Mária Tomková

On a local property of the unoriented graph of a modular multilattice

Mathematica Slovaca, Vol. 30 (1980), No. 3, 289--298

Persistent URL: http://dml.cz/dmlcz/136246

Math. Slovaca 30, 1980, No. 3, 289—298

ON A LOCAL PROPERTY OF THE UNORIENTED GRAPH OF A MODULAR MULTILATTICE

MÁRIA TOMKOVÁ

L. R. Alvarez [1] investigated unoriented graphs of modular lattices of finite length. In this note an analogous question for modular multilattices will be studied.

A partially ordered set P is said to be of locally finite length if each bounded chain in P is finite. Throughout this paper P will be a partially ordered set of locally finite length. For the elements $a, b \in P$ we write $a \succ b$ or $b \prec a$ (a covers b, or b is covered by a) if $a > b$ and there does not exist any element $c \in P$ with $a > c > b$. If two elements $a, b \in P$ are noncomparable, we write $a \| b$.

A subset S of a partially ordered set P is called a saturated subsystem of P if for each $x, y \in S$ we have $x \prec y$ in P, whenever $x \prec y$ in S.

By a graph $G(S)$ of a subset $S \subset P$ there is meant the unoriented graph (without multiple edges and loops) whose vertices are elements of S; two vertices a, b are joined by the edge (a, b) iff either a covers b or b covers a. In such a case we also say that a and b are neighours.

A circuit in the graph $G(S)$ is a sequence $\{(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)\}$ of distinct edges such that $x_0 = x_n$.

A graph isomorphic with the graph in Fig. 1a is called a graph of type K'. A graph isomorphic with the graph in Fig. 1b is said to be a cube.

L. R. Alvarez [1] proved the following result:

(A) *Let L be a modular lattice of finite length. Let $F \subset L$ and let $G(F)$ be of type K'. Then there exists an element $x \in L$ such that $G(F \cup \{x\})$ is a cube.*

In this paper we shall show that for modular multilattices of locally finite length the analogous assertion fails to be true. We shall investigate the question under what additional conditions for F the assertion remains valid.

We recall the basic definition concerning multilattices [2].

A multilattice is a poset M in which condition (i) and its dual (ii) are satisfied: (i) If $a, b, h \in M$ and $a \leqq h$, $b \leqq h$, then there exists $v \in M$ such that (a) $v \leqq h$, $v \geqq a$, $v \geqq b$ and (b) $z \in M$, $z \geqq a$, $z \geqq b$, $z \leqq v$ implies $z = v$.

The symbol $(a \vee b)_h$ designates the set of all elements $v \in M$ satisfying (i); the symbol $(a \wedge b)_d$ has a dual meaning. We denote

$$a \vee b = \bigcup_{\substack{h \geqq a \\ h \geqq b}} (a \vee b)_h, \quad a \wedge b = \bigcup_{\substack{d \leqq a \\ d \leqq b}} (a \wedge b)_d.$$

Recall that the sets $a \vee b$ or $a \wedge b$ may be empty. If M is directed, then $a \vee b \neq \emptyset \neq a \wedge b$ for each pair $a, b \in M$.

A multilattice M is said to be modular iff for every $a, b, c, d, h \in M$ satisfying $d \leqq a, b, c \leqq h, a \leqq b$ $h \in (a \vee c)_h$, $d \in (b \wedge c)_d$ we have $a = b$.

We shall describe all nonisomorphic types of partially ordered sets F, card $F = 7$ satisfying the following conditions:

(a) $G(F)$ is of type K';

(b) there exists a directed modular multilattice M_F of locally finite length such that F is a saturated subsystem of M_F.

