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ON COMPACT GROUP-VALUED MEASURES

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ABSTRACT. Various concepts of compactness for partially ordered group-valued measures are investigated. Relations between different types of regularities of partially ordered groups are discussed; in main results the lattice structure of po-groups is not assumed.

Introduction

Suppose that μ is an additive set function with values in partially ordered group G. Let σ-additivity of μ be defined via the order structure of G. In his series of papers in the 1970's, J. D. M. Wright studied measures with values in partially ordered vector spaces and vector lattices (=Riesz spaces) [17], [18], [19], [20]. Some of his results were generalized, and related problems were solved, for lattice group valued measures [8], [9], [13], [15] and also for partially ordered group valued measures [10], [12].

One of the main results of the above mentioned Wright's papers is his complete characterization of vector lattices having the extension property. He proved that weak σ-distributivity of a vector lattice V is a necessary and sufficient condition for V-valued μ to be extended from a ring to a σ-ring. He showed that weak σ-distributivity, i.e., an algebraic property of V, is equivalent to regularity of V-valued Baire measures. In [19], the relations between regularity and countable additivity of μ are studied.

Following Marczewski's concept of compact measures (see [7]) authors in [4] and [11] proved Alexandroff and Kolmogoroff theorems for Riesz space valued measures. The goal of this paper is to suggest a concept of compactness which works also in a nonlattice range case and to study the relations between various properties of regularity of partially ordered groups. We point out that abandoning of the lattice structure of a group brings some peculiarities that cannot arise in the lattice case.

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Po-group valued measures

The range space of a set function μ is an ordered commutative group G, i.e., a commutative group $(G,+)$ partially ordered by reflexive, antisymmetric and transitive relation $\leq$ which is consistent with the group structure, i.e., $a \leq b$ implies $a+c \leq b+c$ for any $c \in G$. Of course, if, moreover, G is a real vector space, then a $\leq b$ implies $\lambda \cdot a \leq \lambda \cdot b$ for any positive λ. Although we do not assume that $(G,\leq)$ is a lattice, we suppose that $(G,\leq)$ is directed upwards (i.e. for any $a,b \in G$, there is some $c \in G$ with $c \geq a$ and $c \geq b$) and monotone σ-complete (i.e. every increasing bounded sequence has a least upper bound). Since G is a group (i.e. a $\leq b$ if and only if $-a \geq -b$), it is easy to see that G is also directed downwards, and every decreasing bounded sequence has a largest lower bound. The set of non-negative elements in G will be denoted by G^+; note that $(G,\leq)$ is directed if and only if $G = G^+ - G^+$.

Let $\mathcal{R}$ be a ring of subsets of a nonempty set X and μ be a positive, finitely additive G-valued mapping on $\mathcal{R}$, i.e., $\mu\colon \mathcal{R} \to G^+$, $\mu(A) \geq 0$, for any $A \in \mathcal{R}$, and $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A \cap B = \emptyset$ (due to the additivity, $\mu(\emptyset) = 0$). The countable additivity of μ is defined in terms of the order in G, i.e., μ is countably additive G-valued measure if

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigvee\left\{ \sum_{i=1}^{n} \mu(A_i) \mid n = 1,2,\ldots \right\}$$

whenever (A_i) is a sequence of disjoint sets in $\mathcal{R}$. Instead of $\bigvee\left\{ \sum_{i=1}^{n} a_i \mid n = 1,2,\ldots \right\}$ we will write $\sum_{i=1}^{\infty} a_i$.

It is easy to verify that $(G,\leq)$ is a convergence group, i.e., the mapping $(x,y) \mapsto x-y$ is order continuous. If (x_n) is a decreasing sequence of non negative elements in G, then, for any $a \in G$, $a - \bigwedge x_n = \bigvee(a - x_n)$. From this it is easy to verify that, when $\mu\colon \mathcal{R} \to G^+$ is finitely additive, then the following conditions are equivalent:

(i) μ is countably additive.
(ii) μ is continuous from above at $\emptyset$.
(iii) μ is continuous from below at every $A \in \mathcal{R}$.

