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FUNDAMENTAL VECTOR FIELDS ON TYPE FIBRES OF JET PROLONGATIONS OF TENSOR BUNDLES

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1. Introduction. In this paper the problem of constructing the fundamental vector fields [9] on the type fibres of the jet prolongations of tensor bundles is considered. This is one of the most important problems of the theory of invariant Lagrangian structures [3], [4], [7].

The well-known direct method of finding these vector fields follows from their definition, and is based on the formulas for the action of the structure group of tensor bundles on their type fibres. Examples [4] show, however, that the corresponding computations are rather complicated and in the cases of higher jet prolongations practically non-realizable. In this paper we determine the fundamental vector fields by a simple differentiation procedure which makes use of their relation to the jet prolongations of the so called *induced* vector fields on tensor bundles [2], [5], [6].

Throughout these notes the standard summation convention will be used unless otherwise stated. The symbol $*$ will denote the composition of jets, and R^n will denote the n-dimensional real Euclidean space; we shall write $R=R^1$. For the sake of simplicity all our manifolds and maps will belong to the category C^∞.

2. The jet prolongations of projectable vector fields. The purpose of this section is to recall the definition and main properties of the jet prolongations of projectable vector fields on fibred manifolds [2], [5], [6]. Throughout, $\pi\colon Y\to X$ denotes a locally trivial, finite-dimensional fibred manifold. We set $n=\dim X$, $m=\dim Y-\dim X$.

Let Ξ be a π-projectable vector field on Y and ξ its π-projection. Denote by α_t and α_{0t} the local one-parameter groups of Ξ and ξ, respectively. For every local section γ of π and every sufficiently small t, $\alpha_t\gamma\alpha_{0t}^{-1}$ is a local section of π, and the r-jet prolongation $j^r\alpha_t\gamma\alpha_{0t}^{-1}$ of $\alpha_t\gamma\alpha_{0t}^{-1}$ is a local section of the r-jet prolongation $\pi_r\colon \mathcal{J}^rY\to X$ of π. Setting

$$j^r\Xi(j^r_x\gamma)=\left\{\frac{\mathrm{d}}{\mathrm{d}t}\,j^r_{\alpha_{0t}(x)}\alpha_t\gamma\alpha_{0t}^{-1}\right\}_0$$

we obtain a vector field on $\mathcal{J}^rY$, $j^r\Xi$, called the r-jet prolongation of Ξ.

There exists a simple description of this vector field. To obtain it, let us introduce some special charts on $\mathscr{J}^r Y$. To every $y_0 \in Y$ there exist a chart (V, ψ) on Y with centre y_0 and a chart (U, φ) on X such that $U = \pi(V)$ and the map ψ: $V \to R^n \times R^m$ is of the form

$$\psi(y) = (\varphi\pi(y), \psi_0(y)),$$

where ψ_0: $V \to R^m$ is a map. The coordinate functions defined by such a chart are called the *fibre coordinates* on Y.

Let (V, ψ) and (U, φ) be as above, let x_i, y_σ, where $1 \leqslant i \leqslant n$, $1 \leqslant \sigma \leqslant m$, be the corresponding fibre coordinates. We set for every $j_x^r \gamma \in \pi_r^{-1}(U)$

$$\begin{aligned} x_i(j_x^r\gamma) &= x_i(x), \\ y_\sigma(j_x^r\gamma) &= y_\sigma\gamma(x), \\ z_{i\sigma}(j_x^r\gamma) &= \mathrm{D}_i y_\sigma \gamma\varphi^{-1}(\varphi(x)), \\ &\dots \\ z_{i_1 \dots i_r\sigma}(j_x^r\gamma) &= \mathrm{D}_{i_1} \dots \mathrm{D}_{i_r}(y_\sigma\gamma\varphi^{-1})(\varphi(x)), \end{aligned}$$

where D_i means the i-th partial derivative operator. The functions x_i, y_σ, ..., $z_{i_1 \dots i_r \sigma}$, $1 \leqslant i \leqslant n$, $1 \leqslant \sigma \leqslant m$, $1 \leqslant i_1 \leqslant \dots \leqslant i_r \leqslant n$, are some fibre coordinates on $\mathscr{J}^r Y$ which are said to be associated to the fibre coordinates x_i, y_σ.