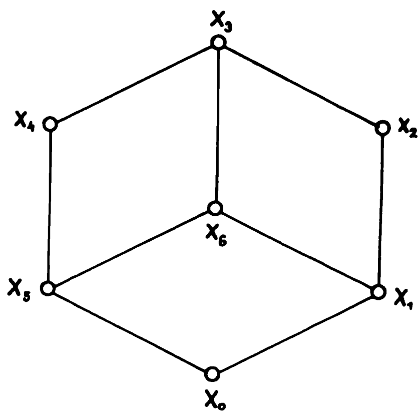

Fig. 1a

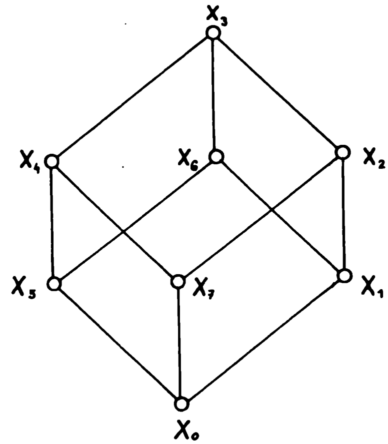

Fig. 1b

We denote by $\mathscr{F}$ the set of all nonisomorphic types of partially ordered sets as above. Let $F \in \mathscr{F}$. We shall say that F can be extended to a cube when it fulfils the following condition: Whenever M is a directed modular multilattice of locally finite length such that F is a saturated subsystem of M, then there exists an element $x \in M$, such that $G(F \cup \{x\})$ is a cube.

We shall determine the set of all partially ordered sets $F \in \mathscr{F}$ that can be extended to a cube (Theorems 3, 4).

Assume that M is a directed modular multilattice of locally finite length and F is a saturated subsystem of M such that the graph $G(F)$ contains a circuit $K_6 = \{(x_0, x_1), (x_1, x_2), \ldots, (x_4, x_5), (x_5, x_0)\}$ and the vertex x_6, which is joined by edges with three vertices of the circuit K_6 such that there exist no triangles in $G(F)$. Let us denote $K = \{x_0, x_1, \ldots, x_5\}$. In what follows the elements of K will be denoted by x_i, where i is an integer taken modulo 6.

In the Lemmas 1—4 we suppose that $G(K)$ is a circuit. (We do not use the property of the set F.)

Lemma 1. *The partially ordered set K has at most three minimal elements.*

Proof. If we assume that K has more than three minimal elements, then some two of them would be comparable and this is a contradiction.

Lemma 2. *Let the saturated subsystem K have the least element. Then K is isomorphic to the partially ordered set in Fig. 2.*

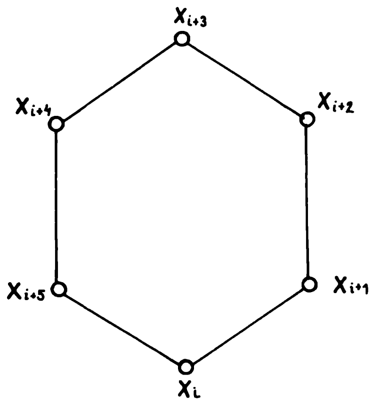

Fig. 2

Proof. Assume that x_i is the least element of K. Since x_i is a neighbour of the vertices x_{i+1}, x_{i+5}, we have $x_{i+1} > x_i$, $x_{i+5} > x_i$. The element x_{i+1} is joined by an edge with the vertex x_{i+2} and we shall show that $x_{i+2} > x_{i+1}$ is valid. Let $x_{i+2} < x_{i+1}$. Since x_i is the least element of K, we have $x_{i+2} > x_i$ contradicting $x_{i+1} > x_i$. By the same reasoning we get $x_{i+4} > x_{i+5}$. Now we will show that $x_{i+1} | x_{i+4}$. Assume that we would have $x_{i+1} < x_{i+4}$. Then there exists an element $z \in M$ such that $x_{i+1} < z < x_{i+4}$. This yields a contradiction, because the multilattice M is modular and hence all maximal chains from x_i to x_{i+4} must be of the same length. Similarly we cannot have $x_{i+1} > x_{i+4}$; thus $x_{i+1} | x_{i+4}$. Analogously we werify that the relation $x_{i+2} | x_{i+5}$ holds. This implies that $x_{i+4} | x_{i+2}$ and hence for the element x_{i+3}, which is joined by edges with x_{i+4}, x_{i+2}, we have the following possibility:

a) x_{i+3} covers x_{i+4} and x_{i+2}.

b) x_{i+3} is covered by x_{i+4} and x_{i+2}.

