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PERIODIC BOUNDARY VALUE PROBLEMS FOR THIRD ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. There are studied the questions of existence of periodic solutions of the equation $u''' = f(t, u, u', u'')$ by means of topological degree methods.

In this paper there are found some new conditions for the existence of solutions of the problem

$$u''' = f(t, u, u', u''), \tag{1.1}$$

$$u(a) = u(b), \quad u'(a) = u'(b), \quad u''(a) = u''(b), \tag{1.2}$$

where $-\infty < a < b < +\infty$.

The problems of such type have been already solved in many works, for example [1–7]. Here, the proof of the main result is based on *Mawhin's continuation theorem* [6] (see Lemma 1).

1. Notations, definitions and auxiliary results

Let X, Y be real vector normed spaces and $\operatorname{dom} L \subset X$ a vector subspace.

Definition 1. *A linear mapping*

$$L\colon \operatorname{dom} L \to Y$$

will be called a Fredholm mapping of index zero iff

(i) $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$;

(ii) $\operatorname{Im} L$ *is closed in* Y.

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It follows from the definition above and from basic results of linear functional analysis that there exist continuous projectors

$$P\colon X \to X \quad \text{and} \quad Q\colon Y \to Y$$

such that

$$\operatorname{Im} P = \operatorname{Ker} L \quad \text{and} \quad \operatorname{Ker} Q = \operatorname{Im} L$$

so that

$$X = \operatorname{Ker} L \oplus \operatorname{Ker} P\,, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q$$

as topological direct sums.

Consequently, the restriction L_p of L to $\operatorname{dom} L \cap \operatorname{Ker} P$ is one-to-one and onto $\operatorname{Im} L$, so that its (algebraic) inverse $K_p\colon \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ is defined. [6, p. 6]

Definition 2. *Let $L\colon \operatorname{dom} L \to Y$ be a Fredholm mapping of index zero and let $\Omega \subset X$ be an open bounded set. A (not necessarily linear) mapping $N\colon X \to Y$ will be called L-compact on $\overline{\Omega}$ iff the mappings $QN\colon \overline{\Omega} \to Y$ and $K_p(I-Q)N\colon \overline{\Omega} \to Y$ are compact, i.e. continuous on $\overline{\Omega}$ and such that $QN(\overline{\Omega})$ and $K_p(I-Q)N(\overline{\Omega})$ are relatively compact.*

Note. $\overline{\Omega}$ and $\partial\Omega$ is the *closure* and the *boundary* of $\Omega \subset X$, respectively.

Definition 3. *We shall say that $A\colon X \to Y$ is L-completely continuous if it is L-compact on every bounded $\overline{\Omega} \subset X$.*

One can show that Definitions 2,3 do not depend upon the choice of the continuous projectors P and Q, which justifies the terminology. [6, p. 12]

Lemma 1. ([6, Theorem IV.5, p. 44]). *Let $L\colon \operatorname{dom} L \to Y$ be a linear Fredholm mapping of index zero and let $\Omega \subset X$ be an open bounded set. Let $N\colon \overline{\Omega} \to Y$ be L-compact on $\overline{\Omega}$ and let $A\colon X \to Y$ be L-completely continuous and such that*

(i) $\operatorname{Ker}(L-A) = \{0\}$;

(ii) *for every* $(x,\lambda) \in (\operatorname{dom} L \cap \partial\Omega) \times]0,1[$

$$Lx - (1-\lambda)Ax - \lambda Nx \neq 0\,,$$

and assume that $0 \in \Omega$.

Then equation

$$Lx = Nx$$

has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

$AC^i(a,b)$ $[C^i(a,b)]$ is the set of all real functions having absolutely continuous [continuous] i-th derivatives on $[a,b]$, $i = 0,1,2$.

$L^p(a,b)$ is the set of all real functions f with $|f|^p$ Lebesgue integrable on $]a,b[$, $p \in [1,+\infty[$.

