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COHERENCE IN DUAL DISCRIMINATOR VARIETIES

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D. Geiger [5] (see also [6]) proved that every coherent variety is regular and permutable. W. Taylor [6] gave an example of a variety which is regular and permutable but not coherent (see also [2]). This stimulated our aim to investigate what condition has to be added to regularity and permutability to obtain coherence. This problem was solved in [2] for a general case. However, the recent results in dual discriminator varieties by H. Draškovičová [3] enable us to essentially simplify the characterization of [2] in the case of such varieties.

Let us first recall the concepts. A variety $\mathcal{V}$ of algebras is:

regular if every two congruences on each $\mathfrak{A} \in \mathcal{V}$ coincide whenever they have a congruence class in common;

coherent if for each subalgebra $\mathfrak{B}$ of every $\mathfrak{A} \in \mathcal{V}$; if $\mathfrak{B}$ contains a class of some $\Theta \in Con\,\mathfrak{A}$, then $\mathfrak{B}$ is a union of classes of Θ;

permutable if $\Theta_1 \cdot \Theta_2 = \Theta_2 \cdot \Theta_1$ for each two $\Theta_1, \Theta_2 \in Con\,\mathfrak{A}$ and each $\mathfrak{A} \in \mathcal{V}$;

discriminator if $\mathcal{V}$ has a ternary polynomial $p(x, y, z)$ such that $p(x, y, z)$ is the discriminator on each subdirectly irreducible member $\mathfrak{A}$ of $\mathcal{V}$, i.e. $p(x, y, z) = x$ if $x \neq y$ and $p(x, x, z) = z$ for each $x, y, z \in \mathfrak{A}$;

dual discriminator if $\mathcal{V}$ has a ternary polynomial $q(x, y, z)$ such that $q(x, y, z)$ is the dual discriminator on each subdirectly irreducible member $\mathfrak{A} \in \mathcal{V}$, i.e. $q(x, y, z) = z$ if $x \neq y$ and $q(x, x, z) = x$ for each $x, y, z \in \mathfrak{A}$, see [4].

Let $\mathfrak{A}$ be an algebra. By a *unary translation,* briefly *translation,* φ of $\mathfrak{A}$ we mean a unary algebraic function over $\mathfrak{A}$, i.e. φ is a mapping of $\mathfrak{A}$ into itself which arises from some n-ary polynomial $p(x_1, \ldots, x_n)$ $(n \geq 1)$ replacing $n-1$ of variables by fixed elements of $\mathfrak{A}$, thus

$$\varphi(x) = p(a_1, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_n) \text{ for } a_i \in \mathfrak{A}.$$

If $S \neq \emptyset$ is a subset of $\mathfrak{A}$ and φ is a translation of $\mathfrak{A}$, denote by $\varphi(S) = \{\varphi(b);\, b \in S\}$.

Definition. *An algebra $\mathfrak{A}$ has subalgebras closed under translations of principal congruence classes (briefly SCT) if for each subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, each element $x \in \mathfrak{A}$, $y, z \in \mathfrak{B}$ and each translation φ of $\mathfrak{A}$,*

$$[z]_{\Theta(x,y)} \subseteq \mathfrak{B} \text{ and } \varphi(z) = y \text{ imply } \varphi([z]_{\Theta(x,y)}) \subseteq \mathfrak{B}.$$

A variety $\mathcal{V}$ has SCT if each $\mathfrak{A} \in \mathcal{V}$ has SCT.

Example 1. *The variety $\mathcal{A}$ of all Abelian groups has* SCT.