Compactness and regularity of measures

A classical theorem of A. D. Alexandroff states that, when μ is a non-negative, finitely additive, and regular measure on the ring $\mathcal{R}$ of subsets of a compact Hausdorff space X, then μ is countably additive on $\mathcal{R}$. The generalization of Alexandroff theorem for vector lattice valued measures was done in [19], where the concept of regularity is the following: A finitely additive measure μ on a ring $\mathcal{R}$ of subsets of a compact Hausdorff space X is regular if, whenever $E \in \mathcal{R}$, there exist a downward filtering family $\mathcal{G} \subseteq \mathcal{R}$ and an upward filtering family $\mathcal{L} \subseteq \mathcal{R}$ such that

$$\bigwedge\{\mu(G) : G \in \mathcal{G}\} = \mu(E) = \bigvee\{\mu(H) : H \in \mathcal{L}\},$$

where for each $G \in \mathcal{G}$, the interior of G contains E, and, for each $H \in \mathcal{L}$, the closure of H is contained in E.

Different concepts of regularity were used in [8], [11] and [4]. The authors used Marczewski's abstract concept of a compact system (see [7]): A system $\mathcal{C}$ of subsets of X is called *compact* if $\bigcap_{n=1}^{\infty} C_n = \emptyset$ ($C_n \in \mathcal{C}$, $n = 1, 2, \dots$) implies that there exists $n_0 \in \mathbb{N}$ such that $\bigcap_{n=1}^{n_0} C_n = \emptyset$. In [8], μ is inner regular on a ring $\mathcal{R}$ with respect to a compact system $\mathcal{C}$, $\mathcal{C} \subseteq \mathcal{R}$, if to any $E \in \mathcal{R}$ there exists a sequence (C_n) in $\mathcal{C}$ such that $C_n \subseteq E$ for each n and $\mu(E - C_n) \searrow 0$.

It is well known ([7]) that, if $\mathcal{C}$ is compact, then $\mathcal{C}_\delta$ is compact ($\mathcal{C}_\delta$ is the class of all sets of the form $\bigcap_{n=1}^{\infty} C_n$, $C_n \in \mathcal{C}$) and also $\mathcal{C}_s$ is compact ($\mathcal{C}_s$ is the system of all sets of the form $\bigcup_{i=1}^{k} C_i$, $C_i \in \mathcal{C}$, $k \in \mathbb{N}$). Note that it is not necessary to assume that $\mathcal{C} \subseteq \mathcal{R}$ (as it is done in [8]). We shall slightly modify both mentioned concepts of regularity in order to compare these concepts with the next one (in Definition 2). As the term "regularity" we want to reserve for properties of a range space, we prefer the term *compact* measure (instead of "inner regular" measure).

DEFINITION 1. Let $\mathcal{R}$ be a ring of subsets of X, and $\mathcal{C}$ be a compact system of subsets of X. An additive $\mu\colon \mathcal{R} \to G^+$ is called

(i) *sup_compact* if for each $E \in \mathcal{R}$

$$\mu(E) = \bigvee\{\mu(F) : \text{ exists } C \in \mathcal{C},\ F \subseteq C \subseteq E\},$$

(ii) *seq_compact* if for each $E \in \mathcal{R}$ there exist sequences (C_n) in $\mathcal{C}$ and (F_n) in $\mathcal{R}$, such that for each $n \in \mathbb{N}$, $F_n \subseteq C_n \subseteq E$ and $\mu(E - F_n) \searrow 0$.

Relation between these concepts is apparent; if μ is seq_compact, then it is sup_compact, and, moreover, if G is order separable, they are equivalent (recall that $(G, \leq)$ is *order separable* if every non empty subset A possessing a supremum contains an at most countable subset possessing the same supremum as A).

PROPOSITION 2. *Let G be order separable, and $\mathcal{C}$ be a compact system closed under the formation of finite unions. Then μ is sup_compact if and only if μ is seq_compact.*

Authors in [4], [11] imitated the well-known epsilon technique whose modification for vector lattice V was successfully used implicitly in [3] and explicitly in [9], [10] in the context of an extension of a V-valued measure. Briefly speaking, the epsilon technique is substituted for double sequences converging to the zero element. Authors in [4] and [11] used compactness in the sense of (i) in the next definition and although they considered Riesz space valued measures, what they really utilized was the lattice structure of the range space. In the next definition, we suggest a modification which does not require the lattice structure of G.