Let F be a function on $\pi_r^{-1}(U)$, and denote by $\mathrm{D}_k F$, $\mathrm{D}_\sigma F$, ..., $\mathrm{D}_{i_1 \dots i_r \sigma} F$ the partial derivatives of F with respect to x_k, y_σ, ..., $z_{i_1 \dots i_r \sigma}$, respectively. We define a function $\mathrm{d}_k F$ on $\pi_{r+1}^{-1}(U)$ by the formula

$$\mathrm{d}_k F = \mathrm{D}_k F + \mathrm{D}_\sigma F \cdot z_{k\sigma} + \mathrm{D}_{i\sigma} F \cdot z_{ki\sigma} + \dots + \sum_{i_1 \leqslant \dots \leqslant i_r} \mathrm{D}_{i_1 \dots i_r \sigma} F \cdot z_{i_1 \dots i_r \sigma}.$$

The following is proved in [6]:

Lemma 1. *Let Ξ be a π-projectable vector field, let*

$$\Xi = \xi_i \frac{\partial}{\partial x_i} + \Xi_\sigma \frac{\partial}{\partial y_\sigma}$$

be its expression in some fibre coordinates x_i, y_σ on Y. Then in the associated fibre coordinates on $\mathscr{J}^r Y$,

$$j^r \Xi = \xi_i \frac{\partial}{\partial x_i} + \Xi_\sigma \frac{\partial}{\partial y_\sigma} + \Xi_{i\sigma} \frac{\partial}{\partial z_{i\sigma}} + \dots + \\ + \sum_{i_1 \leqslant \dots \leqslant i_r} \Xi_{i_1 \dots i_r \sigma} \frac{\partial}{\partial z_{i_1 \dots i_r \sigma}},$$

where for every s, $0 \leqslant s \leqslant n-1$,

$$\Xi_{ki_1 \dots i_s \sigma} = \mathrm{d}_k\, \Xi_{i_1 \dots i_s \sigma} - z_{ji_1 \dots i_s \sigma} \cdot \mathrm{D}_k \xi_j.$$

3. Fundamental vector fields on left L_n^r-spaces. In this section X will denote an n-dimensional manifold.

We begin by recalling some notation and definitions. Let L_n^r be the Lie group of all invertible r-jets with source and target at the origin $0 \in R^n$. An *r-frame* at a point $x \in X$ is an invertible r-jet with source $0 \in R^n$ and target x. The set $\mathscr{F}^r X$ of all r-frames carries a natural structure of a principal L_n^r-bundle [1] called the *bundle of r-frames* on X. The projection of this bundle will be denoted by π_r. The right action of L_n^r on $\mathscr{F}^r X$ is defined by the map $(y, g) \to y * g$.

Let Q be a left L_n^r-space. We shall denote by $\mathscr{F}_Q^r X$ the fibre bundle with type fibre Q, associated to $\mathscr{F}^r X$. The equivalence class of a pair $(y, q) \in \mathscr{F}^r X \times Q$ with respect to the standard equivalence relation in $\mathscr{F}^r X \times Q$ [8] will be denoted by $[y, q]$.

Every local diffeomorphism α of X gives rise to a local automorphism $\mathscr{F}^r\alpha$ of $\mathscr{F}^r X$ defined by

$$\mathscr{F}^r\alpha(y) = j^r_{\pi_r(y)}\alpha * y,$$

where $j_x^r\alpha$ denotes the r-jet of α at x, and to a local automorphism $\mathscr{F}_Q^r\alpha$ of $\mathscr{F}_Q^r X$ defined by

$$\mathscr{F}_Q^r\alpha(z) = [\mathscr{F}^r\alpha(y), q],$$

where $z = [y, q]$. Both $\mathscr{F}^r\alpha$ and $\mathscr{F}_Q^r\alpha$ are said to be *induced* by α. The definition of the induced local automorphisms is naturally extended to vector fields.