In what follows let $X = \{x \in C^2(a,b);\ x \text{ satisfies } (1.2)\}$ be a Banach space with the norm

$$\max\left\{\left(\sum_{i=0}^{2}(x^{(i)}(t))^2\right)^{1/2} : a \le t \le b\right\} \quad \text{for } x \in X\,;$$

$Y = L^1(a,b)$ be a Banach space with the norm

$$\int_a^b |y(t)|\mathrm{d}t, \quad \text{for} \quad y \in Y\,;$$

$\operatorname{dom} L = X \cap AC^2(a,b)$;

$$L\colon \operatorname{dom} L \to Y, \quad x \mapsto x'''. \tag{1.3}$$

Then

$$\operatorname{Ker} L = \{x \in \operatorname{dom} L;\ x \text{ is a constant mapping on } [a,b]\}\,;$$
$$\operatorname{Im} L = \{y \in Y;\ y = x''',\ x \in \operatorname{dom} L\} = \left\{y \in Y;\ \int_a^b y(t)\mathrm{d}t = 0\right\}.$$

Therefore $\operatorname{Im} L$ is closed in Y and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 1$. Thus we have proved

Lemma 2. *L, defined by* (1.3), *is a Fredholm mapping of index zero.*

Definition 4. *A function $u \in \operatorname{dom} L$ which fulfils* (1.1) *for a.e. $t \in [a,b]$ will be called a solution of problem* (1.1), (1.2).

We will say that some property is satisfied on D if it is satisfied for a.e. $t \in [a,b]$ and for every $x, y, z \in \mathbb{R}$.

We will write $f \in \mathrm{Car}_{\mathrm{loc}}(D)$ iff f satisfies the *local Carathéodory conditions* on D i.e.

(i) for every $x, y, z \in \mathbb{R}$, the mapping $t \mapsto f(t,x,y,z)$ is Lebesgue measurable on $[a,b]$;

(ii) for a.e. $t \in [a,b]$, the mapping $(x,y,z) \mapsto f(t,x,y,z)$ is continuous on $\mathbb{R}^3$;

(iii) for each $\varrho > 0$ there exists $h_\varrho \in L^1(a,b)$ such that $(x^2+y^2+z^2)^{1/2} < \varrho \implies |f(t,x,y,z)| \le h_\varrho(t)$ on D.

Lemma 3. *Let* $f \in \mathrm{Car}_{\mathrm{loc}}(D)$. *Then the mapping*

$$N\colon X \to Y, \quad x \mapsto f\big(\cdot, x(\cdot), x'(\cdot), x''(\cdot)\big) \tag{1.4}$$

is L*-completely continuous.*

Proof. [6, p. 13–14].

Note. *If* L *and* N *are defined by* (1.3) *and* (1.4), *respectively, then* x *is a solution of* (1.1), (1.2) *iff* $x \in \operatorname{dom} L$ *and* $Lx = Nx$.

2. The main result

For $h \in L^1(a,b)$ and $r \in \,]0,+\infty[$ we shall put

$$\begin{cases} h_0 = \exp\left(2\int_a^b h(t)\mathrm{d}t\right), & r_0 = r + 3(b-a)^2 h_0\,, \\ \varepsilon \in \,]0, 1/2r_0(b-a)[\,, & \\ r_2 = h_0 \exp\left(2\varepsilon r_0(b-a)\right), & r_1 = \varepsilon + r_2(b-a)\,. \end{cases} \tag{2.1}$$

Theorem. *Let there exist* $\mu \in \{-1, 1\}$, $r \in \,]0,+\infty[$ *and a non-negative function* $h \in L^1(a,b)$ *such that* $f \in \mathrm{Car}_{\mathrm{loc}}(D)$ *satisfies on* D *the conditions*

$$|x| \ge r, \quad |y| \le r_1, \quad |z| \le r_2 \implies \mu f(t,x,y,z) \operatorname{sign} x \ge 0 \tag{2.2}$$

and

$$|x| \le r_0, \quad |y| \le r_1, \quad |z| \ge 1 \implies f(t,x,y,z) \operatorname{sign} z \le h(t)|z|\,, \tag{2.3}$$

where r_0, r_1, r_2 *fulfil* (2.1).

Then the problem (1.1), (1.2) *has at least one solution* u *such that*

$$|u(t)| \le r_0, \quad |u'(t)| \le r_1, \quad |u''(t)| \le r_2 \quad \textit{for} \quad a \le t \le b\,. \tag{2.4}$$

First we shall prove some lemmas.

Lemma 4. *Let* $r \in \,]0,+\infty[$ *and let* $h \in L^1(a,b)$ *be a nonnegative function. Let* $r_0, r_1, r_2, \varepsilon$ *fulfil* (2.1).

