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Mathematica Slovaca, Vol. 26 (1976), No. 3, 207--216

Persistent URL: http://dml.cz/dmlcz/136119

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SEMIGROUPS AND THE STRUCTURE OF CATEGORIES

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§1. In this paper the algebraic structure of categories is examined with emphasis on those categories which have a finite number of arrows (or morphisms). The paper contains as an application the determination of the number and structure of all categories of orders four or less. The approach taken is to view a category as a special kind of semigroup with zero [2, p. 78] and first to establish results for semigroups and then apply these results to categories.

§2. By a *semigroup* we understand a class S of elements together with an associative binary operation such that S contains a zero element 0 (i. e. $0x = x0 = 0$ for all x in S). This definition involves no loss of generality since if S is a semigroup with no zero element under the usual definition, then, of course, a unique semigroup S^0 may be obtained by adjoining a zero. We use the term "class" rather than "set" because we wish to consider those cases in which, as in category theory, the class of elements is too large to form a set. A semigroup in which the class of elements is a set (in the sense of (say) von Neumanns's set theory) will be called an *ordinary semigroup.* By a *homomorphism* we understand a mapping $f\colon A \to B$ of a semigroup A into a semigroup B such that $f(0) = 0$ and $f(xy) = f(x)f(y)$ for all x and y in A. The category whose objects are semigroups and whose arrows (or morphisms) are homomorphisms will be denoted by (*Smgrp*). [3].

We define two other categories whose objects are semigroups. A function $f\colon A \to B$ from a semigroup A to a semigroup B will be called a *functor* provided:

1. $f(x) = 0$ if and only if $x = 0$, and
2. if $xy \neq 0$, then $f(xy) = f(x)f(y)$ for all x and y in A. The idea of functor is related both to the idea of functor in the category theory and the idea of partial homomorphism [1, p. 93]. It is apparent that not all functors are homomorphisms. For example, consider the semigroups $\{e, 0\}$ and $\{e', 0\}$, where e and e' are identity elements. Let S be the free product of $\{e\}$ and $\{e'\}$ and let U be the semigroup $\{e, e', 0\}$ defined by the Cayley table:

	e	e'
e	e	0
e'	0	e'

Then the injection $U \to S^\circ$ is a functor but not a homomorphism. On the other hand some homomorphisms are also not functors since they may map nonzero elements into zero.

The category whose objects are semigroups and whose arrows are functors will be denoted by $(Smgrp)_0$.

A function $f: A \to B$ from a semigroup A to a semigroup B will be called a *partial functor* provided:

1. $f(0) = 0$ and
2. if $xy \neq 0$ then a) $f(xy) = f(x)f(y)$ and
   b) $f(xy) = 0$ only if $f(x) = 0$ or $f(y) = 0$.

(The name "partial functor" is adopted because the analogous concept in category theory is a functor defined on a subcategory. See §4.) The category whose objects are semigroups and whose arrows are partial functors will be denoted by $(Smgrp)*$.

Since every functor is a partial functor it follows that $(Smgrp)_0$ is a subcategory of $(Smgrp)*$ and therefore not every partial functor is a homomorphism. Moreover, the following example shows that not all homomorphisms are partial functors. Let A and B be the semigroups given by the tables:

A	0	x	y	z
0	0	0	0	0
x	0	x	x	x
y	0	x	y	x
z	0	x	x	z

B	0	a	b
0	0	0	0
a	0	a	0
b	0	0	b

and define $f: A \to B$ by $f(0) = 0$, $f(x) = 0$, $f(y) = a$, and $f(z) = b$. Then f is a semigroup homomorphism but not a partial functor.