We shall show that the case b) is impossible. In fact, if $x_{i+3} < x_{i+4}$ and $x_{i+3} < x_{i+2}$, then either $x_{i+3} | x_i$ (and hence x_i would not be the least element of K), or $x_{i+3} > x_i$, which would imply that the maximal chains from x_i to x_{i+2} are not of the same length. Thus $x_{i+3} > x_{i+2}$, $x_{i+3} > x_{i+4}$ and the proof is complete.

Lemma 3. *Let the subsystem K have two minimal elements. Then it is isomorphic with some of the partially ordered sets in Figs. 3a, 3b, 3c.*

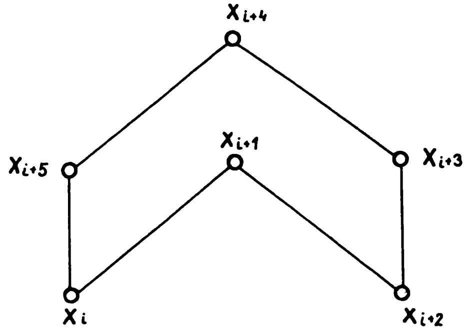

Fig. 3a

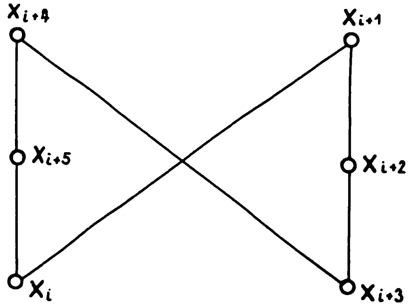

Fig. 3b

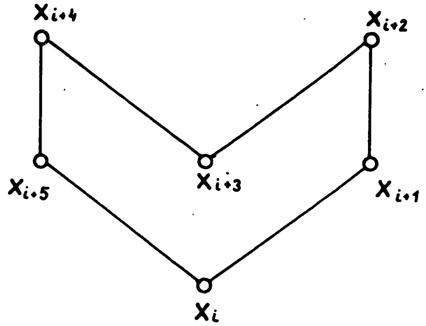

Fig. 3c

Proof. We may suppose that one of the pairs $\{x_i, x_{i+2}\}$, $\{x_i, x_{i+3}\}$ is the pair of minimal elements of K.

If x_i, x_{i+2} are minimal, then $x_i < x_{i+1}$, $x_i < x_{i+5}$, $x_{i+2} < x_{i+3}$, $x_{i+2} < x_{i+1}$. We shall show that in this case $x_{i+3} | x_{i+5}$. Since the multilattice M is directed and modular, there exists an element $u \in M$ such that $u < x_i$, $u < x_{i+2}$. If we assume $x_{i+3} < x_{i+5}$, then the maximal chains from u to x_{i+5} are not of the same length, contradicting the modularity of M. For the same reason the relation $x_{i+5} < x_{i+3}$ is impossible. This implies that for the element x_{i+4} we have to investigate the following cases:

3a) x_{i+4} covers the elements x_{i+3}, x_{i+5};

3a′) x_{i+4} is covered by x_{i+3}, x_{i+5}.

The case 3a′) cannot occur because the element x_{i+4} is not minimal. In the case 3a) we have obviously $x_{i+1} | x_{i+3}$, $x_{i+1} | x_{i+5}$. From the Jordan—Dedekind chain condition it follows $x_{i+1} | x_{i+4}$, $x_i | x_{i+3}$, $x_{i+2} | x_{i+5}$. Hence in this case the subsystem K is isomorphic with the partially ordered set in Fig. 3a.

If x_i, x_{i+3} are minimal elements, then $x_i < x_{i+1}$, $x_i < x_{i+5}$, $x_{i+3} < x_{i+2}$, $x_{i+3} < x_{i+4}$. Since x_{i+4}, x_{i+5} are joined by an edge and so are x_{i+1}, x_{i+2}, we have the following four possibilities:

3b$_1$) $x_{i+4} > x_{i+5}$ and $x_{i+1} > x_{i+2}$

3b$_2$) $x_{i+4} > x_{i+5}$ and $x_{i+2} > x_{i+1}$

3b$_1'$) $x_{i+4} < x_{i+5}$ and $x_{i+1} < x_{i+2}$

3b$_2'$) $x_{i+4} < x_{i+5}$ and $x_{i+2} < x_{i+1}$.

