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THE WEAK SUBALGEBRA LATTICE OF A UNARY PARTIAL ALGEBRA OF A GIVEN INFINITE UNARY TYPE

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ABSTRACT. In the present paper we characterize the weak subalgebra lattice of a unary partial algebra of a given infinite unary type. This lattice satisfies the conditions from [BARTOL, W.: *Weak subalgebra lattices*, Comment. Math. Univ. Carolin. **31** (1990), 405–410] and moreover, there exists an algebraic closure operator on the set of all atoms of this lattice which satisfies one special condition concerning its join-irreducible elements. Such characterization for finite unary types was given in the previous part [PIÓRO, K.: *The weak subalgebra lattice of a unary partial algebra of a given finite unary type*, Acta Sci. Math. **65** (1999), 439–460].

In [PIÓRO, K.: *On some non-obvious connections between graphs and unary partial algebras*, Czechoslovak Math. J. **50(125)** (2000), 295–320] we reduced our algebraic problem for infinite unary types to a graph question: Let G be a graph (which may have infinite sets of vertices and edges) and let η be an infinite cardinal number: when can edges of G be directed so that at most η directed edges start from every vertex? In this paper we first solve this graph problem and hence we easily obtain a solution of our algebraic problem for infinite unary types.

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Universal algebra is quite rich in papers which investigate connections between a (total) algebra and its lattice of (also total) subalgebras (see e.g. [Jón] or [Grä]). For example, the full characterization of the subalgebra lattice of a (total) algebra is in [BiFr]. Moreover, [JoSe] (see also [Jón]) contains a characterization of the subalgebra lattice of a unary (total) algebra of a given unary type. Recall that such a characterization for arbitrary algebras (not only unary)

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of a given type is an important problem of universal algebra (see [Jón]), which is not completely solved yet.

In the theory of partial algebras a few different lattices of subalgebras can be considered for an arbitrary algebra and of course investigations of subalgebra lattices are also its important part (see e.g. [BRR] or [Bur]). For example, [Bar] contains a complete characterization of the weak subalgebra lattice of a partial algebra (in particular also of a unary partial algebra). In the present paper and in the previous part [Pió2] we characterize the weak subalgebra lattice of a unary partial algebra of a given unary type. In other words, for a given unary type we describe all lattices $\mathbf{L}$ for which there exists a unary partial algebra of this unary type such that its weak subalgebra lattice is isomorphic to $\mathbf{L}$. Recall that we reduced in [Pió1] this algebraic problem to the following graph question: Let $\mathbf{G}$ be a graph and let η be a cardinal number. When (necessary and sufficient conditions) is there a directed graph $\mathbf{D}$ of type η (i.e. at most η directed edges start from every vertex of $\mathbf{D}$) such that the graph obtained from $\mathbf{D}$ by omitting the orientation of all edges is isomorphic to $\mathbf{G}$?

The above graph problem for finite types of directed graphs (i.e. η is a natural number) was solved in the previous part [Pió2]. Hence we obtained the solution of our algebraic problem for finite unary types.

In the present paper we solve this graph problem for infinite types of directed graphs (i.e. η is an infinite cardinal number) and in this way we obtain the solution of our algebraic problem for infinite unary types. More precisely, we prove that for a graph $\mathbf{G}$ and an infinite cardinal number η the desired directed graph exists if and only if there exists an algebraic closure operator on the set of all vertices of $\mathbf{G}$ which satisfies one special condition concerning its edges. From this result we obtain that a lattice $\mathbf{L}$ is isomorphic to the weak subalgebra lattice of a unary partial algebra of a given infinite unary type K if and only if $\mathbf{L}$ satisfies the conditions from [Bar] and moreover, there exists an algebraic closure operator on the set of all atoms of $\mathbf{L}$ which satisfies one special condition concerning its join-irreducible elements.

We assume knowledge of basic concepts and facts from the theory of partial and total algebras, and also from lattice theory (see e.g. [Bur], [BRR], [Grä] and [Jón]). In the whole paper the cardinality of a set A is denoted by $|A|$, $\mathbb{N}$ is the set of all non-negative integers and $\aleph_0$ is the infinite countable cardinal number, i.e. $\aleph_0 = |\mathbb{N}|$. Further, $\simeq$ denotes simultaneously isomorphism of algebras, lattices, graphs, etc. Moreover, $+$ and $\cdot$ denote, in general, the addition and the multiplication of arbitrary (not only finite) cardinal numbers.