DEFINITION 3. Let $\mathcal{R}$ be a ring of subsets of X, and $\mathcal{C}$ be a compact system of subsets of X. A mapping $\mu\colon \mathcal{R} \to G^+$ is said to be

(i) $\vee$-*compact* if for any $E \in \mathcal{R}$ there exists a bounded double sequence (a_{ij}) in G, $a_{ij} \searrow 0$ $(j \to \infty)$, for each $i \in \mathbb{N}$, such that for any $\varphi\colon \mathbb{N} \to \mathbb{N}$ there exist $C \in \mathcal{C}$ and $F \in \mathcal{R}$ such that $F \subseteq C \subseteq E$ and

$$\mu(E - F) \le \bigvee_{i=1}^{\infty} a_{i\varphi(i)} ,$$

(ii) $\sum$-*compact* if for any $E \in \mathcal{R}$ there exists a double sequence (a_{ij}) in G, $a_{ij} \searrow 0$ $(j \to \infty)$, for $i = 1, 2, \dots$, such that for any $\varphi\colon \mathbb{N} \to \mathbb{N}$ there exist $C \in \mathcal{C}$ and $F \in \mathcal{R}$ such that $F \subseteq C \subseteq E$ and

$$\mu(E - F) \le \sum_{i=1}^{\infty} a_{i\varphi(i)} .$$

PROPOSITION 4. *Let G be a σ-complete lattice group. Then $\mu\colon \mathcal{R} \to G^+$ is $\vee$-compact if and only if it is $\sum$-compact.*

Proof. Since the double sequence (a_{ij}) is a sequence of positive elements, then it is obvious that $\bigvee\limits_{i=1}^{\infty} a_{i\varphi(i)} \le \sum\limits_{i=1}^{\infty} a_{i\varphi(i)}$.

From this we obtain that $\vee$-compactness implies $\sum$-compactness. For the reverse implication, we use the lemma from [15]:

If (a_{ij}) is a double sequence in G such that $a_{ij} \searrow 0$, $(j \to \infty)$ for each $i \in \mathbb{N}$, then to every positive $b \in G$ there exists a bounded double sequence (b_{ij}) in G, $b_{ij} \searrow 0$, $(j \to \infty)$ and such that for any $\varphi\colon \mathbb{N} \to \mathbb{N}$ it holds

$$b \wedge \left(\sum_{i=1}^{\infty} a_{i\varphi(i)} \right) \le \bigvee_{i=1}^{\infty} b_{i\varphi(i)} .$$

Taking $\mu(E)$ in the place of b, the desired implication is proved. □

It is evident that compactness from Definition 3 requires additional assumptions on G. We need to guarantee that $\bigvee_{i=1}^{\infty} a_{i\varphi(i)}$ (and $\sum_{i=1}^{\infty} a_{i\varphi(i)}$) are close to the zero element of G, whenever $a_{ij} \searrow 0$.

Assumptions on range space

We recall that a σ-complete lattice group G is said to be weakly σ-distributive if

$$\bigwedge\left\{ \bigvee_{i=1}^{\infty} a_{i_\varphi(i)} \mid \varphi\colon \mathbb{N} \to \mathbb{N} \right\} = 0$$

whenever (a_{ij}) is a bounded double sequence in G such that $a_{ij} \searrow 0$ $(j \to \infty)$, for each $i \in \mathbb{N}$.

PROPOSITION 5. *If μ is seq_compact, then it is $\vee$-compact. If G is weakly σ-distributive and μ is $\vee$-compact, then it is sup_compact.*

Proof. Let $(F_n) \subseteq \mathcal{R}$ and $(C_n) \subseteq \mathcal{C}$ be due to the assumption. Set $a_{ij} := \mu(E - F_j)$, for each $i \in \mathbb{N}$. Let $\varphi\colon \mathbb{N} \to \mathbb{N}$. Let us define $k = \min\{\varphi(i) : i \in \mathbb{N}\}$ and take $F = F_k$. We have $\mu(E - F) \le \bigvee_{i=1}^{\infty} a_{i\varphi(i)}$.