Let $\mathscr{F}^r X \oplus \mathscr{F}_Q^r X$ denote the Whithey sum of $\mathscr{F}^r X$ and $\mathscr{F}_Q^r X$. To each pair $(y, z) \in \mathscr{F}^r X \oplus \mathscr{F}_Q^r X$ there is uniquely associated a point $q \in Q$ such that $z = [y, q]$. We define the *relative image maps* $\varkappa$ and $\varkappa_y$ by

$$\varkappa(y, z) = \varkappa_y(z) = q.$$

$\varkappa_y$ is obviously a surjective diffeomorphism between the fibre over $x = \pi_r(y)$ in $\mathscr{F}_Q^r X$ and Q.

Let $x \in X$, $y \in \pi_r^{-1}(x)$, let ξ be a vector field defined on a neighbourhood of x such that $\xi(x) = 0$. Denote by $\mathscr{F}_Q^r\xi$ the induced vector field on $\mathscr{F}_Q^r X$, and define a vector field on Q, ξ_Q, by the relation

$$\mathrm{T}\varkappa_y \cdot \mathscr{F}_Q^r\xi = \xi_Q \circ \varkappa_y,$$

where $\mathrm{T}\varkappa_y$ is the tangent map to $\varkappa_y$. The following statement gives us a description of the Lie algebra of fundamental vector fields on Q.

Lemma 2. *Let $x \in X$ be a point. To each fundamental vector field Ξ on the left L_n^r-space Q there exists a vector field ξ defined on a neighbourhood of x, such that $\xi(x) = 0$ and $\Xi = \xi_Q$. Conversely, if ξ is a vector field on a neighbourhood of x and $\xi(x) = 0$, then ξ_Q is a fundamental vector field on Q.*

Proof. Let Ξ be a fundamental vector field on Q. There must exist a one-parameter subgroup g_t of L_n^r such that

$$\Xi(q)=\left\{\frac{\mathrm{d}}{\mathrm{d}t}(g_t\cdot q)\right\}_0$$

for every q. It is easily seen that g_t can be chosen in the form $g_t=j_0^r\alpha_t$, where α_t is a one-parameter group of local transformations of R^n. Let $y\in\pi_r^{-1}(x)$ and choose a map f of a neighbourhood of $0\in R^n$ to X such that $y=j_0^rf$. For every sufficiently small t, $\beta_t=f\alpha_tf^{-1}$ is a local diffeomorphism of X satisfying $\beta_t(x)=x$. The induced local diffeomorphism $\mathscr{F}^r\beta_t$ satisfies

$$\mathscr{F}^r\beta_t(y)=j^r_{\pi_r(y)}\beta_t*j_0^rf=j_0^r(f\alpha_tf^{-1}\circ f)=y*g_t.$$

For $q\in Q$ and $z=[y,q]$ this relation implies that

$$g_t\cdot q=\varkappa_y([y,g_t\cdot q])=\varkappa_x([y*g_t,q])=\varkappa_y\mathscr{F}^r_Q\beta_t(z),$$

which in turn implies that

$$\Xi(q)=\mathrm{T}\varkappa_y\cdot\left\{\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}^r_Q\beta_t(z)\right\}_0=\xi_Q(q).$$

Conversely, let ξ be a vector field on a neighbourhood of x, and assume that $\xi(x)=0$. Let β_t be the local one-parameter group of ξ. Then $\beta_t(x)=x$ for all t and there exists a curve g_t in L_n^r such that

$$\mathscr{F}^r\beta_t(y)=y*g_t.$$

Let $z=[y,q]$ be any point. Then

$$\varkappa_y\mathscr{F}^r_Q\beta_t(z)=\varkappa_y([y*g_t,q])=g_t\cdot q=g_t\cdot\varkappa_y(z).$$

On differentiating this relation with respect to t we directly obtain that ξ_Q is a fundamental vector field on Q.

This completes the proof of Lemma 2.

