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ON SOME CONTRIBUTIONS TO QUANTUM STRUCTURES BY FUZZY SETS

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It is well known that the fuzzy sets theory can be successfully used in quantum models ([5, 26]). In this paper we give first a review of recent development in the probability theory on tribes and their generalizations – multivalued (MV)-algebras. Secondly we show some applications of the described method to develop probability theory on IF-events.

Keywords: probability, fuzzy sets, MV-algebra, IF events

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1. INTRODUCTION

Since the Kolmogorov basic paper [9] (see also [8]) the probability is a mapping

$$P : \mathcal{S} \to [0, 1]$$

defined on a σ-algebra $\mathcal{S}$ of subsets of a set Ω (generally a Boolean σ-algebra can be considered instead of $\mathcal{S}$). Random variable is a mapping

$$\xi : \Omega \to R$$

which is $\mathcal{S}$-measurable, i. e. $\xi^{-1}(A) \in \mathcal{S}$ for any Borel set $A \subset R$. If we denote by $\mathcal{B}(R)$ the family of all Borel subsets of R, then to any random variable ξ there corresponds a mapping

$$x : \mathcal{B}(R) \to \mathcal{S}$$

called observable and defined by the equality $x(A) = \xi^{-1}(A)$. The mapping x preserves all set-theoretical operations, therefore the notion of an observable can be introduced also in the case of general Boolean algebra $\mathcal{B}$. Then $x : \mathcal{B} \to \mathcal{S}$ is a σ-morphism between two Boolean σ-algebras $\mathcal{B}$, $\mathcal{S}$. Recall that the basic property of the Kolmogorov theory – σ-additivity – can be reformulated by the continuity and additivity

$$P(A \cup B) + P(A \cap B) = P(A) + P(B), \quad A, B \in \mathcal{S}$$

of course, added by the conditions $P(\Omega) = 1$, $P(\emptyset) = 0$.

Although the Kolmogorov theory is very successful and useful also in the present time, it was not acceptable in the non-commutative case important in the quantum theory. Therefore von Neumann [15] suggested an alternative theory based on the logic $L(H)$ of all closed subspaces of a Hilbert space H. In the theory a state (= probability) is a mapping

$$m : L(H) \to [0, 1]$$

(continuous and additive) and an observable is a morphism

$$x : \mathcal{B}(R) \to L(H).$$

This theory was generalized for general logics in [5].

Twenty years ago some experiences started using fuzzy sets in quantum models (see [5, 16, 26]). A fuzzy subset A of a space Ω is identified with so-called membership function μ_A, what is a mapping

$$\mu_A : \Omega \to [0, 1].$$

A special case of a fuzzy set is any set $A \subset \Omega$, which can be identified with the characteristic function $\chi_A : \Omega \to \{0, 1\}$. So instead of a σ-algebra $\mathcal{S}$ a tribe $\mathcal{T}$ of fuzzy sets can be considered (or an MV-algebra generally), and a state is a mapping

$$m : \mathcal{T} \to [0, 1],$$

an observable is a mapping

$$x : \mathcal{B}(R) \to \mathcal{T}$$

satisfying some axioms. In Section 2 we present a review of recent results concerning probability theory on MV-algebras. Recall that MV-algebras play the same role in multi-valued logics as Boolean algebras in two-valued logics, hence the mentioned results could have some important consequences.

Section 3 is devoted to *IF*-sets. An IF-set is a pair $A = (\mu_A, \nu_A)$ of fuzzy sets $\mu_A, \nu_A : \Omega \to [0, 1]$ such that $\mu_A + \nu_A \leq 1$. The mapping μ_A is called the membership function of A, the mapping ν_A the non-membership function of A. The theory has been summarized in [1]. It is interesting from the mathematical point of view and it has some remarkable applications. Our main problem is, how to use the results and methods of the probability theory on MV-algebras to families of IF-sets. We present here two ways for realizing this aim in Section 4 and Section 5.