For the second part, set $a = \mu(E) - \bigvee\{\mu(F) : F \subseteq C \subseteq E,\ C \in \mathcal{C}\}$. It is obvious that

$$\begin{aligned} 0 \le a &= \mu(E) - \bigvee\{\mu(F) : F \subseteq C \subseteq E,\ C \in \mathcal{C}\} \\ &= \bigwedge\{\mu(E - F) : F \subseteq C \subseteq E,\ C \in \mathcal{C}\}. \end{aligned}$$

Due to the assumption, there exists bounded (a_{ij}), $a_{ij} \searrow 0$ $(j \to \infty)$ such that for any φ_0, $\varphi_0\colon \mathbb{N} \to \mathbb{N}$, there exist $F_0 \in \mathcal{R}$ and $C_0 \in \mathcal{C}$, $F_0 \subseteq C_0 \subseteq E$, such that

$$\mu(E - F_0) \le \bigvee_{i=1}^{\infty} a_{i\varphi_0(i)}.$$

We obtain $a = \bigwedge\{\mu(E - F) : F \subseteq C \subseteq E,\ C \in \mathcal{C}\} \le \mu(E - F_0) \le \bigvee_{i=1}^{\infty} a_{i\varphi_0(i)}$ for any $\varphi_0\colon \mathbb{N} \to \mathbb{N}$, i.e., $a \le \bigwedge\left\{ \bigvee_{i=1}^{\infty} a_{i\varphi(i)} \mid \varphi\colon \mathbb{N} \to \mathbb{N} \right\}$. Due to the weak σ-distributivity, we get $a \le 0$ so that $\mu(E) = \bigvee\{\mu(F) : F \subseteq C \subseteq E,\ C \in \mathcal{C}\}$.
□

DEFINITION 6. A monotone σ-complete partially ordered group G is called to be *s-regular* if

$$\bigwedge\left\{\sum_{i=1}^{\infty} a_{i\varphi(i)} \mid \varphi\colon \mathbb{N}\to\mathbb{N}\right\} = 0$$

whenever (a_{ij}) is in G and $a_{ij} \searrow 0$, $(j\to\infty)$, for each $i\in\mathbb{N}$.

Note that this is a bit stronger form of regularity than the concept of g-regularity used in [13], [14]. Now we require that elements $\sum\limits_{i=1}^{\infty} a_{i\varphi(i)}$ are "small enough" even if (a_{ij}) is not bounded from above. Note that the set $\left\{\sum\limits_{i=1}^{\infty} a_{i\varphi(i)} \mid \varphi\colon \mathbb{N}\to\mathbb{N}\right\}$ is directed downwards.

There are several types of "regularities" in vector lattices, e.g., the diagonal property, the d-property, the (strong) Egoroff property (see [6; Chap. 10]). An Archimedean Riesz space possessing the strong Egoroff property is sometimes called a *regular Riesz space.* Relations between the (strong) Egoroff property, g-regularity and weak σ-distributivity in lattice groups were investigated in [14]. The following definition presents known concepts [6], [8] in non-lattice case.

DEFINITION 7. Let G be a partially ordered group. G is said to have the *strong Egoroff property* if, given any double sequence (a_{ij}) such that $a_{ij}\searrow 0$ $(j\to\infty)$ for each $i\in\mathbb{N}$, there exist a sequence (b_k) in G, $b_k\searrow 0$, $(k\to\infty)$ and $\varphi\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ with the property that, for every pair $(k,i)\in\mathbb{N}\times\mathbb{N}$, we have $b_k \ge a_{i\varphi(k,i)}$.

G is said to be *weakly regular* if to any a $\in G^+$ and any double (a_{ij}), $a_{ij}\searrow 0$, $(j\to\infty)$, for each $i\in\mathbb{N}$, there exists $\varphi\colon\mathbb{N}\to\mathbb{N}$ such that $a\le\sum\limits_{i=1}^{n} a_{i\varphi(i)}$ holds for no n.

PROPOSITION 8. *If G is a partially ordered vector space and has the strong Egoroff property, then G is s-regular. If G is s-regular group, then it is weakly regular.*

Proof. Let $a_{ij}\searrow 0$ $(j\to\infty)$ and set $b_{ij}=2^i a_{ij}$. Due to the assumption, there exist c_k in G, $c_k\searrow 0$, and $\varphi\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ such that for each $(k,i)\in\mathbb{N}\times\mathbb{N}$, $c_k\ge b_{i\varphi(k,i)}$ holds, i.e., $2^{-i}c_k\ge a_{i\varphi(k,i)}$. Thus we obtain $c_k=\sum\limits_{i=1}^{\infty}2^{-i}c_k\ge \sum\limits_{i=1}^{\infty} a_{i\varphi(k,i)}$, and since $c_k\searrow 0$, this concludes the proof of the first part. The second part is easy when we start indirectly; if G is not weakly regular, then there exist $a\in G^+$ and (a_{ij}), $a_{ij}\searrow 0$ such that for any φ, $\varphi\colon\mathbb{N}\to\mathbb{N}$, there exists $n \in \mathbb{N}$ such that $a \le \sum_{i=1}^{n} a_{i\varphi(i)}$, i.e., positive element $a \in G^+$ bounds from below the set $\left\{ \sum_{i=1}^{\infty} a_{i\varphi(i)} \mid \varphi\colon \mathbb{N} \to \mathbb{N} \right\}$. This contradicts the s-regularity. □

