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INDIVIDUAL ERGODIC THEOREM IN A REGULAR SPACE

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The paper brings a proof of the Individual Ergodic Theorem for Y-valued measurable functions, where Y is a regular boundedly σ-complete vector lattice (Theorem 5).

Preliminaries

Troughout the paper X will denote a non-empty set, $\mathscr{S}$ a σ-algebra of subsets of x, R the set of all real numbers, R^+ the set of all positive real numbers and N the set all positive integers.

Let Y be a boundedly σ-complete vector lattice (see [5]). We say that a sequence $(y_n)_{n\in N}$ of elements of Y r-converges to y if there exists $u\in Y$, $u>0$ such that, for each $\varepsilon\in R^+$, there exists such $n_0\in N$ that for any $n\geqq n_0$

$$|y-y_n|\leqq \varepsilon u,$$

where $|y|=\sup\{y,\ -y\}$ (see [5]).

Whe shall denote the r-convergence by $y_n\xrightarrow{r} y$ or $\lim\limits_{n\to\infty} y_n=y$. If (y_n) is a non-increasing sequence, we shall write $y_n\searrow y$. and dually if (y_n) will be a non-decreasing sequence.

Let (y_n) be a bounded sequence of points in Y. Put

$$z_n=\inf_{k\geqslant n} y_k,\ w_n=\sup_{k\geqslant n} y_k.$$

Then the sequence (z_n) is non-decreasing, (w_n) is non-increasing and $z_n\leqq y_n\leqslant w_n$ for each $n\in N$. Further, the sequence (z_n) is upper bounded and (w_n) is lower bounded and therefore there exists $\inf w_n$ and $\sup z_n$. We shall denote the element $\inf w_n$ by $\limsup\limits_{n\to\infty} y_n$ and $\sup z_n$ by $\liminf\limits_{n\to\infty} y_n$.

In what follows, Y will denote a regular space, i.e., Y will mean a boundedly σ-complete vector lattice with the following properties:

1/ Y has the σ-property, i.e., if (y_n) is any sequence in Y such that $y_n \geqq 0$ each $n \in N$, then there is a bounded sequence (r_n) of positive real numbers such that $(r_n y_n)_{n \in N}$ is a bounded sequence;

2/ $y_n \xrightarrow{r} y$ if and only if $y = \lim_{n \to \infty} \sup y_n = \lim_{n \to \infty} \inf y_n$. Note that the family of all real sequences is regular whereas, e.g., the family of all real sequences converging to 0 is not. We denote the set $\{y \in Y: y > 0\}$ by Y^+ and put

$$I[u] = \{y \in Y: |y| \leqq ru, \text{ for some } r \in R^+\}$$

for $u \in Y^+$. We define a function p: $I[u] \to R$ in the following way:

$$p(y) = \inf\{r \in R: |y| \leqq ru\}. \tag{1}$$

Then $I[u]$ becomes a Banach lattice, i.e., $I[u]$ is a boundedly σ-complete vector lattice with a monotonous norm p (see [5]).

Proposition 1. A vector lattice Y has the σ-property if and only if, for an arbitrary countable subset M of Y, there exists $u \in Y^+$ such that $M \subset I[u]$.

The proof is evident.

Definition 1. We say that a sequence of functions $(f_n)_n$, f_n: $X \to Y$, uniformly r-converges to f if there exists $u \in Y^+$ such that the following condition holds: given $\varepsilon \in R^+$, we can find $n_0 \in N$ such that, for each $n \geqq n_0$ and $x \in X$, we have the inequality: $|f(x) - f_n(x)| \leqq \varepsilon u$.

We shall denote by $\text{u-}\lim_{n \to \infty} f_n = f$ the uniform r-convergence of a sequence of functions $(f_n)_n$ to f. The element $u \in Y^+$ is then called the regulator of the uniform convergence.

Elements of the integration theory

Let $(X, \mathcal{S}, P)$ be a probability space. A function $f: X \to Y$ is said to be a simple measurable function if there exist pairwise disjoint sets $A_1, \ldots, A_n \in \mathcal{S}$ and elements $a_1, \ldots, a_n \in Y$ such that

$$f = \sum_{i=1}^{n} a_i \chi_{A_i}$$

The family of all simple measurable functions $f: X \to Y$ will be denoted by $\mathcal{J}$. We define a function $I: \mathcal{J} \to Y$ by putting

$$I(f) = \sum_{i=1}^{n} a_i P(A_i).$$

Definition 2. The element $I(f)$ is called the integral of a function f.