§3. The ideals [1] in a semigroup S form a modular lattice under the operations of union and intersection. The zero element of the lattice is the semigroup 0 which contains only the zero element. The semigroup 0 is both an initial and terminal object in the categories *Smgrp* and $(Smgrp)*$; that is to say, for every semigroup S there is a unique arrow $S \to 0$ and a unique arrow $0 \to S$. Thus 0 is a zero or null object in these categories. The *kernel* of a homomorphism or partial functor $f: A \to B$ is the subset of elements in A that map into 0. The kernel K of f is an ideal in A and if f is a partial functor, then K is a prime ideal [1, p. 40] in A, i. e. $(A - K)^\circ$ is a subsemigroup of A. The inclusion mapping $K \to A$ is an equalizer of the pair of arrows $A \underset{0}{\overset{f}{\rightrightarrows}} B$ in the respective categories $(Smgrp)$ and $(Smgrp)*$.

Now let I be an ideal of a semigroup S and let S/I be the Rees quotient of S with respect to I. The *natural homomorphism* $p: S \to S/I$ is defined by

$$p(x) = \begin{cases} x & \text{if } x \in (S - I) \\ 0 & \text{if } x \in I. \end{cases}$$

If I is a prime ideal, then the homomorphism p is a partial functor. But p is not a functor unless I is trivial. Obviously, the kernel of p is I. If $I \rightarrow S$ is the inclusion mapping, then $p\colon S \rightarrow S/I$ is a coequalizer of the pair of arrows $I \underset{0}{\rightrightarrows} S$ in $(Smgrp)$ (and also in $(Smgrp)_*$ provided I is a prime ideal).

Analogous to the internal direct product of two groups, we have the following:

Definition. *Let I and J be ideals in a semigroup S. Then S is the direct union of I and J provided* $S = I \cup J$ *and* $I \cap J = 0$. *We shall write* $S = I + J$. *(In Clifford and Preston,* [2 p. 13], $I + J$ *is called a* 0*-direct union.) It should be noted that in the direct union if x is in I and y is in J, then* $xy = yx = 0$. *The direct union is a coproduct in the categories* $(Smgrp)_*$ *and* $(Smgrp)_\circ$ *but not in* $(Smgrp)$.

Clearly the ideals I and J in a direct union $I + J$ of semigroups I and J are prime ideals in $I + J$. Moreover, we have two natural homomorphisms $p_1\colon S \rightarrow I$ and $p_2\colon S \rightarrow J$ in which $I = S/J$ and $J = S/I$, which is analogous to the situation in the direct product of groups.

Definition. *If I is a prime nontrivial proper ideal in a semigroup S, then S is an extension of S/I by I.* [1, *p.* 137]

We shall show in §5 that such extensions are not completely determined by S/I and I.

If S is the extension of S/I by I and S/I is an ideal in S, then $S = I + S/I$. Conversely, if $S = I + J$, then S is both an extension of I by J and of J by I.

Definition. *A semigroup is indecomposable if* $S = I + J$ *only if* $I = 0$ *or* $J = 0$.

The following theorem is the analog for semigroups of the Wedderburn-Remak Theorem on the decomposition of groups, and is a corollary of the corresponding theorem of Ore on the decomposition of the elements of a finite-dimensional modular lattice. [5]

Theorem 1. *(Decomposition Theorem). Let S be a semigroup whose ideals form a finite-dimensional modular lattice. Then if S has two representations as the direct union of indecomposable ideals*

$$S = A_1 + A_2 + \ldots + A_m$$

and

$$S = B_1 + B_2 + \ldots + B_n,$$

it follows that $m = n$ *and every* A_i *is replaceable by some* B_j *and the* A_i*'s and the* B_j*'s are pairwise equal.*

Proof. The ideals of S form a modular lattice and the definitions of direct union agree, hence we may apply the theorem of Ore on the decomposition of the elements of a finite-dimensional modular lattice. Now let $\bar{A}_i = A_1 + \ldots + A_{i-1} + A_{i+1} + \ldots + A_m$. Then if $S = A_i + \bar{A}_i$ and $S = B_i + \bar{A}_i$, it follows that $A_i = S/\bar{A}_i$ and $B_i = S/\bar{A}_i$ and consequently the theorem is proved.