Using the representation theorem for lattice groups ([1; Theorem 4.]), it is possible to prove that the strong Egoroff property of a lattice group G implies s-regularity. But different techniques must be used in a non lattice case. From the definition of an ordered linear space, one can conclude that, if $nx \ge 0$ for some $n \in \mathbb{N}$, then $x \ge 0$. It is known that this implication holds in commutative lattice groups ([2]), but in non lattice case $2x \ge 0$ does not imply $x \ge 0$; for example, let the integers be ordered by the semigroup of all non negative integers except 1. In the proof of the next lemma, we need that $2x \ge 2y$ implies $x \ge y$.

LEMMA 9. *Let G a be partially ordered group, and let for each $x \in G$, $2x \ge 0$ if and only if $x \ge 0$. If $a_k \ge 0$, $c \ge 2^k a_k$, $k = 1, \dots, n$, then $c \ge a_1 + a_2 + \dots + a_n$.*

Proof. From $c \ge 2^n a_n \ge 2^{n-1} a_n$ and $c \ge 2^{n-1} a_{n-1}$, we have $2c \ge 2(2^{n-2} a_n + 2^{n-2} a_{n-1})$ and $c \ge 2^{n-2} a_n + 2^{n-2} a_{n-1}$. Adding the inequality $c \ge 2^{n-2} a_{n-2}$ we have $c \ge 2^{n-3} a_n + 2^{n-3} a_{n-1} + 2^{n-3} a_{n-2}$. After n such steps, we get the desired inequality. □

PROPOSITION 10. *Let G be partially ordered group in which $2x \ge 0$ implies $x \ge 0$. If G has the strong Egoroff property, then G is s-regular.*

Proof. We can proceed as we did in the proof of Proposition 8. Using that notation we have $c_k \ge 2^i a_{i\varphi(k,i)}$, and this is true for $i = 1, 2, \dots, n, \dots$. According to Lemma 9, for each n we have $c_k \ge \sum_{i=1}^{n} a_{i\varphi(k,i)}$ so that $c_k \ge \sum_{i=1}^{\infty} a_{i\varphi(k,i)}$. Since $c_k \searrow 0$, $(k \to \infty)$, $\bigwedge \left\{ \sum_{i=1}^{\infty} a_{i\varphi(i)} \mid \varphi\colon \mathbb{N} \to \mathbb{N} \right\} = 0$. □

In [12], the property of a po-group $(G, \le)$ is formulated via its order dual. According to [12], let $G^{\le}$ be the set of all order continuous additive functionals on G which can be represented as a difference of two monotone additive functionals. G is called a *separative* group if for each non zero $x \in G$ there exists $\xi \in G^{\le}$ such that $\xi(x) \ne 0$. A simple example of a separative, monotone σ-complete group is the set of all polynomials of the degree at most m with pointwise ordering. On the other hand, $L_p[0,1]$ $(0 < p < 1)$ is an example of the ordered vector space having no non zero monotone linear functionals (see [5; p. 21]).

PROPOSITION 11. *If G is monotone σ-complete, separative group, then it is weakly regular.*

Proof. Let us proceed indirectly; let $a \in G^+$ be positive, and (a_{ij}) in G be such that, for each $i \in N$, $a_{ij} \searrow 0$, and for any $\varphi\colon \mathbb{N} \to \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $a \le \sum_{i=1}^{n} a_{i\varphi(i)}$. Due to the assumption, there exists $\xi \in G^{\le}$ such that $\xi(a) > 0$. For the double sequence of reals $\big(\xi(a_{ij})\big)_{ij}$ one can construct ψ, $\psi\colon \mathbb{N} \to \mathbb{N}$, such that, for each $n \in \mathbb{N}$, $\sum_{i=1}^{n} \xi\big(a_{i\psi(i)}\big) < \xi(a)$. This is a contradiction. □

The next Proposition gives two sufficient conditions for an additive set function $\mu\colon \mathcal{R} \to G^+$ to be countably additive. Taking milder formulation of the compactness of μ, a stronger property of the range is needed to be required.