Then for any function $u \in \operatorname{dom} L$ *the inequalities*

$$|u(t)| \le r_0, \quad |u'(t)| \le r_1 \quad \textit{for every} \quad t \in [a,b] \tag{2.5}$$

and

$$u'''(t)\operatorname{sign} u''(t) \le h(t)|u''(t)| + \varepsilon|u(t)| \quad \textit{for a.e. } t \in [a,b] \textit{ and } |u''(t)| \ge 1 \tag{2.6}$$

imply

$$|u''(t)| < r_2 \quad \textit{for every} \quad t \in [a,b]\,. \tag{2.7}$$

P r o o f. Since (1.2), there exists $t_0 \in \,]a,b[$ such that

$$u''(t_0) = 0\,. \tag{2.8}$$

1. Let us suppose that there exists $t^* \in \,]t_0,b[$ such that

$$|u''(t^*)| \ge \sqrt{r_2}\,. \tag{2.9}$$

Then there exists $t_* \in \,]t_0,t^*[$ such that

$$|u''(t_*)| = 1 \quad \text{and} \quad |u''(t)| \ge 1 \quad \text{for} \quad t \in [t_*,t^*]\,. \tag{2.10}$$

a) Let $u''(t) \ge 1$ on $[t_*,t^*]$. Then, by (2.6),

$$\int_{t_*}^{t^*} \frac{u'''(t)\mathrm{d}t}{u''(t)} \le \int_{t_*}^{t^*} (h(t) + \varepsilon r_0)\mathrm{d}t < \int_a^b h(t)\mathrm{d}t + \varepsilon r_0(b-a)\,.$$

Thus $u''(t^*) < \sqrt{r_2}$, a contradiction.

b) Let $u''(t) \le -1$ on $[t_*,t^*]$. Similarly, by (2.6),

$$\int_{t_*}^{t^*} \frac{-u'''(t)\mathrm{d}t}{-u''(t)} \le \int_{t_*}^{t^*} (h(t) + \varepsilon r_0)\mathrm{d}t < \int_a^b h(t)\mathrm{d}t + \varepsilon r_0(b-a)\,.$$

Thus $-u''(t^*) < \sqrt{r_2}$, a contradiction. Therefore we have

$$|u''(a)| < \sqrt{r_2} \quad \text{for every} \quad t \in [t_0,b]\,. \tag{2.11}$$

According to (1.2), $|u''(a)| < \sqrt{r_2}\,.$

2. Supposing the existence of $t^* \in \,]a,t_0[$ satisfying

$$|u''(t^*)| \ge r_2\,, \tag{2.12}$$

we obtain $t_* \in \,]a,t^*[$ such that (2.10) (we write there $\sqrt{r_2}$ instead of 1) is fulfilled. In the same way as in the first part, integrating (2.6) from t_* to t^*, we get

$$|u''(t^*)| < r_2\,,$$

which contradicts (2.12). Thus

$$|u''(t)| < r_2 \quad \text{for every} \quad t \in [a,t_0]\,. \tag{2.13}$$

Inequalities (2.11), (2.13) imply estimate (2.7).

Lemma 5. *Let* $r \in]0, +\infty[$ *and let* $h \in L^1(a,b)$ *be a nonnegative function. Let* $r_0, r_1, r_2, \varepsilon$ *fulfil* (2.1).

Then for any function $u \in \operatorname{dom} L$ *the inequalities*

$$|u''(t)| \le r_2 \qquad \textit{for every} \quad t \in [a,b] \tag{2.14}$$

and

$$|u(t)| \ge r \implies \mu u'''(t) \operatorname{sign} u(t) > 0 \quad \textit{for a.e.} \quad t \in [a,b] \tag{2.15}$$

imply

$$|u(t)| < r_0 \quad \textit{and} \quad |u'(t)| < r_1 \quad \textit{for every} \quad t \in [a,b]\,. \tag{2.16}$$

Proof. Since (1.2) and (2.15), there exist $t_0, t_1 \in]a,b[$ such that

$$|u(t_0)| < r\,, \quad u'(t_1) = 0\,. \tag{2.17}$$

Integrating (2.14), we get by (2.1) and (2.17)

$$|u'(t)| \le r_2(b-a) < r_1\,, \quad |u(t)| < r + r_2(b-a)^2 < r_0\,.$$

The Lemma is proved.