Definition 3. A function $f: X \to Y$ is said to be integrable if there exists a sequence $(f_n)_n$ in $\mathscr{J}$ such that $\text{u-}\lim_{n \to \infty} f_n = f$.

Proposition 2. If $(f_n)_n$ is a sequence of simple measurable functions and this sequence uniformly r-converges to f, then there exists $\lim_{n \to \infty} I(f_n)$.

The proof of Proposition 2 is very easy.

Corollary. Let (f_n), (g_n) be sequences of simple measurable functions such that both (f_n) and (g_n) uniformly r-converges to f, then

$$\lim_{n \to \infty} I(f_n) = \lim_{n \to \infty} I(g_n) \ .$$

Definition 4. Let f be a integrable function. The value $\lim_{n \to \infty} I(f_n)$ from Proposition 3 is called by the integral of the function f and we shall denote it $I(f)$, too. The family of all integrable functions $f: X \to Y$ will be denoted by $\mathscr{F}$.

The proofs of the following Theorems are very easy. They follow from the uniform r-convgence and the σ-property.

Theorem 1. If $f, g \in \mathscr{F}$, $c \in R$, then $f + g$, cf, $|f|$, $\sup\{f, g\}$, $\inf\{f, g\}$ are integrable functions and

$$\begin{aligned} &I(f+g) = I(f) + I(g)\,, \\ &I(cf) = c\,I(f)\,, \\ &|I(f)| \leqq I(|f|)\,. \end{aligned}$$

Further, if $f \leqq g$, then $I(f) \leqq I(g)$.

Theorem 2. If $(f_n)_n$ uniformly r-converges to f and $f_n \in \mathscr{F}$, then $f \in \mathscr{F}$ and $\lim_{\to \infty} I(f_n) = I(f)$.

A transformation T: $X \to Y$ is called measurable if $\mathrm{T}^{-1} A \in \mathscr{S}$ for each $A \in \mathscr{S}$.

Theorem 3. Let $f \in \mathscr{F}$ and T be a measurable ttransformation. Then $f \circ \mathrm{T} \in \mathscr{F}$.

The proof of Theorem 3 is straightforward.

We shall denote by $\int \xi \, \mathrm{d}P$ the integral of a real valued integrable measurable function ξ.

Theorem 4. Let ξ be a bounded real valued integrable measurable function and $c \in Y$, then $c\xi \in \mathscr{F}$ and

$$I(c\xi) = c \int \xi \, \mathrm{d}P\,.$$

Proof. If ξ is a simple real valued measurable function, the proof is evident. Let ξ be a bounded measurable real valued function. Then there exists a sequence $(\xi_n)_n$ of simple real valued measurable functions such that, for each $\varepsilon \in R^+$, there exists $n_0 \in N$ such that

$$|\xi_n(x) - \xi(x)| \leqq \varepsilon$$

for ani $n \leqq n_0$ and $x \in X$. Hende $\text{u-}\lim_{n \to \infty} \text{c} \lim_{n \to \infty_n} = \text{c}\xi$. Also

$$|I(c\xi_n) - c\int \xi \, \mathrm{d}P| \leqq |c| \int |\xi_n - \xi| \, \mathrm{d}P \leqq \varepsilon |c|,$$

whenever $n \geqq n_0$, and thus

$$\lim_{n \to \infty} I(c\xi_n) = c\int \xi \, \mathrm{d}P.$$

Q.E.D.

An individual ergodic theorem

Recall (see [6]) that $(X, \mathscr{S}, P, \mathrm{T})$ is called a dynamical system if $(X, \mathscr{S}, P)$ is a probability space and T is a measure-presreving transformation, i.e., T is measurable and

$$P(A) = P(\mathrm{T}^{-1}A)$$

for each $A \in \mathscr{S}$. A function $f{:}X \to Y$ is called invariant if there exists $A \in \mathscr{S}$ with $P(A) = 0$ and, for each $x \in X - A$ we have

$$f(x) - f(\mathrm{T}x).$$

We are going to prove the individual ergodic theorem for a Y-valued integrable function.