PROPOSITION 12. *Let G be a monotone σ-complete po-group, and let $\mu\colon \mathcal{R} \to G^+$ be additive. Each couple of the following conditions is sufficient for the countable additivity of μ:*

(i) *μ is seq_compact and G is weakly regular.*
(ii) *μ is $\sum$-compact and G is s-regular.*

Proof.

(i) The proof given in [8] can easily be modified for the seq_compact μ as the lattice structure does not play any role in it.

(ii) The proof is given in [16], where the author did not distinguish appropriately the difference between g-regularity and s-regularity of G. □

Let us denote by $\mathcal{R}_\delta$ the smallest class of sets closed under the formation of countable intersections, containing $\mathcal{R}$.

PROPOSITION 13. *Let $\mu\colon \mathcal{S} \to G^+$ be additive and countably subadditive on a σ-ring $\mathcal{S}$. If $\mathcal{R} \subseteq \mathcal{S}$ is a ring, and μ is $\sum$-compact on $\mathcal{R}$ (with respect to $\mathcal{C}$), then μ is $\sum$-compact on $\mathcal{R}_\delta$ (with respect to the compact system $\mathcal{C}_\delta$).*

Proof. Let $A = \bigcap_{n=1}^{\infty} A_n \in \mathcal{R}_\delta$, $A_n \in \mathcal{R}$. By the assumption, for each $n \in \mathbb{N}$ there exists a double (a^n_{ij}), $a^n_{ij} \searrow 0$ $(j \to \infty)$ such that for any φ, $\varphi\colon \mathbb{N} \to \mathbb{N}$, there exist $C^\varphi_n \in \mathcal{C}$ and $B^\varphi_n \in \mathcal{R}$, $B^\varphi_n \subseteq C^\varphi_n \subseteq A_n$, such that

$$\mu(A_n - B^\varphi_n) \le \sum_{i=1}^{\infty} a^n_{i\varphi(i)}\,.$$

Observe that we can conclude the following: for any φ, $\varphi\colon \mathbb{N} \to \mathbb{N}$, there exist $C^\varphi_n \in \mathcal{C}$ and $B^\varphi_n \in \mathcal{R}$, $B^\varphi_n \subseteq C^\varphi_n \subseteq A_n$, such that

$$\mu(A_n - B^\varphi_n) \le \sum_{i=1}^{\infty} a^n_{i\varphi(i+n-1)}\,.$$

Now we need the following lemma (its proof is given in [16]):

LEMMA. *If (a_{ij}^n) is a triple sequence in G such that $a_{ij}^n \searrow 0$ $(j \to \infty)$ for each $n, i \in \mathbb{N}$, then there exists a double sequence (b_{ij}) in G such that $b_{ij} \searrow 0$ $(j \to \infty)$ and*

$$\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} a_{i\varphi(i+n-1)}^n \leq \sum_{k=1}^{\infty} b_{k\varphi(k)}$$

holds for any $\varphi \colon \mathbb{N} \to \mathbb{N}$.

To an arbitrary ψ, $\psi \colon \mathbb{N} \to \mathbb{N}$, we can consider sequences B_n^ψ, C_n^ψ and the double sequence (b_{ij}); take $B = \bigcap_{n=1}^{\infty} B_n^\psi$ and $C = \bigcap_{n=1}^{\infty} C_n^\psi$. We get

$$\begin{aligned} \mu(A - B) &= \mu\left(\bigcap_{n=1}^{\infty} A_n - \bigcap_{n=1}^{\infty} B_n^\psi \right) \leq \mu\left(\bigcup_{n=1}^{\infty} (A_n - B_n^\psi) \right) \\ &\leq \sum_{n=1}^{\infty} \mu(A_n - B_n^\psi) \leq \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} a_{i\psi(i+n-1)}^n \leq \sum_{k=1}^{\infty} b_{k\psi(k)} . \end{aligned}$$

Since $B \in \mathcal{R}_\delta$, $C \in \mathcal{C}_\delta$ and $B \subseteq C \subseteq A$, the compactness of μ on $\mathcal{R}_\delta$ (with respect to $\mathcal{C}_\delta$) is proved. □

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