Lemma 6. *Let* $f \in \mathrm{Car}_{\mathrm{loc}}(D)$ *and* $\mu \in \{-1, 1\}$. *Let* $\varepsilon \in]0, +\infty[$ *be such that equation*

$$u''' = \mu\varepsilon u \tag{2.18}$$

has only the trivial solution in $\operatorname{dom} L$. *Let there exist an open bounded set* $\Omega \subset X$ *such that* $0 \in \Omega$ *and for any* $\lambda \in]0,1[$ *each solution* $u_\lambda \in \operatorname{dom} L$ *of equation*

$$u''' = \lambda f(t, u, u', u'') + (1-\lambda)\mu\varepsilon u \tag{2.19}$$

satisfies

$$u_\lambda \notin \partial\Omega\,.$$

Then problem (1.1), (1.2) *has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

Proof. Let us consider the mappings

$$\begin{aligned} &L\colon \operatorname{dom} L \to Y, \quad x \mapsto x''' \\ &N\colon X \to Y, \quad x \mapsto f\big(\cdot, x(\cdot), x'(\cdot), x''(\cdot)\big) \\ &A\colon X \to Y, \quad x \mapsto \mu\varepsilon x\,. \end{aligned}$$

By Lemma 2, L is a Fredholm mapping of index zero and by Lemma 3, N and A are L-completely continuous, and thus N is L-compact on $\overline{\Omega}$. Since (2.18) has only the trivial solution in $\operatorname{dom} L$, condition (i) of Lemma 1 is valid. Since (2.19) has no solution on $\partial\Omega$, condition (ii) of Lemma 1 is satisfied. Therefore the assertion of Lemma 6 follows from Lemma 1.

Proof of the Theorem. Let us put

$$\Omega = \{x \in X : |x(t)| < r_0,\ |x'(t)| < r_1,\ |x''(t)| < r_2 \text{ for each } t \in [a,b]\}.$$

Then $x \in \partial\Omega$ iff

$$\begin{cases} |x^{(i)}(t)| \le r_i,\ |x^{(k)}(t)| \le r_k \quad \text{and} \\ \max\{|x^{(j)}(t)| : a \le t \le b\} = r_j, \quad \text{for each} \\ \qquad t \in [a,b],\ i,j,k \in \{0,1,2\},\ i \ne j \ne k. \end{cases} \tag{2.20}$$

We can choose $\varepsilon \in]0, 1/2r_0(b-a)[$ so small that problem (2.18), (1.2) has only the trivial solution. Let $\lambda \in]0,1[$ and let u_λ be a solution of a problem (2.19), (1.2). Supposing $u_\lambda \in \overline{\Omega}$, we shall show $u_\lambda \notin \partial\Omega$.

First let

$$|u_\lambda(t)| \le r_0 \quad \text{and} \quad |u'_\lambda(t)| \le r_1 \quad \text{for each } t \in [a,b]. \tag{2.21}$$

Then, by (2.3), $u'''_\lambda \operatorname{sign} u''_\lambda = \lambda f \operatorname{sign} u''_\lambda + (1-\lambda)\mu\varepsilon u_\lambda \operatorname{sign} u''_\lambda \le h(t)|u''_\lambda| + \varepsilon|u_\lambda|$ for a.e. $t \in [a,b]$ and $|u''_\lambda(t)| \ge 1$. Applying Lemma 4, we obtain

$$|u''_\lambda(t)| < r_2 \quad \text{for each } t \in [a,b]. \tag{2.22}$$

Further, according to (2.2), $\mu u''_\lambda \operatorname{sign} u_\lambda = \lambda\mu f \operatorname{sign} u_\lambda + \mu(1-\lambda)\mu\varepsilon u_\lambda \operatorname{sign} u_\lambda > 0$ for a.e. $t \in [a,b]$ and $|u_\lambda(t)| \ge r$. Using Lemma 5, we get

$$|u_\lambda(t)| < r_0 \quad \text{and} \quad |u'_\lambda(t)| < r_1 \text{ for each } t \in [a,b]. \tag{2.23}$$

Thus if $u_\lambda \in \overline{\Omega}$, then u_λ satisfies (2.21), (2.22), (2.23) and so $u_\lambda \in \overline{\Omega} \setminus \partial\Omega$. The Theorem is proved.

Example. The conditions of the Theorem are satisfied for example when $h \in L^1(a,b)$ is non-negative, $r \in]0,+\infty[$, $c \in \mathbf{R}$, $c \ne 0$, $r_0, r_1, r_2 \in \mathbf{R}$ satisfy (2.1) and

$$f(t,x,y,z) = h(t)c|z|x^k/(1+y^n), \quad \text{where } k,n \in \mathbf{N},\ k \text{ is odd},\ n \text{ is even},\ |c| \le r_0^{-k},$$

or

$$f(t,x,y,z) = h(t)c(x+1)e^{xy}(z+r_2), \quad \text{where } |c| \le 1/(r_0+1)e^{r_0 r_1}(1+r_2).$$

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