Bedřich Pondělíček

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Mathematica Slovaca, Vol. 44 (1994), No. 3, 359--364

Persistent URL: http://dml.cz/dmlcz/136613

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Mathematica Slovaca

Dedicated to Academician Štefan Schwarz on the occasion of his 80th birthday

ON GENERALIZED CONDITIONALLY COMMUTATIVE SEMIGROUPS

BEDŘICH PONDELÍČEK

(Communicated by Tibor Katriňák)

ABSTRACT. The purpose of this paper is to show that RC-commutative semigroups and GC-commutative Δ-semigroups satisfying (5) are weakly exponential.

A semigroup whose congruences form a chain with respect to inclusion is called a *Δ-semigroup*. A complete description of commutative Δ-semigroups was given by Schein [1] and Tamura [2] independently. Etterbeek [3] has obtained a generalization of their results for medial Δ-semigroups, Trotter [4] has characterized the exponential Δ-semigroups and Nagy [5] has described the weakly exponential Δ-semigroups.

Recall that a semigroup S is called a *weakly exponential semigroup* if for every $(x,y) \in S \times S$ and every positive integer n there is a positive integer m such that

$$(xy)^{n+m} = x^n y^n (xy)^m = (xy)^m x^n y^n .$$

By [6], a semigroup S is said to be *conditionally commutative* if $ab = ba$ implies $axb = bxa$ for any $a, b, x \in S$. A conditionally commutative semigroup S is called an *RC-commutative semigroup* (see [7]) if for $(a,b) \in S \times S$ there is an element $x \in S^1$ such that $ab = bax$.

In this paper, we shall show that every RC-commutative semigroup is weakly exponential. We shall define a class of semigroups whose Δ-semigroups are weakly exponential. This class contains semigroups which are not weakly exponential.

AMS Subject Classification (1991): Primary 20L99.
Key words: Conditionally commutative semigroup, Weakly exponential semigroup, Δ-semigroup.

DEFINITION. A semigroup S is called a *generalized conditionally commutative semigroup* (briefly a *GC-commutative semigroup*) if

$$x^2yx = xyx^2$$

for every $(x, y) \in S \times S$.

LEMMA 1. *Let S be a GC-commutative semigroup. Then $x^n yx = xyx^n$ for every $(x, y) \in S \times S$ and for every positive integer n.*

LEMMA 2. *Let S be a GC-commutative semigroup. Then*

$$(xy)^m x^n y^n = x^n y^n (xy)^m$$

for every $(x, y) \in S \times S$ and for any positive integers m, n.

Proof. According to Lemma 1, we have $(xy)^m x^n y^n = x\big(y(xy)^{m-1}\big)x^n y^n = x^n\big(y(xy)^{m-1}\big)xy^n = x^n y\big((xy)^{m-1}x\big)y^n = x^n y^n (xy)^{m-1}xy = x^n y^n (xy)^m$, where $(xy)^0$ is the unity in S^1.

LEMMA 3. *Every conditionally commutative semigroup is a GC-commutative semigroup.*

Proof. It follows from $x^2x = xx^2$ that $x^2yx = xyx^2$.

Note 1. It is easy to show that every non-commutative idempotent monoid is a GC-commutative semigroup but not conditionally commutative.

THEOREM 1. *A GC-commutative semigroup S is weakly exponential if and only if for every $(x, y) \in S \times S$ there exists a positive integer m such that*

$$(xy)^{m+2} = x^2y^2(xy)^m . \tag{1}$$

Proof. Let S be a GC-commutative semigroup and x, y be arbitrary elements of S. Suppose that for some positive integer m we have (1).

First we shall show that

$$(xy)^{m+2} = (xy)^m(yx^2y) . \tag{2}$$

Indeed, according to Lemma 1, we obtain $(xy)^{m+2} = x^2y^2(xy)^m = x^2y(xy)^m y = x(xy)^{m+1}y = x(xy)^m(xy)y = x(xy)^m y(xy) = (xy)^m(yx^2y)$.

Now we shall prove that

$$(xy)^{m+2n} = x^n y^n (xy)^{m+n} \tag{3}$$

for every positive integer n.