2. MV–ALGEBRAS

An MV-algebra is a system $(M, \oplus, \odot, \neg, 0, u)$ (where $\oplus, \odot$, are binary operations, $\neg$ is a unary operation, $0, u$ are fixed elements) such that the following identities are satisfied: $\oplus$ is commutative and associative, $a \oplus 0 = a, a \oplus u = u, \neg(\neg a) = a, \neg 0 = u, a \oplus (\neg a) = u, \neg(\neg a \oplus b) \oplus b = \neg(a \oplus \neg b) \oplus a, a \odot b = \neg(\neg a \oplus \neg b)$. Every MV-algebra is a distributive lattice, where $a \vee b = a \oplus (\neg(a \oplus \neg b))$, 0 is the least element, and u is the greatest element of M.

Example 1. An instructive example is the unit interval $[0, 1]$ endowed with the operations $a \oplus b = (a + b) \wedge 1,\ a \odot b = (a + b - 1) \vee 0,\ \neg a = 1 - a,\ u = 1$. Recall that $a \oplus b$ corresponds to the disjunction, $a \odot b$ to the conjunction, and $\neg a$ to the negation in the classical two-valued logic.

Example 2. Another example is the family $\mathcal{T}$ of all measurable (with respect to a σ-algebra $\mathcal{S}$ of subsets of Ω) functions $f : \Omega \to [0, 1]$ again with the Łukasiewicz connectives $f \oplus g = (f + g) \wedge 1_\Omega,\ f \odot g = (f + g - 1) \vee 0_\Omega,\ \neg f = 1_\Omega - f,\ u = 1_\Omega$. Recall that if $f = \chi_A, g = \chi_B$, then $f \oplus g = \chi_{A\cup B}, f \odot g = \chi_{A\cap B}$.

Definition. (Riečan and Mundici [25]) Let $(M, \oplus, \odot, \neg, 0, u)$ be an MV-algebra. A state on the MV-algebra is a mapping $m : M \to [0, 1]$ satisfying the following conditions:

(i) $m(1) = 1, m(0) = 0$;

(ii) $m(a) + m(b) = m(a \oplus b) + m(a \odot b),\ \forall\, a, b \in M$;

(iii) $a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a)$.

An observable is a mapping $x : \mathcal{B}(R) \to M$ satisfying the following properties:

(i) $x(R) = u$;

(ii) $A \cap B = \emptyset \Longrightarrow x(A) \odot x(B) = 0, x(A \cup B) = x(A) \oplus x(B)$;

(iii) $A_n \nearrow A \Longrightarrow\ x(A_n) \nearrow x(A)$.

The main tool in MV-algebra probability theory is the idea of a local representation: to a given sequence $(x_n)_n$ of observables a classical probability space $(\Omega, \mathcal{S}, P)$ and a sequence $(\xi_n)_n$ of random variables on the space are constructed. To the sequence $(\xi_n)_n$ some classical results of the Kolmogorov theory can be applied and the corresponding results are translated to the sequence $(x_n)_n$. Since $(x_n)_n$ is arbitrary, one can obtain a general result for MV-algebras.

The main results of the theory were summarized in [25], and before it in [26]:

Strong and weak laws of large numbers
Central limit theorem
Martingale convergence theorem
Individual ergodic theorem
Isomorphism and entropy of dynamical systems

Of course, there are some strengthening of the previous results achieved in recent time. We mention two of them. The first is the individual ergodic theorem, the second is the entropy theory.

The individual ergodic theorem states that if ξ is integrable random variable on $(\Omega, \mathcal{S}, P)$ and $T : \Omega \to \Omega$ is a measure preserving transformation (i. e. $T^{-1}(A) \in \mathcal{S}$

and $P(T^{-1}(A)) = P(A)$ for any $A \in \mathcal{S}$), then for P-almost every $\omega \in \Omega$ there exists

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \xi(T^i(\omega)) = \xi^*(\omega),$$

ξ^* is integrable, $E(\xi^*) = E(\xi)$, and ξ^* is invariant, i.e. $\xi^*(T(\omega)) = \xi^*(\omega)$ for almost every $\omega \in \Omega$. The mentioned MV-algebra version of the individual ergodic theorem contained all assertions of the classical version besides the last. Recently in [24] the invariance of the limit observable was formulated and proved. Moreover in [12] a remarkable strengthening of [24] is presented.

