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GLEASON THEOREM FOR SIGNED MEASURES WITH INFINITE VALUES

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The Gleason theorem for signed measures with infinite values on the lattice of all closed subspaces of a Hilbert space whose dimension is a nonreal measurable cardinal number $\neq 2$ is proved.

1. Introduction

Let H be a Hilbert space over the field of real or complex numbers. Let $\mathscr{L}(H)$ denote the orthocompleted complete lattice of all closed subspaces of H. A signed measure on $\mathscr{L}(H)$ is a function m: $\mathscr{L}(H) \to R_1 \cup \{-\infty\} \cup \{+\infty\}$ such that (i) m is σ-additive on all sequences of mutually orthogonal elements; (ii) $m(O)=0$; (iii) from the values $\pm\infty$ it attains only one; for the sake of definitness we consider $+\infty$ as the possible value.

The famous Gleason theorem [5] asserts that a positive signed measure m on a separable Hilbert space H, $\dim H \geqslant 3$, is induced by a positive Hermitean trace operator T on H via the formula $m(P)=tr(TP)$ (we identify P with the projector T^P onto P). Sherstnev [8] and the author [2] generalized this theorem to bounded signed measures. Eilers and Horst [4] and Drish [1] proved this theorem for positive signed measures and for bounded signed measures on a non-separable Hilbert space. Sherstnev [7] studied positive measures on ideals of a Hilbert space. Positive signed measures on $\mathscr{L}(H)$ with $m(H)=\infty$ are investigated by Lugovaja and Sherstnev in [6].

Here we show that the Gleason theorem may be generalized to signed measures with $m(H)=\infty$ on a Hilbert space whose dimension is a nonreal measurable cardinal number $\neq 2$. We recall that a cardinal I is said to be real measurable iff there exists a positive measure $\mu \neq 0$ on the power set of I with $\mu(\{\alpha\})=0$ for each $\alpha \in I$.

2. Notations

Let H be a Hilbert space over the field C with the elements $x, y, \ldots$, whose dimension is a nonreal measurable cardinal $\neq 2$. By $\|x\|=(x, x)^{1/2}$ we denote the norm of $x \in H$. We call T an operator if it is linear on H and its domain, $\mathscr{D}(T)$, is dense in H. If $O \neq x \in H$, then by P_x we denote the subspace of H generated by x. By $Tr(H)$ we denote the class of all bounded operators T on H such that for every orthonormal basis $\{x_a, a \in I\}$ of H the series $\sum_{a \in I}(Tx_a, x_a)$ absolutely converges and it is independent of the basis. If $P_a \in L(H)$, $a \in A$, then the equality $P = \bigoplus_{a \in A} P_a$ means that $\{P_a, a \in A\}$ are mutually orthogonal elements with the join P.

A bilinear form is a function t: $\mathscr{D}(t) \times \mathscr{D}(t) \to C$, where $\mathscr{D}(t)$ is a linear submanifold of H (not necessarily dense or closed in H, named the domain of definition of t), such that it is linear in the first argument and antilinear in the second.

If $t(x, y) = \overline{t(y, x)}$ for all $x, y \in \mathscr{D}(t)$, then t is said to be symmetric (SBF); if for a SBF t we have $t(x, x) \geq 0$ for all $x \in \mathscr{D}(t)$, then it is said to be positive. Let t be a SBF and $B \geq 0$ be a selfadjoint operator. Then by $t \circ B$ we mean a SBF defined by $t \circ B(x, y) = t(B^{1/2}x, B^{1/2}y)$ when the corresponding assumptions on the domains of t and $B^{1/2}$ are satisfied.

3. Signed measures on ideals

A nonempty class $\mathscr{M} \subset \mathscr{L}(H)$ is said to be an ideal if

(i) $\mathscr{M} \neq \{0\}$;

(ii) if $Q \leq P$, $P \in \mathscr{M}$, then $Q \in \mathscr{M}$;

(iii) $P \oplus Q \in \mathscr{M}$ whenever P, $Q \in \mathscr{M}$:

(iv) if P_x, $P_y \in \mathscr{M}$, then $P_x \vee P_y \in \mathscr{M}$.