In all these cases we have evidently $x_{i+2}|x_{i+4}$, $x_{i+1}|x_{i+5}$. In the case 3b$_1$) $x_i|x_{i+2}$, $x_{i+5}|x_{i+3}$ is valid and we shall show $x_{i+1}|x_{i+5}$. The assumption $x_{i+1} > x_{i+5}$ would contradict $x_{i+4} > x_{i+3}$ and $x_{i+5} < x_{i+2}$ implies $x_{i+5} < x_{i+1}$, which contradicts $x_{i+5}|x_{i+1}$. Analogously we get $x_{i+1}|x_{i+4}$. Thus in the case 3b$_1$) the subsystem K is isomorphic with the partially ordered set in Fig. 3b. In the case 3b$_1'$) we arrive at the same conclusion.

In the case 3b$_2$) we have: $x_i|x_{i+3}$, $x_{i+3}|x_{i+5}$. From the modularity of the multilattice M it follows that $x_{i+1}|x_{i+4}$, $x_{i+5}|x_{i+2}$. This implies that the subsystem K is isomorphic with the partially ordered set in Fig. 3c. The case 3b$_2'$) yields the same conclusion.

The following assertion is obvious.

Lemma 4. *Let the subsystem K have three minimal elements. Then K is isomorphic with the partially ordered set in Fig. 4.*

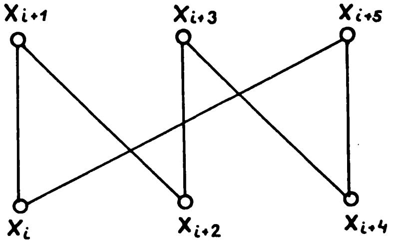

Fig. 4

The lemmas 1—4 may be summarized as follows.

Theorem 1. *Let M be a directed modular multilattice of locally finite length and let $K = \{x_0, x_1, ..., x_5\}$ be a saturated subsystem of M such that $G(K)$ is a circuit. Then K is isomorphic with some of the partially ordered sets in Figures 2, 3a, 3b, 3c, 4.*

Now we shall suppose that x_6 is an element of the multilattice M such that $G(K \cup \{x_6\})$ is of type K' (i.e., $K \cup \{x_6\} = F \in \mathcal{F}$).

Lemma 5. *The subsystem K cannot be isomorphic with the partially ordered set in Fig. 3b.*

Proof. Assume that the K would be isomorphic with the partially ordered set in Fig. 3b. Then there exists exactly one minimal element x_i of K that is a neighbour of x_6. Thus x_6 is a neighbour of x_{i+2}, and x_6 is a neighbour of x_{i+4}. Since $x_{i+4} > x_i$, we have $x_i < x_6 < x_{i+4}$. From $x_i|x_{i+2}$ it follows $x_{i+2} < x_6$. This implies $x_{i+2} < x_{i+4}$ contradicting $x_{i+2}|x_{i+4}$.

Lemma 6. *Let the subsystem K be isomorphic with the partially ordered set in Fig. 2. Then the partially ordered sets F and that in Fig. 5 are isomorphic or dually isomorphic.*

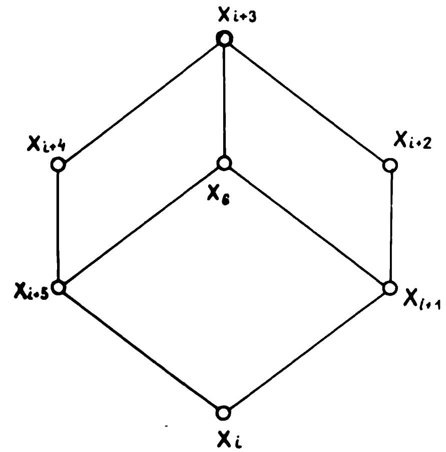

Fig. 5

Proof. Assume that x_i is a minimal element of K and that x_6 is joined by edges with x_{i+1}, x_{i+3}, x_{i+5}. From $x_{i+3} > x_{i+1}$, $x_{i+3} > x_{i+5}$ and from the modularity of M it follows $x_{i+1} < x_6 < x_{i+3}$, $x_{i+5} < x_6 < x_{i+3}$. We have obviously $x_6 | x_{i+2}$, $x_6 | x_{i+4}$. Hence the subsystem F is isomorphic with the partially ordered set in Fig. 5. If we assume that x_6 is joined with the elements x_i, x_{i+2}, x_{i+4} and that x_i is a minimal element of K, then by a similar argument we arrive at the conclusion that the subsystem F is dually isomorphic with the partially ordered set in Fig. 5.