The entropy theory was constructed only in a special type of MV-algebras, so-called MV-algebras with product (see [14, 17, 25]). Now in [21] and [3] the entropy was defined also for an arbitrary MV-algebra. Moreover, in [4] there are some effective rules for the computation of the entropy.

New results are also the proof of the conjugation of a large family of probability MV-algebras to the unit MV-algebra (see [20]), and the construction of the free product of MV-algebras (see [19]).

The successes of the probability theory on MV-algebras justify proposals for applying the results and the methods also in other areas, particularly in the theory of *IF*-events.

3. IF–EVENTS

Consider a measurable space $(\Omega, \mathcal{S}), \mathcal{S}$ be a σ-algebra, $\mathcal{T}$ be the family of all $\mathcal{S}$-measurable functions $f : \Omega \to [0,1]$,

$$\mathcal{F} = \Big\{(\mu_A, \nu_A); \mu_A, \nu_A : \Omega \to [0,1], \mu_A, \nu_A \text{ are } \mathcal{S}\text{-measurable, } \mu_A + \nu_A \le 1\Big\}.$$

The members of $\mathcal{F}$ are called *IF*-events. After some experiences the following definition of *IF*-probability was stated. Denote by $\mathcal{J}$ the family of all compact subintervals of [0,1], and for $A, B \in \mathcal{F}, A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ we put

$$\begin{aligned} A \oplus B &= (\mu_A \oplus \mu_B, \nu_A \odot \nu_B), \\ A \odot B &= (\mu_A \odot \mu_B, \nu_A \oplus \nu_B). \end{aligned}$$

(See Example 2 in Section 2.)

Definition. (Riečan [18]) An IF-probability on $\mathcal{F}$ is a mapping $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ satisfying the following conditions:

(i) $\mathcal{P}((0,1)) = [0,0], \quad \mathcal{P}((1,0)) = [1,1]$;

(ii) $\mathcal{P}((\mu_A,\nu_A)) + \mathcal{P}((\mu_B,\nu_B)) = \mathcal{P}((\mu_A,\nu_A) \oplus (\mu_B,\nu_B)) + \mathcal{P}((\mu_A,\nu_A) \odot (\mu_B,\nu_B))$ for any $(\mu_A,\nu_A), (\mu_B,\nu_B) \in \mathcal{F}$;

(iii) $(\mu_{A_n}, \nu_{A_n}) \nearrow (\mu_A, \nu_A) \Longrightarrow \mathcal{P}((\mu_{A_n}, \nu_{A_n})) \nearrow \mathcal{P}((\mu_A, \nu_A))$.

(Recall that $[a,b] + [c,d] = [a+c, b+d]$, of course $[a+c, b+d]$ need not be a member of $\mathcal{J}$. Further $[a_n, b_n] \nearrow [a,b]$ means that $a_n \nearrow a, b_n \nearrow b$. On the other hand $(\mu_{A_n}, \nu_{A_n}) \nearrow (\mu_A, \nu_A)$ means that $\mu_{A_n} \nearrow \mu_A$, and $\nu_{A_n} \searrow \nu_A$.)

Example 1. (Grzegorzewski and Mrowka [7]) The probability $\mathcal{P}(A)$ of an IF event A is defined as the interval

$$\mathcal{P}(A) = \left[\int_\Omega \mu_A \, \mathrm{d}p, 1 - \int_\Omega \nu_A \, \mathrm{d}p \right].$$

Example 2. (Gersternkorn and Manko [6]) The probability $\mathcal{P}(A)$ of an IF-event A is defined as the number

$$\mathcal{P}(A) = \frac{1}{2} \left(\int_\Omega \mu_A \, \mathrm{d}p + 1 - \int_\Omega \nu_A \, \mathrm{d}p \right).$$

Of course, we want to use some results from MV-algebras. Therefore we must express the function $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ with help of some functions from $\mathcal{F}$ to [0,1].