Proof. Let $\mathfrak{A} \in \mathcal{A}$, $\mathfrak{B}$ be a subgroup of $\mathfrak{A}$ and $x \in \mathfrak{A}$, $y, z \in \mathfrak{B}$ and φ be a translation of $\mathfrak{A}$. Denote by $\mathfrak{N}_{xy}$ the subgroup of $\mathfrak{A}$ induced by $\Theta(x, y)$. Clearly $\mathfrak{N}_{xy} = \{x^k \cdot y^{-k};\ k = 0, \pm 1, \pm 2, \ldots\}$. Further, $\varphi(v) = a \cdot v^r$, where $a \in \mathfrak{A}$ and $r \in \{\pm 1, \pm 2, \ldots\}$. We have $w \in [z]_{\Theta(x,y)}$ if and only if $w \cdot z^{-1} \in \mathfrak{N}_{xy}$, i.e. if $w = z \cdot x^k \cdot y^{-k}$ for some k. Suppose now $\varphi(z) = y$ and $[z]_{\Theta(x,y)} \subseteq \mathfrak{B}$. Then $a \cdot z^r = y$ and $z \cdot x^k \cdot y^{-k} \in \mathfrak{B}$ for $k \in \{0, \pm 1, \pm 2, \ldots\}$. If $c \in [z]_{\Theta(x,y)}$, then $c = z \cdot x^n \cdot y^{-n}$. We compute $\varphi(c) = a \cdot c^r = a \cdot z^r (x^n \cdot y^{-n})^r = y(x^m \cdot y^{-m}) = y \cdot z^{-1} (z \cdot x^m \cdot y^{-m})$. Since $y \in \mathfrak{B}$, $z^{-1} \in \mathfrak{B}$ and $z \cdot x^m \cdot y^{-m} \in [z]_{\Theta(x,y)} \subseteq \mathfrak{B}$, we conclude $\varphi(c) \in \mathfrak{B}$ proving $\varphi([z]_{\Theta(x,y)}) \subseteq \mathfrak{B}$.

Remark. Since the variety of all Abelian groups is permutable and regular, there is the question whether SCT, permutability and regularity are independent properties. The following example shows that the property SCT does not depend on permutability and/or regularity.

Example 2. *The variety $\mathcal{S}$ of semilattices has* SCT.

Proof. Let $\mathfrak{A} \in \mathcal{S}$, $\mathfrak{B}$ be a subsemilattice of $\mathfrak{A}$ and $x \in \mathfrak{A}$, $y, z \in \mathfrak{B}$. Suppose $[z]_{\Theta(x,y)} \subseteq \mathfrak{B}$ and let φ be a translation of $\mathfrak{A}$. Then either $\varphi(v) = v$ or $\varphi(v) = a \vee v$ for some $a \in \mathfrak{A}$. The first case is trivial. Suppose the second case and let $\varphi(z) = y$ and $[z]_{\Theta(x,y)} \subseteq \mathfrak{B}$. Then $a \vee z = y$, hence $a \leqslant y$, $z \leqslant y$, i.e. $\varphi(y) = a \vee y = y$. By the description of $\Theta(x, y)$ (e.g. by the Mal'cev Lemma), $z \leqslant y$ implies either

$$[z]_{\Theta(x,y)} = \{z\} \quad \text{or} \quad z \in [y]_{\Theta(x,y)}.$$

In the first case, $\varphi([z]_{\Theta(x,y)}) = \varphi(\{z\}) = \{y\} \subseteq \mathfrak{B}$. In the second case, $z \in [y]_{\Theta(x,y)}$. If $c \in [z]_{\Theta(x,y)}$, then also $c \in [y]_{\Theta(x,y)}$, whence $\langle c, y \rangle \in \Theta(x, y)$, thus

$$\langle \varphi(c), y \rangle = \langle \varphi(c), \varphi(y) \rangle \in \Theta(x, y).$$

Accordingly, $\varphi(c) \in [y]_{\Theta(x,y)} = [z]_{\Theta(x,y)} \subseteq \mathfrak{B}$ proving the inclusion $\varphi([z]_{\Theta(x,y)}) \subseteq \mathfrak{B}$.

We can now formulate the main result:

Theorem. *Let $\mathcal{V}$ be a dual discriminator variety. The following conditions are equivalent:*

(1) *$\mathcal{V}$ is coherent;*

(2) *$\mathcal{V}$ is regular and $\mathcal{V}$ has* SCT;

(3) *$\mathcal{V}$ is permutable and $\mathcal{V}$ has* SCT.

Before the proof, let us formulate three lemmas.