For an arbitrary cardinal number η and $n \in \mathbb{N}$, $\eta_0 := \eta$ and η_{n+1} is the least cardinal number greater that η_n.

Recall that a type of an algebra is a pair (K, κ), where K is a set (its elements will be called operation symbols) and κ is a map of K into $\mathbb{N}$ (called the arity function). A type (K, κ) is unary if and only if $\kappa(K) \subseteq \{1\}$. Observe that a unary type (K, κ) can be identified with the set K. We say that a unary type K is finite (infinite) if and only if K is a finite (infinite) set.

A unary partial algebra of a unary type K is an algebra $\mathbf{A} = \big(A, (k^A)_{k \in K}\big)$ such that for all $k \in K$, k^A is a partial function from A to A. Let $\mathbf{A}$ and $\mathbf{B}$ be unary partial algebras of the same unary type K. Recall that $\mathbf{B}$ is a weak subalgebra of $\mathbf{A}$ ($\mathbf{B} \leq_w \mathbf{A}$) if and only if $B \subseteq A$ and $k^B \subseteq k^A$ for each $k \in K$. The set $S_w(\mathbf{A})$ of all weak subalgebras of the algebra $\mathbf{A}$ with the relation $\leq_w$ forms a complete, and also algebraic, lattice $\mathbf{S}_w(\mathbf{A}) = \big(S_w(\mathbf{A}), \leq_w\big)$.

In this paper we will use some connections between graphs and unary partial algebras from [Piól]. Recall (see e.g. [Ber]) that an (undirected) graph $\mathbf{G} = (V^G, E^G, I^G)$ is a triplet such that V^G and E^G are sets of vertices and edges respectively, and I^G is a function of E^G into the set $\big\{\{v, w\} : v, w \in V^G\big\}$ of all undirected pairs. A digraph (directed graph) $\mathbf{G} = (V^G, E^G, I^G)$ is a triplet such that V^G and E^G are sets of vertices and edges respectively, and $I^G = (I_1^G, I_2^G)$ is a function of E^G into the product $V^G \times V^G$. Note that we admit loops and multiple edges, similarly as in [Ber]. On the other hand, only finite graphs (i.e. with finitely many vertices and edges) are considered in [Ber], but in the present paper we need infinite graphs and digraphs, and therefore we do not restrict the cardinality of vertex and edge sets.

Observe that with any digraph $\mathbf{G}$ we can associate the graph $\mathbf{G}^*$ by omitting the orientation of all edges, i.e.

$$V^{G^*} := V^G, \quad E^{G^*} := E^G \quad \text{and} \quad I^{G^*}(e) := \{I_1^G(e), I_2^G(e)\} \quad \text{for all } e \in E^G.$$

Let $\mathbf{G}$ and $\mathbf{H}$ be digraphs (graphs). Then $\mathbf{H}$ is a weak subdigraph (subgraph) of $\mathbf{G}$ ($\mathbf{H} \leq_w \mathbf{G}$) if and only if $V^H \subseteq V^G$, $E^H \subseteq E^G$, $I^H \subseteq I^G$. $\mathbf{H}$ is a relative subdigraph (subgraph) of $\mathbf{G}$ ($\mathbf{H} \leq_r \mathbf{G}$) if and only if $\mathbf{H} \leq_w \mathbf{G}$ and for every $e \in E^G$, if $I^G(e) \in V^H \times V^H$ ($I^G(e) \subseteq V^H$), then $e \in E^H$. We proved in [Piól] that for every digraph (graph) $\mathbf{G}$, the set $S_w(\mathbf{G})$ of all weak subdigraphs (subgraphs) of $\mathbf{G}$ with the relation $\leq_w$ forms an algebraic lattice $\mathbf{S}_w(\mathbf{G}) = \big(S_w(\mathbf{G}), \leq_w\big)$.

Let $\mathbf{G}$ be a digraph. Then for every $v \in V^G$ we define the set of edges: $E_s^G(v) := \big\{e \in E^G : v = I_1^G(e)\big\}$, and the cardinal number: $s^G(v) := |E_s^G(v)|$.