Definition. A function $p : \mathcal{F} \to [0, 1]$ will be called a state if the following conditions are satisfied:

(i) $p((0, 1)) = 0, p((1, 0)) = 1$;

(ii) $p((\mu_A, \nu_A) \oplus (\mu_B, \nu_B)) + p((\mu_A, \nu_A) \odot (\mu_B, \nu_B)) = p((\mu_A, \nu_A)) + p((\mu_B, \nu_B))$ for any $(\mu_A, \nu_A), (\mu_B, \nu_B) \in \mathcal{F}$;

(iii) $(\mu_{A_n}, \nu_{A_n}) \nearrow (\mu_A, \nu_A) \Longrightarrow p((\mu_{A_n}, \nu_{A_n})) \nearrow p((\mu_A, \nu_A))$.

Theorem 1. Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ be a mapping. Denote $\mathcal{P}(A) = [\mathcal{P}^\flat(A), \mathcal{P}^\sharp(A)]$ for any $A \in \mathcal{F}$. Then $\mathcal{P}$ is a probability if and only if $\mathcal{P}^\flat, \mathcal{P}^\sharp$ are states.

Of course, $\mathcal{F}$ is not an MV-algebra. Therefore we must look for some ways how to construct the probability theory on $\mathcal{F}$. The first one is to consider a structure (so-called L-lattice) more general than MV-algebra is, and to repeat some methods inspired by MV-algebras. The second one is an embedding of $\mathcal{F}$ to a convenient MV-algebra.

4. L–LATTICE

All basic facts in this section have been presented in the paper [11] and the thesis [10]. The used methods are surprising generalizations of the methods used in [26].

Definition. (Lendelová [10, 11]) An L-lattice (Łukasiewicz lattice) is a structure $L = (L, \leq, \oplus, \odot, 0_L, 1_L)$, where $(L, \leq)$ is a lattice, 0_L is the least and 1_L the greatest element of the lattice L, and $\oplus, \odot$ are binary operations on L.

It is quite surprising that there are given no conditions about the binary operations $\oplus, \odot$. Of course, here it is used the fact that the main importance in the probability theory has the probability distribution of a random variable (see Theorem 2). On the other hand many concrete structures can be considered as examples of an L-lattice.

Example 1. Any MV-algebra is an L-lattice.

Example 2. The set $\mathcal{F}$ of all *IF*-events defined on a measurable space $(\Omega, \mathcal{S})$ is an L-lattice.

Example 3. Let H be a Hilbert space, $L(H)$ the family of all closed subspaces of H ordered by the inclusion. The family $L(H)$ is a lattice, where $A \wedge B = A \cap B$, and $A \vee B$ is the closed subspace of H generated by $A \cup B$. Define further $A \oplus B = A \vee B$, if $A \perp B$, $A \oplus B = H$ otherwise, and $A \odot B = \{0\}$, if $A \perp B$, $A \odot B = H$ otherwise. Then $(L(H), \subset, \oplus, \odot, \{0\}, H)$ is an L-lattice.

Definition. (Lendelová [10, 11]) A probability on an L-lattice L is a mapping $p : L \to [0, 1]$ satisfying the following three conditions:

(i) $p(1_L) = 1, p(0_L) = 0$;

(ii) if $a \odot b = 0_L$, then $p(a \oplus b) = p(a) + p(b)$;

(iii) if $a_n \nearrow a$, then $p(a_n) \nearrow p(a)$.