Lemma 1. *Let $\mathcal{V}$ be a discriminator variety, $\mathfrak{A} \in \mathcal{V}$ and x, y, z be elements of $\mathfrak{A}$. Let $p(x, y, z)$ be the discriminator on subdirectly irreducible members of $\mathcal{V}$. Then $\Theta(x, y) = \Theta(z, p(x, y, z))$.*

Proof. By Theorem 3.1 in [4], $p(x, x, z) = z$ and hence

$$\langle z, p(x, y, z)\rangle = \langle p(x, x, z), p(x, y, z)\rangle \in \Theta(x, y)$$

giving $\Theta(z, p(x, y, z)) \subseteq \Theta(x, y)$. Prove the converse inclusion. If $x = y$, then this inclusion is trivially true. Suppose $x \neq y$ and $p(x, y, z) = z$. Since $\mathfrak{A}$ is a subdirect product of subdirectly irreducible algebras, say $\{\mathfrak{A}_i; i \in I\}$, then $\mathfrak{A} \subseteq \prod_{i \in I} \mathfrak{A}_i$. Denote by w_i the projection of w onto the ith coordinate $\mathfrak{A}_i$. Clearly $p(x, y, z) = z$ gives $p(x_i, y_i, z_i) = z_i$ onto each $\mathfrak{A}_i$. Since $x \neq y$, there exists $j \in I$ with $x_j \neq y_j$. Since p is the discriminator on $\mathfrak{A}_j$, we have $p(x_j, y_j, z_j) = x_j$. This implies $z_j = x_j$; however, Theorem 3.1 in [4] then implies $p(x_j, y_j, z_j) = p(x_j, y_j, x_j) = y_j$, thus $x_j = y_j$, which is a contradiction. Accordingly, $x \neq y$ implies $p(x, y, z) \neq z$, thus $p(x, y, z) = z$ if and only if $x = y$. Suppose now the factor algebra $\mathfrak{A}/\Theta = \mathfrak{B}$ for $\Theta = \Theta(z, p(x, y, z))$. Then

$$[z]_\Theta = [p(x, y, z)]_\Theta = p([x]_\Theta, [y]_\Theta, [z]_\Theta).$$

Since $\mathfrak{B} \in \mathcal{V}$, the foregoing result yields

$$[x]_\Theta = [y]_\Theta,$$

i.e. $\langle x, y\rangle \in \Theta = \Theta(z, p(x, y, z))$, proving the converse inclusion.

Let $\mathfrak{A} = (A, F)$ be an algebra and R be a binary relation on A. R is called a *diagonal relation on* $\mathfrak{A}$ if it is reflexive and R has the substitution property with respect to F, i.e. if R is a subalgebra of the direct product $\mathfrak{A} \times \mathfrak{A}$. Since diagonal relations on $\mathfrak{A}$ form a complete lattice with respect to the set inclusion, there exists the *least diagonal relation on* $\mathfrak{A}$ *containing the given pair* $\langle a, b\rangle \in \mathfrak{A} \times \mathfrak{A}$; denote it by $D(a, b)$.

Lemma 2. *Let $\mathfrak{A}$ be an algebra and a, b be elements of $\mathfrak{A}$. $\langle x, y\rangle \in D(a, b)$ if and only if there exists a translation φ of $\mathfrak{A}$ with $x = \varphi(a)$, $y = \varphi(b)$.*

The proof is clear and hence omitted (it is also implicitely contained in the proof of the Theorem in [2]).

Lemma 3. *Let $\mathcal{V}$ be a dual discriminator variety. The following conditions are equivalent:*

(a) *$\mathcal{V}$ is regular;*
(b) *$\mathcal{V}$ is permutable;*
(c) *$\mathcal{V}$ is discriminator variety;*
(d) *$\mathcal{V}$ is arithmetical.*

Proof. (a) is equivalent to (b) by Corollary 2.3 in [3] and (b) is equivalent to (c) by (iii) of Lemma 2.2 in [4]. Moreover, (i) of Lemma 2.2 in [4] gives (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (b) is trivial.