4. Fundamental vector fields on the type fibres of jet prolongations of tensor bundles. In this section, X is an n-dimensional manifold, Q a vector space endowed with a tensor representation of the group $\mathrm{Gl}_n(R)$, π_Q: $\mathscr{F}^1_QX\to X$ the bundle of tensors of type Q on X, and $\pi_{Q,r}$: $\mathscr{J}^r\mathscr{F}^1_QX\to X$ its r-jet prolongation. Recall that $\mathscr{J}^r\mathscr{F}^1_QX$ is a fibre bundle with type fibre T_n^rQ, the space of all r-jets with source $0\in R^n$ and target in Q, associated to the bundle $\mathscr{F}^{r+1}X$ of $(r+1)$-frames on X [3]. In our previous notation, $\mathscr{J}^r\mathscr{F}^1_QX=\mathscr{F}^{r+1}_PX$, where $P=\mathrm{T}_nQ$.

Let $x\in X$, let (U,φ) be a chart on X such that $x\in U$ and $\varphi(x)=0$. Denote by x_i the coordinate functions defined by this chart, and by q_σ, where σ runs over an appropriate set of multi-indices, some global coordinates on Q. Let $\varkappa^{(1)}$:

$\mathscr{F}^1X \oplus \mathscr{F}^1_Q X \to Q$ be the relative image map introduced in Section 3. Fibre coordinates $\tilde{x}_i$, $\tilde{q}_\sigma$ on $\mathscr{F}^1_Q X$, associated to the coordinates x_i, are defined by

$$\tilde{x}_i(\tilde{q}) = x_i \pi_Q(\tilde{q}), \quad \tilde{q}_\sigma(\tilde{q}) = q_\sigma \circ \varkappa^{(1)}(j^1_0 \varphi^{-1}, \tilde{q}),$$

where $\tilde{q} \in \pi_Q^{-1}(x)$.

Let now ξ be a vector field on U,

$$\xi = \xi_i \frac{\partial}{\partial x_i}.$$

It is known that in the associated local coordinates $\tilde{x}_i$, $\tilde{q}_\sigma$,

$$\mathscr{F}^1_Q \xi = \xi_i \frac{\partial}{\partial x_i} + C^{\sigma k}_{\lambda i} \cdot \mathrm{D}_k \xi_i \cdot \tilde{q}_\sigma \frac{\partial}{\partial \tilde{q}_\lambda},$$

where $C^{\sigma k}_{\lambda i} \in R$ are some constants determined by the tensor character of Q. Using Lemma 1 we obtain

$$j^r \mathscr{F}^1_Q \xi = \xi_i \frac{\partial}{\partial x_i} + \mathrm{d}_k \xi_i \cdot \tilde{\Theta}^i_k + \ldots + \mathrm{D}_{k_1} \ldots \mathrm{D}_{k_{r+1}} \xi_i \cdot \tilde{\Theta}^i_{k_1 \ldots k_{r+1}},$$

where $\tilde{\Theta}^i_k$, $\tilde{\Theta}^i_{k_1 k_2}$, $\ldots$, $\tilde{\Theta}^i_{k_1 \ldots k_{r+1}}$ are some uniquely determined vector fields on $\pi^{-1}_{Q,r}(U)$ symmetric in the subscripts.

Let $\varkappa^{(r+1)}$: $\mathscr{F}^{r+1}X \oplus \mathscr{J}^r \mathscr{F}^1_Q X \to \mathrm{T}^r_n Q$ be the relative image map, let $y = j^{r+1}_0 \varphi^{-1} \in \mathscr{F}^{r+1}X$ be the $(r+1)$-frame at $x = \varphi^{-1}(0)$ defined by the chart (U, φ) on X. We shall prove the following

Theorem. *The vector fields* $\mathrm{T}\varkappa^{(r+1)}_y \cdot \tilde{\Theta}^i_k$, $\mathrm{T}\varkappa^{(r+1)}_y \cdot \tilde{\Theta}^i_{k_1 k_2}$, $\ldots$, $\mathrm{T}\varkappa^{(r+1)}_y \cdot \tilde{\Theta}^i_{k_1 \ldots k_{r+1}}$, $1 \leq k_1 \leq \ldots \leq k_{r+1} \leq n$, *span the Lie algebra of fundamental vector fields on the left* L^{r+1}_n*-space* $\mathrm{T}^r_n Q$.