An observable is a mapping $x : \mathcal{B}(R) \to L$ satisfying the following conditions:

(i) $x(R) = 1_L$;

(ii) if $A \cap B = \emptyset$, then $x(A) \odot x(B) = 0_L$ and $x(A \cup B) = x(A) \oplus x(B)$;

(ii) if $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

Theorem 2. Let $x : \mathcal{B}(R) \to L$ be an observable, $p : L \to [0, 1]$ a probability. Then the composite mapping $p \circ x : \mathcal{B}(R) \to [0, 1]$ is a probability measure on $\mathcal{B}(R)$.

The key to the possibility to successfully create the probability theory on L-lattices is in the notion of independence.

Definition. (Lendelová [10, 11]). Observables $x_1, \ldots, x_n$ are independent, if there exists and n-dimensional observable $h_n : \mathcal{B}(R^n) \to L$ such that

$$(p \circ h_n)(A_1 \times \cdots \times A_n) = (p \circ x_1)(A_1) \cdot \ldots \cdot (p \circ x_n)(A_n)$$

for all $A_1, \ldots, A_n \in \mathcal{B}(R)$.

The existence of the joint observable $h_n (n = 1, 2, \ldots)$ can be used in two directions. First for the defining of functions of observables $x_1, \ldots, x_n$, e. g. $\frac{1}{n} \sum_{i=1}^{n} x_i (n = 1, 2, \ldots)$, second for local representation of the sequence $(x_n)_n$ by a probability algebra. Namely

$$\frac{1}{n} \sum_{i=1}^{n} x_i = h_n \circ g_n^{-1}$$

where $g_n(u_1, \dots, u_n) = \frac{1}{n}\sum_{i=1}^n u_i$ The motivation is taken from random vectors, where $(\frac{1}{n}\sum_{i=1}^n \xi_i)^{-1}(A) = (g_n \circ T_n)^{-1}(A) = T_n^{-1}(g_n^{-1}(A)), T_n(\omega) = (\xi_1(\omega), \dots, \xi_n(\omega))$.

Secondly, consider the space R^N, the σ-algebra $\boldsymbol{S}$ generated by the family of all cylinders, and the infinite product $\boldsymbol{P} : \mathcal{S} \to [0,1]$ of the measures $p \circ x_1, p \circ x_2, \dots$ defined by the equality

$$\boldsymbol{P}\left(\{(u_i)_{i=1}^{\infty}; u_1 \in A_1, \dots, u_n \in A_n\}\right) = p \circ x_1(A_1) \cdot \ldots \cdot p \circ x_n(A_n).$$

If we consider $\xi_n : R^N \to R$ as the projection to the nth coordinate, then the observable

$$y_n = g_n(x_1, \dots, x_n) : \mathcal{B}(R) \to L$$

and the random variable

$$\eta_n = g_n(\xi_1, \dots, \xi_n) : R^N \to R$$

have the same probability distribution, i.e.

$$p(g_n(x_1, \dots, x_n)(A)) = \boldsymbol{P}(\eta^{-1}(A)), \quad A \in \mathcal{B}(R).$$

Therefore the convergence of the sequence $(\eta_n)_n$ in some sense, implies the convergence of $(y_n)_n$ in the same sense.

5. EMBEDDING

Recently a new method has been discovered for the construction of the probability theory on *IF*-events: an embedding to an *MV*-algebra.

Theorem 3. (Riečan [22], Th. 1.2) Define $\mathcal{M} = \{(\mu_A, \nu_A); \mu_A, \nu_A$ are $\mathcal{S}$-measurable, $\mu_A, \nu_A : \Omega \to [0,1]\}$ together with operations

$$\begin{aligned}(\mu_A, \nu_A) \oplus (\mu_B, \nu_B) &= (\mu_A \oplus \mu_B, \nu_A \odot \nu_B),\\ (\mu_A, \nu_A) \odot (\mu_B, \nu_B) &= (\mu_A \odot \mu_B, \nu_A \oplus \nu_B),\\ \neg(\mu_A, \nu_A) &= (1 - \mu_A, 1 - \nu_A).\end{aligned}$$

Then the system $(\mathcal{M}, \oplus, \odot, \neg, (0_\Omega, 1_\Omega), (1_\Omega, 0_\Omega))$ is an *MV*-algebra.

