{aligned} \tag{76}$$

where $\mu = \lambda_{\min}(Q_f)/\lambda_{\max}(P_f)$. Integrating both sides over the interval $[0, t]$ and then multiplying both sides by $e^{-\mu t}$ yields

$$e^T P_f e \leq e^T(0)\, P_f e(0)\, e^{-\mu t} + 2b_u M||P_f E||\lambda_{\max}(P_f)\left(1 - e^{-\mu t}\right)/\lambda_{\min}(Q_f) \tag{77}$$

since $||e|| \leq b_u$. Since after a finite time $||e|| \leq 2||P_f E||M/\theta\lambda_{\min}(Q_f)$, equation (77) shows that the state error $e(t)$ converges to the compact set $\mathcal{E}_E$ (71), i. e.

$$\lim_{t\to\infty} e^T P_f e \leq 4M^2||P_f E||^2 \frac{\lambda_{\max}(P_f)}{\theta\lambda^2_{\min}(Q_f)}\sqrt{\frac{\lambda_{\max}(P_f)}{\lambda_{\min}(P_f)}}$$

and then

$$\lim_{t\to\infty} d(e(t), \mathcal{E}_E) = 0 \tag{78}$$

where d denotes the Euclidean metric on $I\!R^n$ and

$$d(e(t), \mathcal{E}_E) = \inf_{\alpha\in\mathcal{E}_E} d\left(e(t), \alpha\right)).$$

The result (78) shows that the state error trajectory enters $\mathcal{E}_E$ at finite time t_e and after this time remains inside $\mathcal{E}_E$. Hence, boundedness is generally guaranteed in the presence of a bounded disturbance with possibly unknown bound. But the size of $\mathcal{E}_E$ cannot be estimated *a priori* if no bound on the disturbance input is given [7].

Equation (77) also yields

$$||e||^2 \leq \left(||e(0)||^2 e^{-\mu t} + \frac{2b_u M ||P_f E||}{\lambda_{\min}(Q_f)} \left(1 - e^{-\mu t}\right) \right) \frac{\lambda_{\max}(P_f)}{\lambda_{\min}(P_f)}.$$

Then

$$\begin{aligned} ||e|| &\leq \left(||e(0)|| e^{-\frac{1}{2}\mu t} + \sqrt{\frac{2b_u M ||P_f E||}{\lambda_{\min}(Q_f)} \left(1 - e^{-\mu t}\right)} \right) \sqrt{\frac{\lambda_{\max}(P_f)}{\lambda_{\min}(P_f)}} \\ &= ||e(0)|| e^{-\frac{1}{2}\mu t} \sqrt{\frac{\lambda_{\max}(P_f)}{\lambda_{\min}(P_f)}} + \frac{2M||P_f E||}{\sqrt{\theta}\lambda_{\min}(Q_f)} \sqrt{1 - e^{-\mu t}} \left(\frac{\lambda_{\max}(P_f)}{\lambda_{\min}(P_f)} \right)^{3/4} \end{aligned}$$

and for $t \to \infty$

$$||e|| \leq \frac{2M||P_f E||}{\sqrt{\theta}\lambda_{\min}(Q_f)} \left(\frac{\lambda_{\max}(P_f)}{\lambda_{\min}(P_f)} \right)^{3/4} = \hat{b}_u = \sqrt{\theta} \left(\frac{\lambda_{\max}(P_f)}{\lambda_{\min}(P_f)} \right)^{1/4} b_u.$$

So

$$\hat{b}_u \begin{cases} > b_u & \text{if } 1 \geq \theta > \eta \\ = b_u & \text{if } \theta = \eta \\ < b_u & \text{if } \theta < \eta \end{cases} \tag{79}$$

where $\eta = \sqrt{\frac{\lambda_{\min}(P_f)}{\lambda_{\max}(P_f)}}$. On the other hand, from (70),

$$\hat{b}_u \begin{cases} > r_\Omega & \text{if } 1 \geq \theta > \eta^3 \\ = r_\Omega & \text{if } \theta = \eta^3 \\ < r_\Omega & \text{if } \theta < \eta^3. \end{cases} \tag{80}$$

However, $b_u \geq r_\Omega$ and the equality holds if $P = pI$ $(p > 0)$. Moreover,

$$\hat{b}_u \begin{cases} > \max\{b_u, r_\Omega\} & \text{if } 1 \geq \theta > \eta \\ < \min\{b_u, r_\Omega\} & \text{if } \theta < \eta^3. \end{cases} \tag{81}$$

Although Ω_E is not the smallest ultimately bounded set, the concept of uniform ultimate boundedness is tantamount to 'practical' asymptotic stability, when Ω_E is a small neighbourhood about the origin.