It is clear for $n = 1$. Assume that (3) is fulfilled for a positive integer n. From (2) and Lemma 1 it follows that

$$\begin{aligned}(xy)^{m+2(n+1)} &= (xy)^{m+2n}(xy)^2 = x^n y^n (xy)^{m+n+2} = x^n y^n (xy)^{m+n}(yx^2y)\\ &= x^{n-1}y^n(xy)^{m+n}(yxy) = x^{n+1}y^{n+1}(xy)^{m+n}(xy)\\ &= x^{n+1}y^{n+1}(xy)^{m+n+1}\,.\end{aligned}$$

Finally, Lemma 2 and (3) imply that S is weakly exponential.

THEOREM 2. *If every right ideal of a GC-commutative semigroup S is a two-sided ideal, then S is weakly exponential.*

P r o o f. Suppose that every right ideal of a GC-commutative semigroup S is a two-sided ideal. Let $x, y \in S$. We shall show that

$$(xy)^4 = x^2y^2(xy)^2\,.$$

Clearly, yS^1 is a right ideal of S, and so, by hypothesis, yS^1 is a two-sided ideal of S. Thus we have $xy \in SyS^1 \subset yS^1$. If $xy = y$, then $(xy)^4 = y^4 = x^2y^2(xy)^2$. Suppose that $xy = yz$ for some $z \in S$. By Lemma 1, we obtain $(xy)^4 = xy(xy)^2yz = xy^2(xy)^2z = xy^2(xy)x(yz) = xy^2(xy)x^2y = x^2y^2(xy)^2$. The rest of the proof follows from Theorem 1.

THEOREM 3. *Every RC-commutative semigroup is weakly exponential.*

The proof follows from Lemma 3, Theorem 2, and [7; Lemma 6].

N o t e 2. According to Theorem 3, the result [7; Theorem 20] follows from [5; Theorem 4.1].

Recall that a semigroup S is called *t-archimedean* if for every $(x, y) \in S \times S$ there is a positive integer n such that

$$x^n \in ySy\,. \tag{4}$$

THEOREM 4. *A GC-commutative semigroup S is a band of t-archimedean semigroups if and only if for every $(x, y) \in S \times S$ there is a positive integer n such that*

$$(xy)^n \in x^2yS\,. \tag{5}$$

P r o o f . According to [8; Lemma 1], a semigroup S is a band of t-archimedean semigroups if and only if for every $(x,y) \in S \times S$ we have

$$\begin{aligned} \langle xy\rangle \cap x^2yS &\neq \emptyset \neq Sxy^2 \cap \langle xy\rangle\,, \\ \langle x^2y\rangle \cap xyS \ &\neq \emptyset \neq Sxy \ \cap \langle xy^2\rangle\,, \end{aligned} \tag{6}$$

where, by $\langle z\rangle$, we denote the subsemigroup of S generated by $z \in S$.

Suppose that S is a band of t-archimedean semigroups, and let $(x,y) \in S\times S$. It follows from (6) that for some positive integers n we obtain (5).

Conversely, assume that for arbitrary pair (x,y) of elements of a GC-commutative semigroup S there are positive integers n, m such that we have (5) and

$$(yx)^m \in y^2xS\,.$$

Then there is an element $u \in S$ such that $(yx)^m = y^2xu$. It follows from Lemma 1 that $(xy)^{m+2} = x(yx)^m yxy = xy^2(xuyx)y = xy(xuyx)y^2 \in Sxy^2$. By Definition, we have $(x^2y)^2 = x^2yx^2y = xyx^3y \in xyS$ and $(xy^2)^2 = xy^2xy^2 = xy^3xy \in Sxy$. According to (6), S is a band of t-archimedean semigroups.

N o t e 3. The following example shows that a GC-commutative semigroup S (which is a band of t-archimedean semigroups) need not be weakly exponential.