If $\mathscr{M}$ is an ideal, then $\mathscr{D}_{\mathscr{M}} = \{x \in H \colon P_x \in \mathscr{M}\}$ is a linear submanifold of H. Define $P_{\mathscr{M}} = \vee\{P \colon P \in \mathscr{M}\}$. A map $m \colon \mathscr{M} \to R_1$ is a signed measure if $m\left(\bigoplus_{i=1}^{\infty} P_i\right) = \sum_{i=1}^{\infty} m(P_i)$.

A signed measure m is $\mathscr{M}$-bounded if $\sup_{Q \leq P} |m(Q)| < \infty$ for every $P \in \mathscr{M}$.

Theorem 3.1. *The σ-additivity of an $\mathscr{M}$-bounded signed measure m on an ideal $\mathscr{M}$ of H whose dimension is a nonreal measurable cardinal implies the total additivity of m, i.e.,*

$$m\left(\bigoplus_{a \in A} P_a\right) = \sum_{a \in A} m(P_a)$$

whenever $P_a \in \mathscr{M}$, $a \in A \subset I$, and $\bigoplus_{a \in A} P_a \in \mathscr{M}$.

Proof. If we put $P = \bigoplus_{a \in A} P_a$, then our theorem follows from the results of Drish [1] applied to the restriction of m to $\mathscr{L}(P)$. Q.E.D.

Theorem 3.2. *Let m be an $\mathcal{M}$-bounded measure on an ideal $\mathcal{M}$ with* $\dim P_{\mathcal{M}} \geqslant 3$ *of a Hilbert space H whose dimension is a nonreal measurable cardinal number. Then there exists a unique SBF t determined on $\mathcal{D}_{\mathcal{M}}$ such that $t \circ P \in Tr(H)$ whenever $P \in \mathcal{M}$, and*

$$m(P) = tr(t \circ P). \tag{3.1}$$

Conversely, every SBF t defined on $\mathcal{D}_{\mathcal{M}}$ for which $t \circ P \in Tr(H)$ whenever $P \in \mathcal{M}$, determines by (3.1) *an $\mathcal{M}$-bounded signed measure on $\mathcal{M}$.*

Proof. Let $O \neq P \in \mathcal{M}$. Then there is $P' \in \mathcal{M}$, $\dim P' \geqslant 3$, such that $P \leqslant P'$. Hence we may assume that $\dim P \geqslant 3$. Due to the theorem of Drish [1] we conclude that $m_P = m|\mathcal{L}(P)$ is a bounded signed measure on $\mathcal{L}(P)$ and therefore there is a Hermitean operator $\bar{T}_P \in Tr(P)$ such that $m_P(Q) = tr(\bar{T}_P Q)$, $Q \leqslant P$. The operator $\bar{T}_P$ may be extended to a Hermitean trace operator $\bar{T}_P$ on H with $m(Q) = tr(\bar{T}_P Q)$ if $Q \leqslant P$.

Define a SBF t on $\mathcal{D}_{\mathcal{M}}$ via the identity $t(x, y) = (\bar{T}_P x, y)$ where $P \geqslant P_x \vee P_y$, $x, y \in \mathcal{D}_{\mathcal{M}}$, $\dim P \geqslant 3$, $P \in \mathcal{M}$. The corectness of this definition can be verified as follows. If $x \in \mathcal{D}_{\mathcal{M}}$ and $x \in P_i$, $i = 1, 2$, then

$$(\bar{T}_{P_1} x, x) = \|x\|^2 m(P_x) = (\bar{T}_{P_2} x, x).$$

Now let $P \in \mathcal{M}$. Then

$$\sup_{\|x\|=1} |t \circ P(x, x)| = \sup_{\|x\|=1} |t(Px, Px)| = \sup_{\|x\|=1} \|P_x\| m(P_x) \leqslant \sup_{Q \leqslant P} |m(Q)| < \infty.$$

Therefore there exists a unique Hermitean operator T_P on H such that

$$(T_P x, y) = t \circ P(x, y), \quad x, y \in H. \tag{3.2}$$

Let $\{x_a, a \in I\}$ be an orthonormal basis of H whose subset, $\{y_b\}$, is a basis of P. Then

$$\sum_{a \in I} (T_P x_a, x_a) = \sum_b t(y_b, y_b) = \sum_b m(P_{y_b}) = m(P).$$

Hence the series on the left-hand side of the above equality converges absolutely by Theorem 3.1 and $T_P \in Tr(H)$.