Theorem 5. (Individual ergodic theorem.) Let $(X, \mathscr{S}, P\ \mathrm{T})$ be a dynamical system and f be an integrable function, then there exists an invariant function $f^* \in \mathscr{F}$ and $A \in \mathscr{S}$ with $P(A) = 0$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\mathrm{T}^i x) = f^*(x)$$

for each $x \in X - A$. Also for f^* there holds

$$I(f^*) = I(f).$$

Proof. This Theorem will be proved in a few steps.
i) Let $f = c\chi_B$, where $B \in \mathscr{S}$. Since χ_B is a bounded real valued measurable

function, then by [6] there is an invariant set $A \in \mathcal{S}$ ($A = \mathrm{T}^{-1}A$) with $P(A) = 0$ and a bounded invariant integrable measurable function ξ such that, for any $\varepsilon \in R^+$ and $x \in X - A$, there is n_0 so that

$$\left|\frac{1}{n}\sum_{i=0}^{n-1} \chi_B(\mathrm{T}^i x) - \xi(x)\right| \leqq \varepsilon \tag{2}$$

for $n \geqq n_0$, and $P(B) = \int \xi \, \mathrm{d}P$. According to (2)

$$\lim_{n \to \infty} \frac{1}{n}\sum_{i=0}^{n-1} f(\mathrm{T}^i x) = c\xi(x)$$

for each $x \in X - A$. Since ξ is invariant then $c\xi$ is also invariant. By Theorem 4 $c\xi \in \mathcal{F}$ and

$$I(c\xi) = c\int \xi \, \mathrm{d}P = c\int \chi \, \mathrm{d}P = I(f).$$

ii) According to i) we have that this Theorem holds in the case of a simple Y-valued integrable function.
iii) Let $f \in \mathcal{F}$ be arbitrary. Then there exists a sequence (f_n) of simple measurable functions and $u \in Y^+$ such that, for any $\varepsilon \in R^+$, there is $n_0 \in N$ such that

$$f_n(x) - \varepsilon u \leqq f(x) \leqq f_n(x) + \varepsilon u$$

for each $n \geqq n_0$ and $x \in X$. For each $k \in N$ put

$$S_k(f, x) = \frac{1}{k}\sum_{i=0}^{n-1} f(\mathrm{T}^i x).$$

Then

$$S_k(f_n, n) - \varepsilon u \leqq S_k(f, x) \leqq S_k(f_n, x) + \varepsilon u$$

whenever $n \geqq n_0$, $k \in N$ and $x \in X$. By ii) there exist invariant sets $A_n \in \mathcal{S}$ with $P(A_n) = 0$ and integrable functions $f_n^* \in \mathcal{F}$ such that

$$\lim_{n \to \infty} S_k(f_n, x) = f_n^*(x),$$

for each $x \in X - A_n$. Putting $A = \bigcup_{i=1}^{\infty} A_i$, we have $P(A) = 0$, A is invariant and, for each $x \in X - A$ and $n \geqq n_0$ we obtain

$$f_n^*(x) - \varepsilon u \leqq \liminf_{n \to \infty} S_k(f, x) \leqq f_n^*(x) + \varepsilon u, \tag{3}$$

$$f_n^*(x) - \varepsilon u \leqq \limsup_{n \to \infty} S_k(f, x) \leqq f_n^*(x) + \varepsilon u. \tag{4}$$

Hence for each $x \in X - A$ there is $\lim_{n \to \infty} f_n^*(x) = f_n^*(x)$. For $x \in A$ we define $f_n^*(x) = 0$. According to (3) or (4),

$$|f_n^*(x)\chi_A(x) - f_n^*(x)| \leqq \varepsilon u,$$

whenever $n \geqq n_0$ and $x \in X$, and therefore $f^* \in \mathscr{F}$.
By ii) f_n^* are invariant and $I f_n^*) = I(f_n)$. Then

$$I(f^*) = \lim_{n \to \infty} I(f_n^* \chi_{X-A}) = \lim_{n \to \infty} I(f_n^*) = \lim_{n \to \infty} I(F_n) = I(f).$$

Q.E.D.

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ИНДИВИДУАЛЬНАЯ ЭРГОДИЧЕСКАЯ ТЕОРЕМА В РЕГУЛЯРНЫХ ПРОСТРАНСТВАХ

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Резюме

В статье иследуется индивидуальная эргодическая теорема в регулярных пространствах. Она доказывается с помощью этой же теоремы для случая действительной случайной величины. Случайная величина определяется как предел последовательности простых случайных величин.