Lemma 7. *Let the subsystem K be isomorphic with the partially ordered set in Fig. 3a (3c). Then the subsystem F is isomorphic (dually isomorphic) with the some of the partially ordered sets in Figs. 6a, 6b, 6c.*

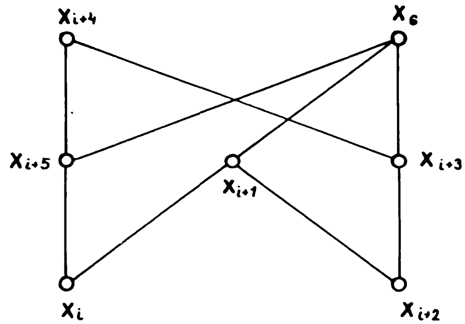

Fig. 6a

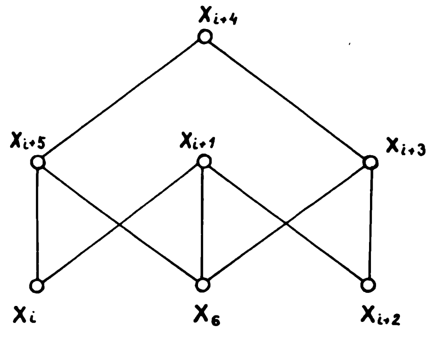

Fig. 6b

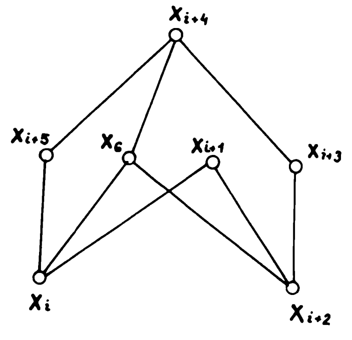

Fig. 6c

Proof. Let x_i, x_{i+2} be minimal elements of K. First we suppose that the element x_6 is joined by edges with the elements x_{i+1}, x_{i+3}, x_{i+5}. Since $x_{i+1}|x_{i+3}$, $x_{i+3}|x_{i+5}$, either x_6 covers x_{i+1}, x_{i+3}, x_{i+5} or x_6 is covered by x_{i+1}, x_{i+3}, x_{i+5}. In the first case we get $x_6|x_{i+4}$. Hence the subsystem F is isomorphic with the partially ordered set in Fig. 6a. In the second case we have $x_6|x_i$, $x_6|x_{i+2}$ and hence the subsystem F is isomorphic with the partially ordered set in Fig. 6b. If we assume that x_6 is joined by edges with the elements x_i, x_{i+2}, x_{i+4}, then from $x_{i+4}>x_i$, $x_{i+4}>x_{i+2}$ and from the modularity of M it follows $x_i<x_6<x_{i+4}$. Moreover $x_6|x_{i+5}$, $x_6|x_{i+1}$, $x_6|x_{i+3}$. Hence the subsystem F is isomorphic with the partially ordered set in Fig. 6c.

Since the partially ordered sets in Fig. 3c, 3a are dually isomorphic, the assertion concerning the partially ordered set in Fig. 3c can be proved by the dual way.

Lemma 8. *Let the subsystem K be isomorphic with the partially ordered set in Fig. 4a. Then the subsystem F is isomorphic or dually isomorphic with some of the partially ordered sets in Figs. 7a, 7b.*