Theorem 4. (Riečan [22], Th. 2.4) To any state $p : \mathcal{F} \to [0,1]$ there exists exactly one state $\bar{p} : \mathcal{M} \to [0,1]$ such that $\bar{p}|\mathcal{F} = p$.

The proof of Theorem 4 is based on the following simple fact:

$$\begin{aligned}(\mu_A, \nu_A) \oplus (0, 1 - \nu_A) &= (\mu_A, 0)\\ (\mu_A, \nu_A) \odot (0, 1 - \nu_A) &= (0, 1).\end{aligned}$$

Therefore it is natural to define $\bar{p}$ by the equality

$$\bar{p}((\mu_A, \nu_A)) + p((0, 1 - \nu_A)) = p((\mu_A, 0)).$$

Definition. (Riečan [22]) An IF-observable is a mapping $x : \mathcal{B}(R) \to \mathcal{F}$ satisfying the following properties:

(i) $x(R) = (1_\Omega, 0_\Omega)$;

(ii) $A, B \in \mathcal{B}(R), A \cap B = \emptyset \Longrightarrow x(A) \odot x(B) = (0_\Omega, 1_\Omega), x(A \cup B) = x(A) \oplus x(B)$;

(iii) $A_n \nearrow A \Longrightarrow x(A_n) \nearrow x(A)$.

Of course, since $\mathcal{F} \subset \mathcal{M}$, any *IF*-observable is an observable in the MV-algebra $\mathcal{M}$. Moreover, *IF*-observables have some further properties.

Definition. (Riečan [23]) The joint IF-observable of IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ is a mapping $h : \mathcal{B}(R^2) \to \mathcal{F}$ satisfying the following conditions

(i) $h(R^2) = (1_\Omega, 0_\Omega)$;

(ii) $A, B \in \mathcal{B}(R^2), A \cap B = \emptyset \Longrightarrow x(A) \odot h(B) = (0, 1), h(A \cup B) = h(A) \oplus h(B)$;

(iii) $A_n \nearrow A \Longrightarrow h(A_n) \nearrow h(A)$;

(iv) $h(C \times D) = x(C) \cdot y(B)$

for any $C, D \in \mathcal{B}(R)$. (Here $(f, g) \cdot (h, k) = (f \cdot h,\ g \cdot k)$.)

Theorem 5. (Riečan [23], Th. 2.3) To any two IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ there exists their joint IF observable.

Definition. (Montagna [14], Riečan [17]) An MV-algebra with product is a pair $(M, *)$, where M is an MV-algebra and $*$ is a commutative and associative binary operation on M satisfying the following conditions:

(i) $u * a = a$ for any $a \in M$;

(ii) if $a \odot b = 0$ then $(c * a) \odot (c * b) = 0$ and $c * (a \oplus b) = (c * a) \oplus (c * b)$.

Theorem 6. (Lendelová [13]) Define $(\mu_A, \nu_A) * (\mu_B, \nu_B) = (\mu_A \mu_B, \nu_A + \nu_B - \nu_A \nu_B)$. Then the family $\mathcal{M}$ is an MV-algebra with product.

By Theorem 6 many results of [25] can be applied immediately for probabilities on $\mathcal{M}$ and therefore for probabilities on $\mathcal{F}$, too.

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REFERENCES

[1] K. Atanassov: Intuitionistic Fuzzy Sets: Theory and Applications. Physica–Verlag, New York 1999.

[2] L. O. Cignoli, M. L. D'Ottaviano, and D. Mundici: Algebraic Foundations of Many–valued Reasoning. Kluwer, Dordrecht 2000.

[3] A. Di Nola, A. Dvurečenskij, M. Hyčko, and C. Manara: Entropy of effect algebras with the Riesz decomposition property I: Basic properties. Kybernetika *41* (2005), 143–160.