§4. In this article we extend the results of §3 to categories. We also prove that for categories every ideal is a prime ideal.

Definition. *Let C be a semigroup with a zero element. A local left (right) identity in C is a nonzero element e such that for all x in C, if* $ex \neq 0$ $(xe \neq 0)$, *then* $ex = x$ $(xe = x)$.

Definition. *A semigroup C is a category provided:* 1. *for every nonzero element a in C, there is both a left local identity e and a right local identity e′ such that* $ea = ae' = a$; *and* 2. *if a, b, and c are elements in C, then* $abc = 0$ *implies* $ab = 0$ *or* $bc = 0$. *As usual, when there is no danger of misunderstanding, we shall refer to local identities in a category as simply identities.*

The second axiom in the definition of a category is called the *categorical law.* When the category is an ordinary semigroup Clifford and Preston ([2 p. 78]) call it a *small category with zero.* Clearly the nonzero elements of a category C form a category in the usual sense, and every category C (in the usual sense) corresponds to a semigroup C° obtained by adjoining a zero arrow 0 such that for any arrows a and b, if ab is undefined in C, then $ab = 0$ in C°. Consequently all categories in the following discussion will be assumed to be semigroups unless otherwise specified.

It is easily established that every left identity in a category is also a right identity and conversely. Distinct identities have a zero product, since $ee' \neq 0$ implies $e = ee' = e'$. Of course, the left and right identities of an element are unique.

In order to define the concept of subcategory we first define a *subsemigroup* $(T, \cdot)$ of a semigroup S as a subclass T of the elements of S such that $0 \in T$ and $(T, \cdot)$ is a semigroup under the operation $\cdot$ in S. A *subcategory S* of a category C is a subsemigroup of C with the properties:

1. S is a category.
2. The local left and right identities of an element in S are the same as in C.

The intersection of two subcategories of a category C is clearly a subcategory of C, but the following example shows that the union of two subcategories may fail to be a subcategory.

C	e	a	c	e'	b	e''
e	e	a	c	0	0	0
a	0	0	0	a	c	0
c	0	0	0	0	0	c
e'	0	0	0	e'	b	0
b	0	0	0	0	0	b
e''	0	0	0	0	0	e''

A	e	a	e'
e	e	a	0
a	0	0	a
e'	0	0	e'

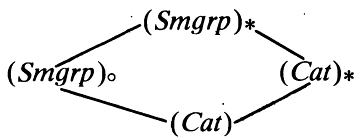
B	e'	b	e''
e'	e'	b	0
b	0	0	b
e''	0	0	e''.

A and B are subcategories of C, but $A \cup B$ is not a subcategory of C since the product $ab = c$ will not be in $A \cup B$.

A *categorical functor* from a category A to a category B is a functor $f\colon A \to B$ (see §2) such that if e is an identity in A, then $f(e)$ is an identity in B. Thus, in addition to being a functor in the usual categorical sense we require that $f(0) = 0$. The category whose objects are categories and whose arrows are categorical functors will be denoted by (Cat). If A and B are categories, then by a functor $f\colon A \to B$ we shall always understand a categorical functor.

We need to generalize the concept of functor in order to extend some of the results on semigroups to category theory. We speak of a "partial functor" when a functor in the usual sense is defined on a subcategory but not on the whole category itself. The existence of a zero allows us to define a partial func or on the whole category by stipulating that elements not in the domain of definition be mapped into zero. The category in which the objects are categories and the arrows are partial functors will be denoted by $(Cat)_*$.

The relationship among four of the categories discussed in this paper are given by the inclusion diagram:



The inclusion of (Cat) as a subcategory of $(Smgrp)_\circ$ is obvious. We shall prove below that $(Cat)_*$ is a subcategory of $(Smgrp)_*$, and consequently results proved for $(Smgrp)_\circ$ and $(Smgrp)_*$ apply to categories.