In the case of the matching condition not being satisfied ($\epsilon > 0$), the gain matrix W (57) should be chosen so that

$$\lambda_{\min}(W) \geq \frac{M||D||\lambda_{\max}(CP_f^{-1}C^T) + M||CE||}{\lambda_{\min}(CP_f{}^{-1}C^T)}. \tag{82}$$

In this case the sliding region is

$$\mathcal{S}_r = \{e \in I\!R^n \mid \|e\| \leq r_2\} \tag{83}$$

where

$$r_2 \leq r_s - \frac{M\|CE\|}{\sigma_M(C(A-HC))} \tag{84}$$

with r_s as defined in (64).

By comparing (64) with (84), one can conclude that in the presence of a disturbance, the radius of the sliding ball obtained from (83), may be smaller than that with the matched uncertainty (64). However, for small values of the unmatched disturbance and the disturbance bound, r_2 approximately equals r_s. In this case it may be assumed that the distance of unmatched disturbance is nearly zero.

If the disturbance norm $\|\xi(t)\|$ converges to zero, $\lim_{t\to\infty} \xi(t) = 0$. The proof is similar to that in [7].

4. EXAMPLE

The example below illustrates our results regarding the sliding mode, stability of the error system and observer design. Consider the system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 2.25 & -0.4 \\ 0 & -1.50 & -2.0 \\ 0 & 0.50 & -1.0 \end{bmatrix} x + \begin{bmatrix} 0.30 & 0 \\ 0 & 1 \\ 0.12 & 0 \end{bmatrix} u + \begin{bmatrix} -0.0007 & 0.0015 \\ 0.0103 & 0.0041 \\ 0.0188 & 0.0188 \end{bmatrix} \xi \\ y &= \begin{bmatrix} 0.2 & 0.4 & 0.89 \\ 0.2 & 0 & 1.00 \end{bmatrix} x. \end{aligned}$$

Suppose ξ is a bounded random signal satisfying $\|\xi\| < 0.1$. Using the LQ method with $R = I_2$ and $Q = I_3$, the u.p.d.s. solution of ARE (54) is

$$P = \begin{bmatrix} 1.8831 & 0.6396 & -0.1829 \\ 0.6396 & 0.5534 & -0.1750 \\ -0.1829 & -0.1750 & 0.3449 \end{bmatrix}.$$

From (55)

$$H = \begin{bmatrix} 0.4697 & 0.1938 \\ 0.1936 & -0.0470 \\ 0.2004 & 0.3083 \end{bmatrix}.$$

The eigenvalues of $A - HC$ are $-1.7043 \pm 1.0365i$, -0.7881. Let $Q_f = 5I_3$. The u.p.d.s. solution matrix of the Lyapunov equation (56) is

$$P_f = \begin{bmatrix} 8.6608 & 3.0202 & -1.0694 \\ 3.0202 & 2.6000 & -0.7952 \\ -1.0694 & -0.7952 & 1.5302 \end{bmatrix}.$$

Let $W = \text{diag}\ (2.5, 2.5)$ and

$$v = W\frac{Ce}{\|Ce\|}. \tag{85}$$

From (59)

$$\Lambda = \begin{bmatrix} -0.0117 & 0.0244 \\ 0.1709 & 0.0676 \\ 0.3133 & 0.3136 \end{bmatrix}$$

The reduced order error system is independent of ξ since $\Gamma = 0.06\Lambda$. In fact $D = 0.06I_2$. So the error system is quadratically stable which means the estimated state error converges quadratically to the actual state.