Proof. By definition, $\mathscr{J}^r \mathscr{F}^1_Q X = \mathscr{F}^{r+1}_P X$, where $P = \mathrm{T}^r_n Q$. We shall verify that for every local diffeomorphism β of x, $\mathscr{F}^{r+1}_P \beta = j^r \mathscr{F}^1_Q \beta$. Choose $z = [j^{r+1}_0 \varphi^{-1}, j^r_0 f] \in \mathscr{F}^{r+1}_P X$, and write z in the form

$$z = j^r_{\varphi^{-1}(0)}(\mathscr{F}^1_Q \varphi^{-1} \circ (\mathrm{id} \times f) \circ \varphi).$$

Then by definition,

$$\begin{aligned}
\mathscr{F}^{r+1}_P \beta(z) &= [\mathscr{F}^{r+1}\beta(j^{r+1}_0 \varphi^{-1}), j^r_0 f] = [j^{r+1}_0(\beta\varphi^{-1}), j^r_0 f] \\
&= j^r_{\beta\varphi^{-1}(0)}(\mathscr{F}^1_Q \beta\varphi^{-1} \circ (\mathrm{id} \times f) \circ \varphi\beta^{-1}) \\
&= j^r_{\beta\varphi^{-1}(0)}(\mathscr{F}^1_Q \beta \circ \mathscr{F}^1_Q \varphi^{-1} \circ (\mathrm{id} \times f) \circ \varphi \circ \beta^{-1}) \\
&= j^r \mathscr{F}^1_Q \beta (j^r_{\varphi^{-1}(0)}(\mathscr{F}^1_Q \varphi^{-1} \circ (\mathrm{id} \times f) \circ \varphi)),
\end{aligned}$$

which shows that $\mathscr{F}^{r+1}_P \beta = j^r \mathscr{F}^1_Q \beta$. Consequently, for every vector field ξ on X the relation $\mathscr{F}^{r+1}_P \xi = j^r \mathscr{F}^1_Q \xi$ holds, and we see that we may apply the results of Section 3 to the bundle $\mathscr{J}^r \mathscr{F}^1_Q X$.

Let $x \in X$ be a point. According to Lemma 2, for every vector field ξ defined on a neighbourhood of x and such that $\xi(x)=0$,

$$\mathrm{T}\varkappa_y^{(r+1)} \cdot j^r \mathscr{F}_Q^1 \xi = \mathrm{D}_k \xi_i \cdot \mathrm{T}\varkappa_y^{(r+1)} \cdot \tilde{\Theta}_k^i + \ldots$$

$$+ \sum_{k_1 \leq \ldots \leq k_{r+1}} \mathrm{D}_{k_1} \ldots \mathrm{D}_{k_{r+1}} \xi_i \cdot \mathrm{T}\varkappa_y^{(r+1)} \cdot \tilde{\Theta}^i_{k_1 \ldots k_{r+1}},$$

where the derivatives of ξ_i are considered at the point $\varphi(x)=0$. Since the numbers $\mathrm{D}_k\xi_i, \ldots, \mathrm{D}_{k_1} \ldots \mathrm{D}_{k_{r+1}}\xi_i$ are independent, each vector field from the collection $\mathrm{T}\varkappa_y^{(r+1)} \cdot \tilde{\Theta}_k^i$, $\mathrm{T}\varkappa_y^{(r+1)} \cdot \tilde{\Theta}^i_{k_1 k_2}, \ldots, \mathrm{T}\varkappa_y^{(r+1)} \cdot \tilde{\Theta}^i_{k_1 \ldots k_{r+1}}$, $1 \leq k_1 \leq \ldots \leq k_{r+1} \leq n$, is a fundamental vector field on $\mathrm{T}_n^r Q$. Lemma 2 also shows that this collection of vector fields spans the Lie algebra of fundamental vector fields on $\mathrm{T}_n^r Q$.