Definition. *A partial functor $f\colon C \to C'$ from a category C to a category C' is a mapping such that:*

1. $f(0) = 0$.
2. *If $ab \neq 0$ then $f(ab) = f(a)f(b)$.*
3. *For all identities e in C, if $f(e) \neq 0$, then $f(e)$ is an identity in C'.*

Theorem 2. *$(Cat)_*$ is a subcategory of $(Smgrp)_*$.*

Proof. We need only to show that a partial functor for categories is also a partial functor for semigroups. Let $f\colon C \to C'$; be a partial functor from a category C to a category C'. Let a and b be elements of C such that $ab \neq 0$ and $f(ab) = 0$. We must show that $f(a) = 0$ or $f(b) = 0$. Let e be the right identity for a. Then we have $aeb \neq 0$, $a = ae \neq 0$, and $b = eb \neq 0$. It follows that $f(a) = f(ae) = f(a)f(e)$, $f(b) = f(eb) = f(e)f(b)$, and finally $f(ab) = f(a)f(b) = f(a)f(e)f(b) = 0$. Consequently, by the categorical law, $f(a) = f(a)f(e) = 0$ or $f(b) = f(e)f(b) = 0$.

Definition. *An ideal I in a category C is a subcategory of C such that if a is in I and b is in C then ab and ba are in I.*

Thus an ideal in a category is a semigroup ideal, but as the following example shows, not every subcategory that is a semigroup ideal in a category is also an ideal in the category. Let C and I be the monoids defined by the following tables:

C	e	a	b
e	e	a	b
a	a	a	b
b	b	b	a

I	a	b
a	a	b
b	b	a

Then I is a category and a semigroup ideal of C *but is not a subcategory of* C since the identity a of I is not the identity of C.

Analogous to the situation for semigroups, we have the following:

Definition. *An ideal I in a category C is a prime ideal if* $(C-I)^\circ$ *is a subcategory of C.*

Theorem 3. *Every ideal in a category is a prime ideal.*

Proof. Let C be a category and let I be an ideal in C. First of all, the identities for each element in $C-I$ must be in $C-I$ since I is an ideal. Now let x and y be in $C-I$, and suppose that xy is in I and $xy \neq 0$. Then there is an identity e in I such that $e(xy) \neq 0$. This means that $ex \neq 0$; whence an element e in I is the left identity of the element x in $C-I$, which is impossible. Thus $(C-I)^\circ$ is a subcategory of C and the theorem is proved.

Theorem 4. *If A and B are subcategories of C such that* $C = A \cup B$ *and* $A \cap B = 0$, *then* $C = A + B$. *Moreover,* $A + B$ *is a coproduct in* (Cat) *and* $(Cat)_*$.

Proof. We must show that A and B are ideals. Suppose x is in A and y is in B. Then either xy is in A, or xy is in B. Suppose xy is in B and let e be the left identity for x. Then $exy = xy$ and if $xy \neq 0$, it follows that e is an identity in B. Consequently $xy = 0$, since $A \cap B = 0$ and we conclude that xy is in A and that A is an ideal in C. Similarly B is an ideal in C and therefore $C = A + B$. The fact that C is a coproduct in (Cat) and $(Cat)_*$ follows almost immediately from the corresponding result for semigroups.

Corollary 4.1. *If I is ideal in a category C, then C/I is an ideal and* $C = I + C/I$.

As a consequence of Theorem 3 or Corollary 4.1, given an ideal I in a category C, it follows that C is an extension of the category C/I by I.

Definition. *A category C is simple if its only ideals are C and* 0.

In view of Corollary 4.1, "simple" and "indecomposable" are equivalent conditions for categories. We note that the decomposition theorem (Theorem 1) applies to categories. In addition, the analog for semigroups of the Jordan—Hold-er—Schreier Theorem, which was stated and proved by Rees [6 p. 388] is also true for categories, as are several theorems analogous to the isomorphism theorems for groups. (Cf. [7 pp. 24—28]).