The minimum eigenvalue of $C\Lambda W$ is 0.0510. The value of the right-hand side of (64) is $r_s = 0.0156$ and the value of the right-hand side of (65) is $\hat{r}_s = 0.0119$. After respective short times τ_1 and τ_2 conditions (64) and (65) are both true. Noting that $\tau_1 \leq \tau_2$, when (64) is valid for $t \in [\tau_1, \tau_2]$ (65) may not be valid, i.e. the condition (64) is weaker than (65). Simulation results are shown in Figure 1.

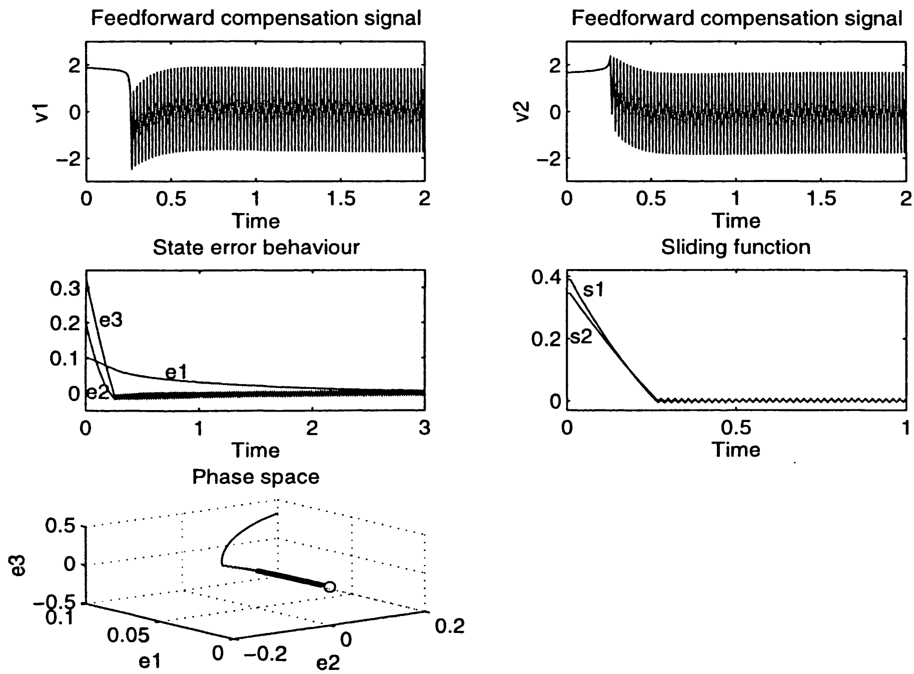

Fig. 1. Responses for the LQ method.

Alternatively the eigenvalues of $A - HC$ can be specified using pole assignment technique. Suppose the desired eigenvalues of $A - HC$ are -1, $-1 \pm 0.15i$. The existence of H is guaranteed by the observability of (A, C). Using the MATLAB Control Toolbox

$$H = \begin{bmatrix} 5.7656 & -5.5452 \\ -1.3486 & -0.7121 \\ 1.2296 & -1.0990 \end{bmatrix}.$$

For v (85) and $Q_f = 3I_3$, the u.p.d.s. matrix solution of the Lyapunov equation (56) is

$$P_f = \begin{bmatrix} 1.4234 & 0.2455 & -0.0052 \\ 0.2455 & 1.6728 & -0.0650 \\ -0.0052 & -0.0650 & 1.5066 \end{bmatrix}.$$

Equation (59) gives

$$\Lambda = \begin{bmatrix} 0.0400 & 0.0568 \\ 0.0991 & 0.0020 \\ 0.2407 & 0.2658 \end{bmatrix}.$$

So

$$D = \begin{bmatrix} 0.1044 & 0.0407 \\ -0.0238 & 0.0339 \end{bmatrix}$$

and then

$$E = \Gamma - \Lambda D = \begin{bmatrix} -0.0035 & -0.0021 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

with norm $||E|| = \epsilon = 0.0041$. So the matching condition is not satisfied. However, the state error trajectory enters a small sphere about the origin with radius

$$r_\Omega = 2M||P_f E||/\lambda_{\min}(Q_f) = 0.00039$$

and $b_u = 0.00047$. So the stability of the error system is nearly achieved. Note that the value of the right-hand side of (64) is 0.0329 and the value of the right-hand side of (65) equals 0.0271. In the time interval $[0, 3]$ the maximum value of $||e||$ is 0.3945 and the minimum value is 0.02. Hence the conditions (64) and (65) hold after short times τ_1 and τ_2, respectively. It is clear that $\tau_1 \le \tau_2$. The time until the start of the sliding mode, t_s, should be smaller than the time τ_1 when (64) is satisfied, i.e. $t_s \le \tau_1$. The sliding region radius (83) is $r_2 = 0.0328$. Simulation results are shown in Figure 2.