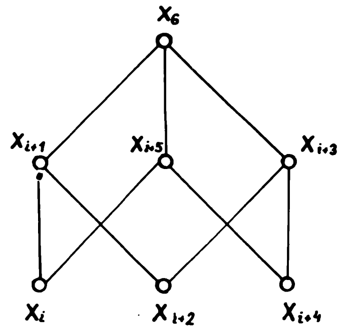

Fig. 7a

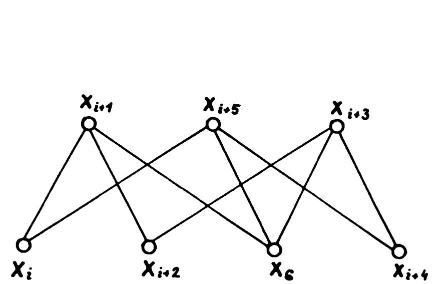

Fig. 7b

Proof. Let x_i, x_{i+2}, x_{i+4} be minimal elements of K. The element x_6 can be joined by edges with either x_i, x_{i+2}, x_{i+4} or with $x_{i+1}, x_{i+3}, x_{i+5}$. Since $x_i|x_{i+2}|x_{i+4}|x_i$ and $x_{i+1}|x_{i+3}|x_{i+5}|x_{i+1}$, in the first case the element x_6 either covers x_i, x_{i+2}, x_{i+4} or x_6 is covered by x_i, x_{i+2}, x_{i+4}. If x_6 is joined with $x_{i+1}, x_{i+3}, x_{i+5}$, we obtain an analogous result. Thus the proof is complete.

The lemmas 5—8 may be summarized as follows.

Theorem 2. *Let M be a directed modular multilattice of locally finite length. If M contains the saturated subsystem $F \in \mathcal{F}$, then F is isomorphic or dually isomorphic with some of the following partially ordered sets in Figs. 5, 6a, 6b, 6c, 7a, 7b.*

A graph which we obtain from the graph in Fig. 5 by adding of one vertex x_7 and edges (x_7, x_{i+2}), (x_7, x_{i+4}), (x_7, x_i), (x_7, x_6) will be called a cube with a diagonal.

Theorem 3. *Let M be a directed modular multilattice of locally finite length. If the saturated subsystem $F \in \mathcal{F}$ belonging to M is isomorphic or dually isomorphic with the partially ordered set in Fig. 5, then F can be extended either to a cube or to a cube with a diagonal.*

Proof. Let us assume that F is isomorphic with the partially ordered set in Fig. 5. Since $x_{i+2} < x_{i+3}$, $x_{i+4} < x_{i+3}$ and $x_i < x_{i+2}$, $x_i < x_{i+4}$, from the modularity of M it follows that there exists an element $x_7 \in M$ such that $x_7 \in (x_{i+2} \wedge x_{i+4})_{x_i}$, $x_7 < x_{i+2}$, $x_7 < x_{i+4}$, $x_i < x_7$. Moreover x_7 is not identical with any element of F, because F is a saturated subsystem of M. If $x_7 > x_6$, then from $x_i < x_6$ and from $x_i < x_7$ we obtain a contradiction. Thus either x_7 is noncomparable with x_6 or $x_7 < x_6$. In the first case $G(F \cup \{x_7\})$ is a cube and in the second one $G(F \cup \{x_7\})$ is a cube with a diagonal. In the case when F is dually isomorphic with the set in Fig. 5 the proof is analogous.

A multilattice M is said to be distributive [2] iff for every $a, b, b', d, h \in M$ satisfying $d \leq a, b, b' \leq h$, $h = (a \vee b)_h = (a \vee b')_h$ $d = (a \wedge b)_d = (a \wedge b')_d$ we have $b = b'$.

Corollary 1. *Let M be a directed distributive multilattice of locally finite length. If the saturated subsystem $F \in \mathcal{F}$ belonging to M is isomorphic or dually isomorphic with the partially ordered set in Fig. 5, then F can be extended to a cube.*

The proof follows directly from the Theorem 3 because from the distributivity of M we obtain $x_6|x_7$.