§5. This article concerns extensions of categories by a semigroup ideal.

Theorem 5. *A finite monoid is either a group or an extension of a group by a semigroup ideal.*

Proof. Let *M be a monoid with identity e* and assume M is not a group. Let I be the set of noninvertible elements of M. To show that I is a semigroup ideal let $a \in I$ and $b \in M$ and suppose that ab is invertible. Then $ab = c$ implies $a(bc^{-1}) = e$. To show that a has a left inverse let the mapping $f\colon M \to M$ be defined by $f(x) = xa$. Now f is injective and hence is surjective since M is finite. Thus a is invertible and $a \notin I$, contrary to hypothesis. Consequently, I is an ideal and moreover I is a prime ideal since $M - I$ is group.

Definition. *Let M and M' be monoids such that $M \cap M' = 0$ and let a be an element not in $M + M'$. Let $M \to M'$ be the category obtained by adjoining a to $M + M'$ in the following manner: for all $x \neq 0$ in M and $y \neq 0$ in M' we define $xa = ay = a$ and $ax = ya = 0$. We shall call $M \to M'$ and $M' \to M$ the simple extensions of $M + M'$.*

Simple Extension Lemma: *The category $M \to M'$ defined above exists and is unique up to isomorphism. Moreover, $M \to M'$ is simple and $M \to M'$ and $M' \to M$ are the only simple categories C such that*

(1) $C = (M + M') \cup \{a\}$

and

(2) *$M + M'$ is a subcategory of C.*

Proof. If e and e' are identity elements in a category, we write $[e, e']$ for the set of all elements x in the category such that $ex = xe' = x$. (In other words $[e, e']$ is the *Hom* set together with zero.) Let e in M and e' in M' be the identities of $M + M'$. We construct $M \to M'$ as follows: since a is not in $M + M'$ we have $a \neq 0$ and since, by definition, $ea = ae' = a$, it follows that we must put a in $[e, e']$ and we must define $aa = 0$. All other nontrivial products of a with elements of $M + M'$ are given in the definition of $M \to M'$. It is routine to verify that $M \to M'$ is a category. It follows that $M \to M'$ is unambiguous up to isomorphism.

To prove that $M \to M'$ is simple, suppose I is a nontrivial ideal in $M \to M'$. Since I is nontrivial it contains a nonzero element x. If x is in M, then $xa = a$ is in I and if $x \in M'$, then $ax = a$ is in I. Hence a is in I and therefore the left and right identities (e and e') are in I; whence $I = M \to M'$. This establishes that $M \to M'$ is simple.

Finally, suppose C is any simple category such that $C = M + M' \cup \{a\}$ and $M + M'$ is a subcategory of C. Suppose $a \in [e, e]$. Then $[e, e]$ is a nontrivial ideal in C which is not equal to C since $e' \notin [e, e]$ But this is impossible since C is simple. For the same reason we cannot have $a \in [e', e']$. If a is an identity in C, then $ea = ae = ae' = e'a = 0$ and consequently $[a, a]$ is a nontrivial ideal in C not equal to C. This again contradicts the simplicity of C. Thus we have $a \in [e, e']$ or $a \in [e', e]$.

Assume $a \in [e, e']$. Then a is the only nonzero element in $[e, e']$ and $[e', e] = \{0\}$. Hence if $x \in M$ and $y \in M'$, we have $xa = ay = a$ and $ax = ya = 0$. Thus $C = M \rightarrow M'$. Similarly $C = M' \rightarrow M$ if $a \in [e', e]$. //

Note that $M \rightarrow M'$ *is an extension of* $M + M'$ *by the semigroup ideal* $\{0, a\}$.

We shall adopt the convention that a nonzero element a in $[e, e']$ has the diagram $e \overset{a}{\rightarrow} e'$. This is consistent with the notation $M \rightarrow M'$. Given a monoid M we shall refer to $M \rightarrow I$ as the *right extension* of M and $I \rightarrow M$ as the *left extension* of M, where I is the monoid with one (non-zero) element.