5. Example. The purpose of this section is to study the fundamental vector fields on the 2-jet prolongation of the bundle of second order, covariant symmetric tensors. Our method should be compared with the direct approach [4].

Let $\pi_Q : \mathscr{F}_Q^1 X \to X$ be the bundle of second order, covariant symmetric tensors on an n-dimensional manifold X. Then the type fibre $Q = R^{n*} \odot R^{n*}$ is the symmetric tensor product of the dual vector spaces of R^n. Let e_i be the canonical basis of R^n, e^i the dual basis. To every tensor $g \in R^{n*} \odot R^{n*}$ there are uniquely associated the numbers $g_{ij}(g)$, where $g_{ij}(g) = g_{ji}(g)$, by the formula

$$g = g_{ij}(g)\, e^i \odot e^j.$$

The functions g_{ij}, where $1 \leq i \leq j \leq n$, are canonical coordinates on Q, associated to the canonical basis e_i of R^n. The corresponding canonical coordinates g_{ij}, $g_{k,ij}$, $g_{kl,ij}$, where $1 \leq i \leq j \leq n$, $1 \leq k \leq l \leq n$, are defined as follows. For $j_0^2 f \in \mathrm{T}_n^2 Q$ we set

$$g_{ij}(j_0^2 f) = g_{ij} f(0), \quad g_{k,ij}(j_0^2 f) = \mathrm{D}_k g_{ij} f(0), \quad g_{kl,ij}(j_0^2 f) = \mathrm{D}_k \mathrm{D}_l g_{ij} f(0).$$

The group $\mathrm{Gl}_n(R)$ acts on Q in the obvious way. Let a_k^i be the standard canonical coordinates on $\mathrm{Gl}_n(R)$, let $(a, g) \to a \cdot g$ denote the left action of $\mathrm{Gl}_n(R)$ on Q. In our coordinates,

$$g_{ij}(a \cdot g) = a_i^k(a^{-1}) \cdot a_j^l(a^{-1}) \cdot g_{kl}(g).$$

Let now (U, φ) be a chart on X, x_i its coordinate functions. To every $\tilde{g} \in \pi_Q^{-1}(U)$ one can associate the numbers $\tilde{g}_{ij}(\tilde{g})$, where $\tilde{g}_{ij}(\tilde{g}) = \tilde{g}_{ji}(g)$, by the relation

$$\tilde{g} = \tilde{g}_{ij}(\tilde{g})\, \mathrm{d}x_i \odot \mathrm{d}x_j.$$

The functions $\tilde{x}_i = x_i \pi_Q$, $\tilde{g}_{ij}$, where $1 \leq i \leq j \leq n$, are some local coordinates on $\mathscr{F}_Q^1 X$ which are said to be associated to the local coordinates x_i on X.

Let us describe the induced vector fields on $\mathscr{F}_Q^1 X$. Let $x \in X$, let (U, φ) be a chart on X such that $x \in U$ and $\varphi(x)=0$, let x_i be the corresponding local coordinates. Assume that ξ is a vector field on U such that $\xi(x)=0$, and denote by

β_t its local one-parameter group. In what follows we identify the groups $\mathrm{Gl}_n(R)$ and L^1_n. By definition,

$$\mathscr{F}^1\beta_t(j^1_0\varphi^{-1})=j^1_0\beta_t\varphi^{-1}=j^1_0\varphi^{-1}*j^1_0\varphi\beta_t\varphi^{-1},$$

where $j^1_0\varphi\beta_t\varphi^{-1}\in L^1_n$. Let $\tilde{g}\in\pi_Q^{-1}(x)$,

$$\tilde{g}=\tilde{g}_{ij}(\tilde{g})\,\mathrm{d}x_i\odot\mathrm{d}x_j=[j^1_0\varphi^{-1},\,\tilde{g}_{ij}(\tilde{g})e^i\odot e^j].$$