Corollary 2. *Let M be a directed modular multilattice of locally finite length and let a saturated subsystem $F \in \mathcal{F}$ belonging to M be isomorphic or dually isomorphic with the partially ordered set in Fig. 5. If $G(M)$ does not contain any subgraph isomorphic to a cube with a diagonal, then F can be extended to a cube.*

Theorem 4. *Let $F \in \mathcal{F}$ be a partially ordered set isomorphic or dually isomorphic to some of the partially ordered sets in Figs. 6a, 6b, 6c, 7a, 7b. Then there exists* *a directed modular multilattice M such that F is a saturated subsystem of M and F can be extended neither to a cube nor to a cube with a diagonal.*

Proof. The multilattices M_1, M_2 and M_3 in Fig. 8a, 8b, 8c are modular (the modularity of these multilattices can be proved analogously as in the case of the

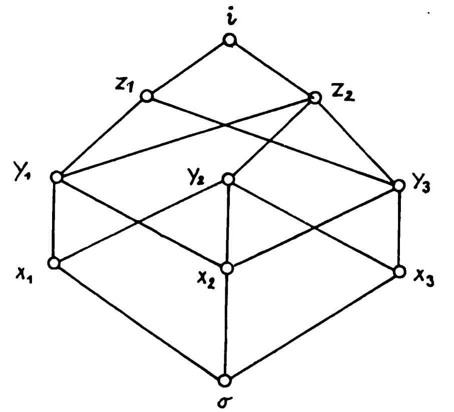

Fig. 8a

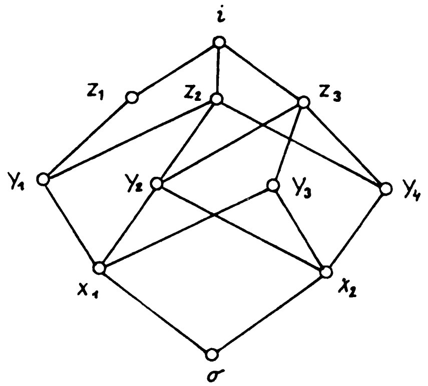

Fig. 8b

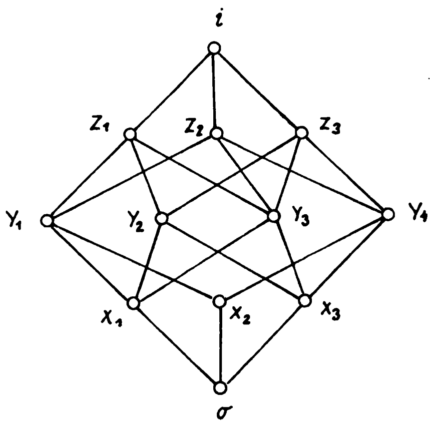

Fig. 8c

multilattice M_1 in [3]). The following saturated subsystems of M_i $(i=1, 2, 3)$ can be extended neither to a cube nor to a cube with a diagonal:

$$\{x_1, x_3, y_1, y_2, y_3, z_1, z_2\} \subset M_1$$
$$\{x_1, x_2, x_3, y_1, y_2, y_3, z_1\} \subset M_1$$
$$\{x_1, x_2, y_1, y_2, y_3, y_4, z_2\} \subset M_2$$
$$\{y_1, y_2, y_4, z_1, z_2, z_3, i\} \subset M_3$$
$$\{y_1, y_2, y_3, y_4, z_1, z_2, z_3\} \subset M_3.$$

REFERENCES

[1] ALVAREZ, L. R.: Undirected graphs realizable as graphs of modular lattices, Canadian Journal of Mathematics, XVII, 1965, 923—932.

[2] BENADO, M.: Les ensembles partiellement ordonnés et le théorème de raffinement de Schreier, II Théorie des multistructures, Czech. Math. J., 5 (80), 1955, 308—344.

[3] TOMKOVÁ, M.: Graph isomorphisms of modular multilattices, Math. Slovaca (to appear).

Received September 23, 1978

Katedra matematiky
Strojníckej fakulty VŠT
Švermova 9
040 01 Košice

ОД ОДНОМ ЛОКАЛЬНОМ СВОЙСТВЕ НЕОРИЕНТИРОВАННОГО ГРАФА МОДУЛЯРНЫХ МУЛЬТИСТРУКТУР

Мария Томкова

Резюме

Л. Р. Алварез доказал что если подграф $G(F)$ куба, который получится из куба удалением одной вершины и ребер с ней инцидентных является подграфом неориентированного графа $G(L)$ модулярной структуры L, которая имеет конечную длину, потом весь куб является подграфом графа этой структуры. В этой статье доказано, что для модулярных направленных мультиструктур аналогическое утверждение не имеет место а изучается вопрос, при которых дополнительных условиях для частично упорядоченного множества F утверждение правдиво.