[4] A. Di Nola, A. Dvurečenskij, M. Hyčko, and C. Manara: Entropy of effect algebras with the Riesz decomposition property II: MV-algebras. Kybernetika *41* (2005), 161–176.

[5] A. Dvurečenskij and S. Pulmannová: New Trends in Quantum Structures. Kluwer, Dordrecht 2000.

[6] T. Gerstenkorn and J. Manko: Probabilities of intuitionistic fuzzy events. In: Issues in Intelligent Systems: Paradigms (O. Hryniewicz et al., eds.). EXIT, Warszawa, pp. 63–58.

[7] P. Grzegorzewski and E. Mrowka: Probability of intuitionistic fuzzy events. In: Soft Methods in Probability, Statistics and Data Analysis (P. Grzegorzewski et al., eds.). Physica–Verlag, New York 2002, pp. 105–115.

[8] P. R. Halmos: Measure Theory. Van Nostrand, New York 1950.

[9] A. N. Kolmogorov: Foundations of the Theory of Probability. Chelsea Press, New York 1950 (German original appeared in 1933).

[10] K. Lendelová: Measure Theory on Multivalued Logics and its Applications. Ph.D. Thesis. M. Bel University, Banská Bystrica 2005.

[11] K. Lendelová: Probability on L-posets. In: Proc. Fourth Conference of the European Society for Fuzzy Logic and Technology and 11 Rencontres Francophones sur la Logique Floue et ses Applications (EUSFLAT-LFA 2005 Joint Conference), Technical University of Catalonia, Barcelona, pp. 320–324.

[12] K. Lendelová: A note on invariant observables. Internat. J. Theoret. Physics *45* (2006), 915–923.

[13] K. Lendelová: Central Limit Theorem for L-posets. J. Electr. Engrg. *12/S* (2005), 56, 7–9.

[14] F. Montagna: An algebraic approach to propositional fuzzy logic. J. Logic. Lang. Inf. *9* (2000), 91–124.

[15] J. von Neumann: Grundlagen der Quantenmechanik. Berlin 1932.

[16] B. Riečan: A new approach to some notions of statistical quantum mechanics. BUSEFAL *36* (1988), 4–6.

[17] B. Riečan: On the product MV-algebras. Tatra Mt. Math. Publ. *16* (1999), 143–149.

[18] B. Riečan: Representation of probabilities on IFS events. In: Advances in Soft Computing, Soft Methodology and Random Information Systems (M. Lopez–Diaz et al., eds.) Springer–Verlag, Berlin 2004, pp. 243–246.

[19] B. Riečan: Free products of probability MV-algebras. Atti Sem. Mat. Fis. Univ. Modena *50* (2002), 173–186.

[20] B. Riečan: The conjugacy of probability MV-σ-algebras with the unit interval. Atti Sem. Mat. Fis. Univ. Modena *52* (2004), 241–248.

[21] B. Riečan: Kolmogorov–Sinaj entropy on MV-algebras. Internat. J. Theoret. Physics *44* (2005), 1041–1052.

[22] B. Riečan: On the probability on IF-sets and MV-algebras. Notes on IFS *11* (2005), 6, 21–25.

[23] B. Riečan: On the probability and random variables on IF events. In: Applied Artificial Intelligence (Proc. 7th FLINS Conf. Genova, Da Ruan et al., eds.), World Scientific 2006, pp. 138–145.
[24] B. Riečan and M. Jurečková: On invariant observables and the individual ergodic theorem. Internat. J. Theoret. Physics *44* (2005), 1587–1597.
[25] B. Riečan and D. Mundici: Probability on MV-algebras. In: Handbook on Measure Theory (E. Pap, ed.), Elsevier, Amsterdam 2002.
[26] B. Riečan and T. Neubrunn: Integral, Measure, and Ordering. Kluwer, Dordrecht 1997.
[27] E. Schmidt and J. Kacprzyk: Probability of intuitionistic fuzzy events and their applications in decision making. In: Proc. EUSFLAT'99, Palma de Mallorca 1999, pp. 457–460.

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