With the aid of the definition of the action of $\mathrm{Gl}_n(R)$ on Q,

$$\begin{aligned}\mathscr{F}^1_Q\beta_t(\tilde{g})&=[\mathscr{F}^1\beta_t(j^1_0\varphi^{-1}),\,\tilde{g}_{ij}(\tilde{g})e^i\odot e^j]\\&=[j^1_0\varphi^{-1}*j^1_0\varphi\beta_t\varphi^{-1},\,\tilde{g}_{ij}(\tilde{g})e^i\odot e^j]\\&=[j^1_0\varphi^{-1},\,a^k_i(j^1_0\varphi\beta_t^{-1}\varphi^{-1})a^l_j(j^1_0\varphi\beta_t^{-1}\varphi^{-1})\tilde{g}_{kl}(\tilde{g})e^i\odot e^j]\\&=[j^1_0\varphi^{-1},\,\mathrm{D}_ix_k\beta_t^{-1}\cdot\mathrm{D}_jx_l\beta_t^{-1}\cdot\tilde{g}_{kl}(\tilde{g})e^i\odot e^j]\\&=\mathrm{D}_ix_k\beta_t^{-1}\cdot\mathrm{D}_jx_l\beta_t^{-1}\cdot\tilde{g}_{kl}(\tilde{g})\,\mathrm{d}x_i\odot\mathrm{d}x_j,\end{aligned}$$

which means that in our local coordinates

$$\begin{aligned}&\tilde{x}_i(\mathscr{F}^1_Q\beta_t(\tilde{g}))=x_i\beta_t\pi_Q(\tilde{g}),\\&\tilde{g}_{ij}(\mathscr{F}^1_Q\beta_t(\tilde{g}))=\mathrm{D}_ix_k\beta_t^{-1}\cdot\mathrm{D}_jx_l\beta_t^{-1}\cdot\tilde{g}_{kl}(\tilde{g}).\end{aligned}$$

Writing

$$\xi=\xi_i\frac{\partial}{\partial\tilde{x}_i}$$

we immediately obtain

$$\mathscr{F}^1_Q\xi=\xi_i\frac{\partial}{\partial\tilde{x}_i}+\sum_{i\leqslant j}\Xi_{ij}\frac{\partial}{\partial\tilde{g}_{ij}},$$

where

$$\Xi_{ij}=\left\{\frac{\mathrm{d}}{\mathrm{d}t}\,\tilde{g}_{ij}\mathscr{F}^1_Q\beta_t\right\}_0=-\mathrm{D}_i\xi_k\cdot\tilde{g}_{kj}-\mathrm{D}_j\xi_k\cdot\tilde{g}_{ki}.$$

Our aim is to determine the 2-jet prolongation of $\mathscr{F}^1_Q\xi$. Using Lemma 1 we obtain

$$\begin{aligned}j^2\mathscr{F}^1_Q\xi=\xi_i\frac{\partial'}{\partial\tilde{x}_i}-\mathrm{D}_r\xi_s\cdot\tilde{\Theta}^s_r-\sum_{q\leqslant r}\mathrm{D}_q\mathrm{D}_r\xi_s\cdot\tilde{\Theta}^s_{qr}-\\-\sum_{p\leqslant q\leqslant r}\mathrm{D}_p\mathrm{D}_q\mathrm{D}_r\xi_s\cdot\tilde{\Theta}^s_{pqr},\end{aligned}$$

where the vector fields $\tilde{\Theta}^s_r$, $\tilde{\Theta}^s_{qr}$, $\tilde{\Theta}^s_{pqr}$ depend on $\tilde{g}_{ij}$, $\tilde{g}_{k,ij}$, $g_{kl,ij}$, and $\xi^i(0)=0$. Consider the relative image map $\varkappa^{(3)}$: $\mathscr{F}^3X\oplus\mathscr{J}^2\mathscr{F}^1_QX\to\mathrm{T}^2_nQ$ and put

$$\Theta^s_r=\mathrm{T}\varkappa^{(3)}_y\cdot\tilde{\Theta}^s_r,\quad\Theta^s_{qr}=\mathrm{T}\varkappa^{(3)}_y\cdot\tilde{\Theta}^s_{qr},\quad\Theta^s_{pqr}=\mathrm{T}\varkappa^{(3)}_y\cdot\tilde{\Theta}^s_{pqr},$$