§6. In this article we determine all categories of order four or less.

A category which contains n non-zero members will be called a category of *order* n. Clearly, the monoid I is the only category of order one. Note that every monoid is simple.

We now establish that *there are exactly three categories of order two.* To prove this we first note that by the decomposition theorem the only nonsimple category of order two is the category $I + I$. Next, we claim that all simple categories of order two are monoids For if A is a category of order two which is not a monoid, then A contains two identities, whence $A = I + I$. The two monoids of order two will be denoted by M_1 and M_2.

We claim that *there are eleven categories of order three.* To prove this we shall show that there are eight simple categories of order three and three non-simple ones. To show that there are three non-simple ones let C be a non-simple category of order three. Then C must be factorable into simple categories whose orders add up to three; the only such factorizations possible are $I + I + I$, $M_1 + I$, and $M_2 + I$.

According to Forsythe [4, p. 446] there are seven monoids of order three. Hence we need only to show that there is only one simple category of order three that is not a monoid. Now if C is of order three and simple and not a monoid, then it must have two identities. Thus it is a simple extension of $I + I$, whence $C = I \rightarrow I$ Following Lawvere [8] we may regard $I \rightarrow I$ as the ordinal number 2 and hence we shall denote it by 2 (to distinguish it from $I + I$).

Theorem 6. *There are six simple categories of order four that are not monoids. Four of these are the right and left extensions of the two monoids of order two.*

Proof. A simple category of order four can have no more than three identities.

Suppose C is a category of order four with three identities and suppose $a \in C$ is not an identity. Let e and e' be the left and right identities for a. If $e \neq e'$, then $C = \bar{2} + l$, while if $e = e'$, then $C = M_1 + l + l$ or $C = M_2 + l + l$. Consequently simple categories of order four that are not monoids must have exactly two identities.

Now let C be a simple category of order four with distinct identities e and e' and let a and b be the other nonzero elements in C. If a and be are in $[e, e]$ or if a is in $[e, e]$ and b is in $[e', e']$, it follows that C is not simple. Therefore one of the following situations prevails:

1. a is in $[e, e']$ and b is in $[e, e']$
2. a is in $[e, e']$ and b is in $[e', e]$
3. a is in $[e, e]$ and b is in $[e, e']$
4. a is in $[e, e]$ and b is in $[e', e]$

Cases (1) and (2) each define a unique category. The category in case (1) is given by the Cayley table:

	e	a	b	e'
e	e	a	b	0
a	0	0	0	a
b	0	0	0	b
e'	0	0	0	e'

It has the diagram:

$$e \underset{\overrightarrow{b}}{\overset{\overrightarrow{a}}{\quad}} e'$$

The category in case (2) is given by:

$$e \underset{\overleftarrow{b}}{\overset{\overrightarrow{a}}{\quad}} e' .$$

	e	a	b	e'
e	e	a	0	0
a	0	0	e	a
b	b	e'	0	0
e'	0	0	b	e'

It may also be noted that the latter category is the only category of order less than or equal to four which is neither a group nor an extension of a subcategory by a semigroup ideal.

The remainder of the theorem follows directly from cases (3) and (4) and the simple extension lemma.

Corollary 6.1. *There are fifty-five categories of order four, of which thirty five are monoids, six are simple but not monoids, eight are of the form $C+I$ where C is a simple category of order three, three are of the form C_1+C_2, where C_1 and C_2 are monoids of order two, two are of the form $C+I+I$, where C is a simple category of order two, and one is of the form $I+I+I+I$.*

Proof. It will follow that there are fifty-five categories of order four if we prove the other assertions in the corollary. The fact that there are thirty-five monoids of order four follows from [4, p. 446—447]. The next assertion is established by Theorem 6 and the rest of the assertions follows directly from the decomposition theorem together with our results on categories of order less than four

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Received May 5, 1974

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