By $\mathbb{N}$, we denote the set of all positive integers, and $I = \{0,1\}$. Define a mapping $\pi\colon \mathbb{N} \to I$ by

$$\pi(n) = \begin{cases} 0 & \text{if } n \text{ is even}\,, \\ 1 & \text{if } n \text{ is odd}\,. \end{cases}$$

Put $S = I \times \mathbb{N}$, and let a multiplication on S be defined as follows:

$$(i,m)(j,n) = \big(i,\, m+n+\pi(i+j+m)\big)\,.$$

First we shall show that S is a GC-commutative semigroup. Let $x,y,z \in S$ and $x = (i,m)$, $y = (j,n)$, $z = (k,p)$. It can be easily verified that

$$(xy)z = \big(i,\, m+n+p+\pi(i+j+m)+\pi(j+k+n)\big) = x(yz)$$

and

$$x^2yx = \big(i,\, 3m+n+\pi(i+j+m)+\pi(i+j+n)+\pi(m)\big) = xyx^2\,.$$

Now we shall show that S is a band of t-archimedean semigroups. Let $x, y \in S$ and $x = (i, m)$, $y = (j, n)$. It is easy to show that

$$x^2y^2 = \big(i, 2m + 2n + \pi(m) + \pi(n) + \pi(i + j + n)\big)\,.$$

Further we have the following two possibilities:

Case 1. $i = j$. Then $(xy)^2 = x^2y^2 \in x^2yS$.

Case 2. $i \neq j$. It is easy to show that

$$(xy)^4 = \big(i,\, 4m + 4n + 4\pi(1 + m) + 3\pi(1 + n)\big)\,.$$

Therefore $(xy)^4 = x^2y^2u$, where $u = \big(i,\, 2m + 2n + 3\pi(1 + m) + 2\pi(1 + n) - \pi(m) - \pi(n)\big)$, and so $(xy)^4 \in x^2yS$.

It follows from Theorem 4 that S is a band of t-archimedean semigroups.

Finally we shall show that S is not weakly exponential. Denote $e = (0, 1)$ and $f = (1, 1)$. By way of contradiction, assume that there exists a positive integer n such that $(ef)^{n+2} = e^2f^2(ef)^n$. It can be easily verified that $(0,\, 2n + 4) = (ef)^{n+2} = e^2f^2(ef)^n = (0, 6)(0, 2n) = (0,\, 2n+6)$, which is impossible. Therefore S is not weakly exponential.

THEOREM 5. *Let S be a GC-commutative semigroup satisfying the conditions* (5). *If S is a Δ-semigroup, then S is weakly exponential.*

P r o o f. Assume that S is a GC-commutative semigroup satisfying (5) and a Δ-semigroup. It follows from Theorem 4 that S is a band of t-archimedean semigroups. By $\sim$ we denote the corresponding congruence. According to [1; Lemma 2], every homomorphic image of a Δ-semigroup is also a Δ-semigroup. This implies that the band $S/\!\sim$ is a Δ-semigroup. It follows from [4] that $S/\!\sim$ is isomorphic to G or G^0 or B or B^0 or B^1, where $\operatorname{card} G = 1$, B is either a left zero semigroup of order 2 or a right zero semigroup of order 2. It is easy to show that for every $(x, y) \in S \times S$ we have

$$xy \sim x \qquad \text{or} \qquad xy \sim y\,. \tag{7}$$

Now we shall prove that S is weakly exponential. Suppose that $x, y \in S$. Assume that $xy \sim x$. Then there is a t-archimedean subsemigroup A of S such that $xy, x \in A$. It follows from (4) that $(xy)^m = xux$ for some $u \in A$ and some positive integer m. According to Definition, we have

$$x(xy)^m = (xy)^m x\,. \tag{8}$$

We shall show that

$$(xy)^{m+n} = x^n y^n (xy)^m \tag{9}$$

for all positive integers n. Suppose that (9) is true for some positive integer n. Then, by (8), (9) and Lemma 1, we have $(xy)^{m+n+1} = x^n y^n (xy)^m xy = x^n y^n x (xy)^m y = x^{n+1} y^n (xy)^m y = x^{n+1} y^{n+1} (xy)^m$. It follows from Lemma 2 that

$$(xy)^{m+n} = (xy)^m x^n y^n \,. \tag{10}$$

If $xy \sim y$, then dually we can show that (10) and (9) are true.

Consequently, S is weakly exponential.

Note 4. A description of weakly exponential Δ-semigroup was given by Nagy in [5].

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Received October 25, 1993

Department of Mathematics
Faculty of Electrical Engineering
Czech Technical University
Suchbátorova 2
CZ-166 27 Praha 6
Czech Republic