where $y = j_0^3\varphi^{-1}$. Let $g_{ij}, g_{k,ij}, g_{kl,ij}$ be the canonical coordinates on T_n^2Q. Obviously

$$\tilde{g}_{ij} = g_{ij} \circ \varkappa_y^{(3)}, \quad \tilde{g}_{k,ij} = g_{k,ij} \circ \varkappa_y^{(3)}, \quad \tilde{g}_{kl,ij} = g_{kl,ij} \circ \varkappa_y^{(3)},$$

which implies that

$$T\varkappa_y^{(3)} \cdot \frac{\partial}{\partial \tilde{g}_{ij}} = \frac{\partial}{\partial g_{ij}}, \quad T\varkappa_y^{(3)} \cdot \frac{\partial}{\partial \tilde{g}_{k,ij}} = \frac{\partial}{\partial g_{k,ij}}, \quad T\varkappa_y^{(3)} \cdot \frac{\partial}{\partial \tilde{g}_{kl,ij}} = \frac{\partial}{\partial g_{kl,ij}}.$$

Taking into account these remarks one immediately obtains the following formulas, with necessary symmetrization on the right-hand side understood:

$$\Theta_r^s = \sum_{i \leq j} (\delta_{ir} g_{sj} + \delta_{jr} g_{si}) \frac{\partial}{\partial g_{ij}} + \sum_{i \leq j} (\delta_{ir} g_{p,sj} + \delta_{jr} g_{p,si} + \delta_{pr} g_{s,ij}) \frac{\partial}{\partial g_{p,ij}}$$

$$+ \sum_{p \leq q,\, i \leq j} (\delta_{ir} g_{pq,sj} + \delta_{jr} g_{pq,si} + \delta_{pr} g_{sq,ij} + \delta_{qr} g_{ps,ij}) \frac{\partial}{\partial g_{pq,ij}},$$

$$\Theta_{qr}^s = \sum_{i \leq j} (\delta_{ir} q_{sj} + \delta_{jr} q_{si}) \frac{\partial}{\partial g_{q,ij}} + \sum_{k \leq l,\, i \leq j} (\delta_{lq} \delta_{ir} g_{k,sj} + \delta_{kq} \delta_{ir} g_{l,sj}$$

$$+ \delta_{lq} \delta_{jr} g_{k,si} + \delta_{kq} \delta_{jr} g_{l,si} + \delta_{kq} \delta_{lr} g_{s,ij}) \frac{\partial}{\partial g_{kl,ij}},$$

$$\Theta_{pqr}^s = \sum_{i \leq j} (\delta_{ir} g_{sj} + \delta_{jr} g_{si}) \frac{\partial}{\partial g_{pq,ij}}.$$

In these formulas δ_{ij} denotes the Kronecker symbol.

These are the desired vector fields spanning the Lie algebra of fundamental vector fields on the left L_n^3-space $T_n^2(R^{n*} \odot R^{n*})$.

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ФУНДАМЕНТАЛЬНЫЕ ВЕКТОРНЫЕ ПОЛЯ НА ТИПОВЫХ СЛОЯХ ДЖЕТ-ПРОДОЛЖЕНИЙ ТЕНЗОРНЫХ РАССЛОЕНИЙ

Деметер Крупка

Резюме

Пусть $\mathscr{F}^1_Q X$ тензорное расслоение типа Q на многообразии X, где Q некоторое векторное пространство тензоров, $\mathscr{J}^r\mathscr{F}^1_Q X$ его r-джет продолжение. $\mathscr{J}^r\mathscr{F}^1_Q X$ рассматривается как расслоение с типовым слоем $T^r_n Q$ и структурной группой L^{r+1}_n, ассоцированное с расслоением $(r + 1)$-реперов на X. В этой работе обсуждается проблема нахождения фундаментальных векторных полей на $T^r_n Q$ связанных с действием группы L^{r+1}_n на $T^r_n Q$. Показывается, что эти векторные поля можно получить из джет-продолжений индуцированных векторных полей на $\mathscr{F}^1_Q X$.