Now let t be a SBF in $\mathcal{D}_{\mathcal{M}}$. Define a function m on $\mathcal{M}$ by $m(P) = tr(t \circ P)$, $P \in \mathcal{M}$. Then m is an $\mathcal{M}$-bounded signed measure on $\mathcal{M}$. Q.E.D.

Corollary 3.3. *Under the conditions of Theorem 3.2 an $\mathcal{M}$-bounded signed measure on an ideal $\mathcal{M}$ exists iff there is a unique system of Hermitean operators $\{T_P, P \in \mathcal{M}\}$ such that*

$$tr(T_P Q) = tr(T_Q), \quad Q \leqslant P. \tag{3.3}$$

In this case $m(P) = tr(T_P)$, $P \in \mathcal{M}$.

Note 1. The assumption $\dim P_{\mathcal{M}} \geqslant 3$ is not superfluous.

Note 2. An $\mathcal{M}$-bounded signed measure m is not necessarily bounded. Indeed, let $\dim H \geq \aleph_0$, $\mathcal{M} = \{P: \dim P < \infty\}$ and $m(P) = \dim P$, $P \in \mathcal{M}$. Then m is an $\mathcal{M}$-bounded signed measure and $\sup_{P \in \mathcal{M}} m(P) = \infty$.

A sequence of signed measures $\{m_n\}$ on an ideal $\mathcal{M}$ converges weakly to a signed measure m if $\lim_n m_n(P) = m(P)$ for each $P \in \mathcal{M}$.

Theorem 3.4. *The class of all $\mathcal{M}$-bounded signed measures defined on an ideal $\mathcal{M}$ with* $\dim P_{\mathcal{M}} \geq 3$ *of a Hilbert space H, whose dimension is a nonreal measurable cardinal number, is a weakly sequentially complete real vector space.*

Proof. Let $\{m_n\}$ be a sequence of $\mathcal{M}$-bounded signed measures on an ideal $\mathcal{M}$, and let there exist a finite limit $\lim_n m_n(P) = m(P)$ for each $P \in \mathcal{M}$. We show that m is an $\mathcal{M}$-bounded signed measure on $\mathcal{M}$. Let $P = \bigoplus_{i=1}^{\infty} P_i$. Then the restrictions of m_n to $\mathcal{L}(P)$, $n \geq 1$ satisfy the Nikodym theorem for logics [3]. Therefore m is a signed measure and the σ-additivity of m_n is uniform with respect to n. Corollary 3.3 implies that for any n there is a SBF t_n and a system of trace operators $\{T_P^n, P \in \mathcal{M}\}$. Let $x \in H$, $P \in \mathcal{M}$ and $y = Px$. Then

$$(T_P^n x, x) = t_n(Px, Px) = t_n(y, y) = \|y\|^2 m_n(P_y) \to \|y\|^2 m(P_y).$$

Therefore the limit $T_P = w - \lim_n T_P^n$ exists, and T_P is a Hermitean operator of trace class.

Now if $Q \leq P$ and $\{x_a\}$ is an orthonormal basis in Q, then

$$tr(T_P Q) = \sum_a (T_P x_a, x_a) = \sum_a m(P_{x_a}) = m(Q),$$

and m is an $\mathcal{M}$-bounded signed measure. Q.E.D.

Note 3. If $\mathcal{M} = \mathcal{L}(H)$ and m is a bounded signed measure, then there are two measures $m_1, m_2 \geq 0$ such that $m = m_1 - m_2$. The author does not know whether a similar result is valid for measures on an arbitrary ideal $\mathcal{M} \neq \mathcal{L}(H)$. This problem is related to the decomposition of a SBF t into positive SBFs t_1, t_2 such that $t = t_1 - t_2$.