5. CONCLUSIONS

In this paper the sliding dynamics for MIMO linear systems and the conditions for the existence of the sliding mode in the presence of uncertainty, have been studied. The existence of the sliding mode guarantees that the state trajectories converge to a sliding surface in a finite time and then move along the surface to the origin. However, the system may generally not be stable. To achieve the asymptotic stability of the system some further conditions may be needed. For the system with unmatched uncertainty some relaxed sliding conditions have been studied to guarantee asymptotic stability.

A sliding observer design method has been proposed such that in the presence of uncertainty the estimated state approximately approaches the actual state. The accuracy of this estimate depends upon the unmatched uncertainty distance and the

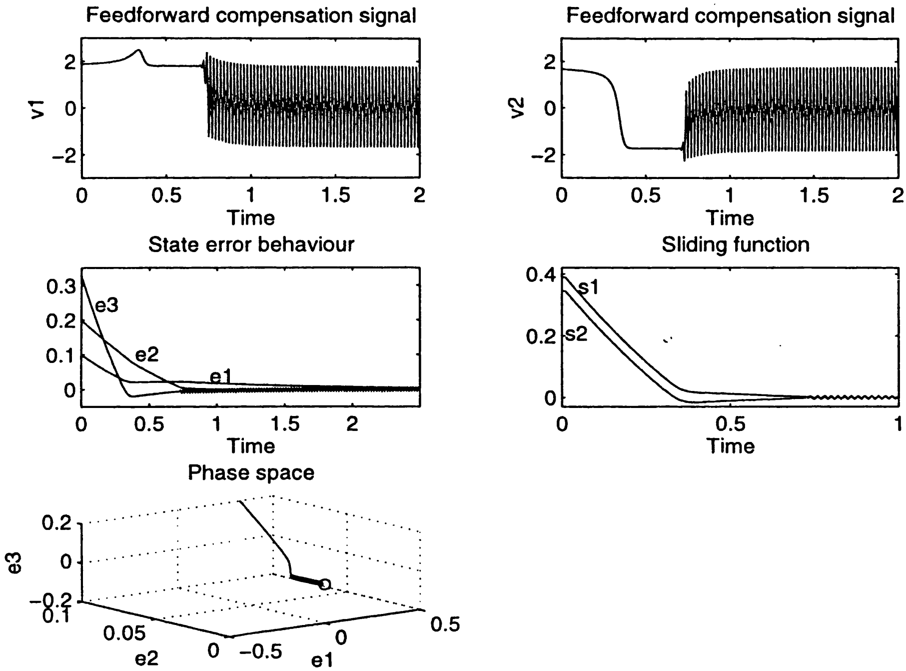

Fig. 2. Responses for the pole assignment technique.

disturbance bound (72). Greater accuracy is obtained if the unmatched uncertainty distance and the disturbance bound take small values. In the presence of unmatched uncertainty, the ultimate boundedness of the error state is generally guaranteed, i. e. the state error trajectory enters a certain ball centred on the origin in finite time and remains within the ball thereafter. The radius of this ball depends upon the accuracy of the unmatched uncertainty distance and the disturbance bound. There exists a finite time after which the state trajectories remain in the 'sliding region' $\mathcal{S}_r$ (83) and move to the origin along the sliding surface. The actual sliding mode domain may be larger than (83). The value r_2 indicates that the reaching time is at most that time when the state error trajectory enters the set $\mathcal{S}_r$.

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Dr. A. Jafari Koshkouei and Prof. Alan S. I. Zinober, Department of Applied Mathematics, The University of Sheffield, Sheffield S10 2TN, U. K.
e-mail: A.Zinober@shef.ac.uk