Proof of the Theorem. (1) $\Rightarrow$ (2): If $\mathscr{V}$ is coherent, then (by [5], see also [2] and [6]) $\mathscr{V}$ is regular and permutable. It remains to prove that $\mathscr{V}$ has SCT. Let $\mathfrak{A} \in \mathscr{V}$, $x \in \mathfrak{A}$, $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$ and $y, z \in \mathfrak{B}$. Let φ be a translation of $\mathfrak{A}$ and let

$$[z]_{\Theta(x,y)} \subseteq \mathfrak{B} \text{ and } \varphi(z) = y.$$

Since $[z]_{\Theta(x,y)}$ is a congruence class, it implies $\langle a, b\rangle \in \Theta(x, y)$ for each $a, b \in \varphi([z]_{\Theta(x,y)})$, i.e. there exists a congruence class C of $\Theta(x, y)$ with $\varphi([z]_{\Theta(x,y)}) \subseteq C$. Since $y \in \varphi([z]_{\Theta(x,y)}) \cap \mathfrak{B}$, we have $y \in C \cap \mathfrak{B}$ and the coherence of $\mathfrak{A}$ implies $C \subseteq \mathfrak{B}$ giving $\varphi([z]_{\Theta(x,y)}) \subseteq \mathfrak{B}$. Accordingly, $\mathfrak{A}$ has SCT.

(2) $\Rightarrow$ (3) follows directly from Lemma 3.

(3) $\Rightarrow$ (1): Permutability of $\mathscr{V}$ implies by Lemma 3 that $\mathscr{V}$ is a regular and discriminator variety. Let $p(x, y, z)$ be the ternary polynomial which is the discriminator on subdirectly irreducible members of $\mathscr{V}$. Denote by $\mathfrak{F}_3(x, y, z)$ the free algebra of $\mathscr{V}$ with free generators x, y, z. By Lemma 1 we have $\Theta(x, y) = \Theta(z, p(x, y, z))$, i.e.

$$\langle y, x\rangle \in \Theta(z, p(x, y, z)).$$

Since $\mathscr{V}$ is permutable, the Theorem of Werner [7] implies that $\Theta(z, p(x, y, z))$ coincides with the diagonal relation $D(z, p(x, y, z))$, i.e.

$$\langle y, x\rangle \in D(z, p(x, y, z)).$$

By Lemma 2, there exists a translation φ over $\mathfrak{F}_3(x, y, z)$ such that $y = \varphi(z)$, $x = \varphi(p(x, y, z))$. Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{F}_3(x, y, z)$ generated by the set $\{y\} \cup [z]_{\Theta(x,y)}$. Since $y \in \mathfrak{B}$, $[z]_{\Theta(x,y)} \subseteq \mathfrak{B}$ and $\varphi(z) = y$, the condition SCT implies $\varphi([z]_{\Theta(x,y)}) \subseteq \mathfrak{B}$, thus also $x = \varphi(p(x, y, z)) \in \mathfrak{B}$. Then there exist an integer $n \geqslant 0$, an $(n+1)$-ary polynomial h and elements $d_1, \ldots, d_n \in \mathfrak{F}_3(x, y, z)$ such that $d_i \in [z]_{\Theta(x,y)}$ and $x = h(y, d_1, \ldots, d_n)$. Since $d_i \in \mathfrak{F}(x, y, z)$, there exist ternary polynomials $t_1, \ldots, t_n$ such that $d_i = t_i(x, y, z)$ and $d_i \in [z]_{\Theta(x,y)}$ gives $t_i(x, x, z) = z$. In the summary,

$$x = h(y, t_1(x, y, z), \ldots, t_n(x, y, z))$$

with $t_i(x, x, z) = z$, which, by [5], is the Mal'cev condition for coherence.

Corollary. *A discriminator variety $\mathscr{V}$ is coherent if and only if $\mathscr{V}$ has* SCT.

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КОГЕРЕНТНОСТЬ В ДАУЛЬНО ДИСКРИМИНАТОРНЫХ МНОГООБРАЗИЯХ

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Резюме

В работе находится условие, которое надо к условиям регулярности и перестановочности конгруэнции, чтобы получить когерентность дуально дискриминаторного многообразия.