4. Signed measures on $\mathcal{L}(H)$

Now we shall investigate signed measures m: $\mathcal{L}(H) \to R_1 \cup \{\infty\}$, such that $m(H) = \infty$. Clearly, if $m(P) < \infty$ and $Q \leq P$, then $m(Q) < \infty$. A signed measure m is said to be (i) f-bounded if $\sup_{Q<P} |m(Q)| < \infty$ whenever $m(P) < \infty$; (ii) σ-finite if there is a sequence of mutually orthogonal elements $\{P_i\}$ such that $m(P_i)<\infty$, $i\geqslant 1$, and $H=\bigoplus_{i=1}^{\infty} P_i$.

As within the proof of the Gleason theorem, the basic role is played by the Hilbert space of dimension three. The next fundamental Lemma 4.1 may be proved in the same way as the Lugovaja—Sherstnev lemma for the positive signed measures with $m(H)=\infty$ [6].

Lemma 4.1. *Let* $\dim H=3$ *and let* m *be a signed measure on* $\mathscr{L}(H)$, $m(H)=\infty$. *For any* $P\in\mathscr{L}(H)$ *with* $\dim P=2$ *and* $m(P)<\infty$, *if a one-dimensional* Q *satisfies* $m(Q)<\infty$, *then* $Q\leqslant P$.

Corollary 4.2. *Let* $\dim H=n\geqslant 3$ *and let* m *be a signed measure on* $\mathscr{L}(H)$ *with* $m(H)=\infty$. *Let* P *satisfy* $\dim P=n-1$ *and* $m(P)<\infty$. *If* Q *is an onedimensional subspaces of* H *with* $m(Q)<\infty$, *then* $Q\leqslant P$.

Proof. If $\dim H=3$, the assertion follows from Lemma 4.1. There remains the case of $\dim H>3$. It is clear that $Q\perp P$. Since $\dim(Q^{\perp}\wedge P)\geqslant 2$, where $Q^{\perp}$ denotes the orthocomplement of Q in H, there is $x_1\in Q^{\perp}\wedge P$. We have $Q\perp P_{x_1}\perp P^{\perp}$ and $\dim(Q\vee P_{x_1}\vee P^{\perp})=3$. From Lemma 4.1 we conclude that

$$m(Q\vee P_{x_1}\vee P^{\perp})=\infty. \tag{4.1}$$

On the other hand, dim $((Q\vee P_{x_1}\vee P^{\perp})\wedge P)\geqslant 2$. Hence there is

$x_2\perp x_1$, $x_2\in(Q\vee P_{x_1}\vee P^{\perp})\wedge P$ and $Q\vee P_{x_1}\vee P_{x_2}=Q\vee P_{x_1}\vee P^{\perp}$.

This implies $m(Q\vee(P_{x_1}\oplus P_{x_2}))<\infty$, which contradicts (4.1).

Denote

$$\mathscr{M}_m=\{P\colon P\in\mathscr{L}(H),\ m(P)<\infty\} \tag{4.2}$$

and

$$P_m=\bigvee\{P\colon P\in\mathscr{M}_m\}, \tag{4.3}$$

where m is a signed measure on $\mathscr{L}(H)$.

The next corollary characterizes the signed measures on finite dimensional Hilbert spaces.

Corollary 4.3. *Let* $\dim H=n\geqslant 2$ *and let* m *be a signed measure on* $\mathscr{L}(H)$ *with* $m(H)=\infty$. *If there is* Q_0 *with* $\dim Q_0=2$ *and* $m(Q_0)<\infty$, *then* $m(Q)<\infty$ *iff* $Q\leqslant P_m$.

Proof. Use Corollary 4.2 successively. Q.E.D.

Theorem 4.4. *Let* m *be a* σ*-finite* f*-bounded signed measure on* $\mathscr{L}(H)$, *with* $m(H)=\infty$, *whose dimension is a nonreal measurable cardinal number* $\neq 2$. *Then there is a unique SBF* t *such that*

$$m(P)=\begin{cases} tr(t\circ P) & \text{if } t\circ P\in Tr(H),\\ \infty & \text{elsewhere.}\end{cases} \tag{4.4}$$

Moreover, m *is totally additive on* $\mathscr{M}_m$.

Proof. We prove that $\mathcal{M}_m$ from (4.2) is an ideal for which $\dim P_m \geqslant 3$. It suffices to prove that if $P_x, P_y \in \mathcal{M}_m$, then $P_x \vee P_y \in \mathcal{M}_m$. Suppose that x and y are linearly independent vectors, $x \perp y$. The σ-additivity of m implies that there exists a P_{k_0}, $\dim P_{k_0} \geqslant 3$ such that $P_{k_0}x \neq 0$, $P_{k_0}y \neq 0$. Let $z \in P_{k_0}$, $\|z\| = 1$, be such that $(z, x) = (z, y) = 0$. Hence, if $m(P_x \vee P_y) = \infty$, then $m(P_z \vee P_x \vee P_y) = \infty$. Due to Lemma 4.1, we have $P_x \vee P_y \in \mathcal{M}_m$ and $\mathcal{M}_m$ is an ideal. Now m restricted to the ideal $\mathcal{M}_m$ is an $\mathcal{M}_m$-bounded σ-additive measure on $\mathcal{M}_m$ and, according to Theorem 3.1, m is totally additive.

Theorem 3.2 guarantees that there is a unique SBF t defined on $\mathcal{D}_m = \{x : x \in H, m(P_x) < \infty\}$ such that $m(P) = tr(t \circ P)$ whenever $P \in \mathcal{M}_m$. Indeed, if $\{x_a\}$ is an orthonormal basis in P, then

$$tr(t \circ P) = \sum_a t(x_a, x_a) = \sum_a m(P_{x_a}) = m(P).$$

Therefore the formula (4.4) holds. Q.E.D.

The notion of a signed measure which attains infinite values may be defined on an ideal, too, and a formula analogous to (4.4) may be proved.

Taking into account Corollary 4.3 and Theorem 4.4, respectively, we see that in order to prove the formula (4.4) it suffices to assume that m is an f-bounded signed measure on $\mathcal{L}(H)$ for which there is a Q such that $\dim Q \geqslant 3$ and $m(Q) < \infty$.

Note 4. It is clear that if $\inf_{P \in \mathcal{L}(H)} m(P) > -\infty$, then m is an f-bounded signed measure. For a finite dimensional Hilbert space these notions are equivalent (Corollary 4.3). The author does not know the answer to the next question: Are there σ-additive signed measures bounded from below, i.e. do we always have

$\inf_{P \in L(H)} m(P) > -\infty$ whenever $m(H) = \infty$?

Example. Let H be a separable Hilbert space, $\dim H = \aleph_0$. Let $\{x_n\}$ be an orthonormal basis in H. Let us put $y_n = (x_1 + \ldots + x_n)/\sqrt{n}$. Then the function m defined by

$$m(P) = \begin{cases} -n & \text{if } P = P_{y_n} \text{ for some } n \geqslant 1, \\ 0 & \text{if } P = 0, \\ \infty & \text{if } P \neq P_{y_n} \text{ for any } n \geqslant 1, \end{cases}$$

is an f-bounded non σ-finite signed measure, which is not bounded from the below.

Also, it would be of interest to examine the Jordan decomposition for signed measures with infinite values, and the total additivity of m on the whole of $\mathcal{L}(H)$.

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ТЕОРЕМА ГЛИЗОНА ДЛЯ ОБОБЩЕННЫХ МЕР ПРИНИМАЮЩИХ БЕСКОНЕЧНЫЕ ЗНАЧЕНИЯ

Анатолий Двуреченский

Резюме

В работе доказана теорема Глизона для обобщенных мер, принимающих бесконечные значения на решетке всех узамкнутых подпространств Гильбертова пространства, размерность которого неизмеримое кардинальное число $\neq 2$.