Let $\mathbf{G}$ be a digraph and let η be a cardinal number. We say that $\mathbf{G}$ is of type η if and only if $s^G(v) \leq \eta$ for all $v \in V^G$.

Let $\mathbf{L} = (L, \leq_L)$ be an arbitrary lattice, recall (see e.g. [CrDi] or [Jón]) that an element $l \in L$ is join-irreducible if and only if for every $k_1, k_2 \in L$, $l = k_1 \vee k_2$ implies $l = k_1$ or $l = k_2$. We will use the notation: $\mathrm{At}(\mathbf{L})$ is the set of all atoms of $\mathbf{L}$, $\mathrm{Ir}(\mathbf{L})$ is the set of all non-zero and non-atomic join-irreducible elements of $\mathbf{L}$ and for every $i \in L$, $\mathrm{At}(i) := \big\{a \in \mathrm{At}(\mathbf{L}) : a \leq_L i\big\}$.

Recall that a complete characterization of the weak subalgebra lattice of a unary partial algebra is given in [Bar]. More precisely, the following result was proved in [Bar] ((a) $\iff$ (c)) and [Pió1] ((a) $\iff$ (b)):

THEOREM 1.1. *Let $\mathbf{L} = (L, \leq_L)$ be an arbitrary lattice. Then the following conditions are equivalent:*

(a) *There is a unary partial algebra $\mathbf{A}$ such that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$.*
(b) *There is a digraph $\mathbf{G}$ such that $\mathbf{S}_w(\mathbf{G}) \simeq \mathbf{L}$.*
(c) *$\mathbf{L}$ satisfies the following four conditions:*
   (c.1) *$\mathbf{L}$ is algebraic and distributive,*
   (c.2) *every element is a join of join-irreducible elements,*
   (c.3) *for every $i \in \mathrm{Ir}(\mathbf{L})$, $1 \leq |\mathrm{At}(i)| \leq 2$,*
   (c.4) *$\mathrm{Ir}(\mathbf{L})$ is an antichain with respect to the lattice ordering $\leq_L$.*

Recall (see [Pió1]) that with every lattice $\mathbf{L}$ which satisfies (c.1) – (c.4) of Theorem 1.1 we can associate the graph $\mathbf{G}(\mathbf{L})$ as follows:

$$V^{G(L)} := \mathrm{At}(\mathbf{L}), \quad E^{G(L)} := \mathrm{Ir}(\mathbf{L}) \quad \text{and} \quad I^{G(L)}(e) := \mathrm{At}(e) \ \text{ for all } e \in E^{G(L)}.$$

Moreover, the following result, which is a basis to investigate our algebraic problem, was proved in [Pió1]:

THEOREM 1.2. *Let a lattice $\mathbf{L}$ satisfy* (c.1) – (c.4) *of Theorem* 1.1 *and let K be a unary type. Then there is a unary partial algebra $\mathbf{A}$ of type K such that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$ if and only if there is a digraph $\mathbf{D}$ of type $|K|$ such that $\mathbf{D}^* \simeq \mathbf{G}(\mathbf{L})$.*

The above theorem reduces our algebraic problem for infinite unary types to the following graph question: Let $\mathbf{G}$ be a graph and let η be an infinite cardinal number. When (necessary and sufficient conditions) is there a digraph $\mathbf{D}$ of the infinite type η such that $\mathbf{D}^* \simeq \mathbf{G}$?

In the sequel we will use the following definition from [Pió2] to simplify formulations and proofs of the results in this paper.

DEFINITION 1.3. Let $\mathbf{G}$ be a graph and let η be a cardinal number. We say that $\mathbf{G}$ *can be directed in the type* η if and only if there exists a digraph $\mathbf{D}$ of type η such that $\mathbf{D}^* = \mathbf{G}$.

It is easy to verify (see [Pió2]) that the following fact holds:

PROPOSITION 1.4. *Let η be a cardinal number and let $\mathbf{G}$ be a graph which can be directed in the type η. Then every weak subgraph $\mathbf{H} \leq_u \mathbf{G}$ can be also directed in the type η.*

It is not difficult to see that for a given cardinal number η there are graphs which cannot be directed in the type η. For example, let $\mathbf{G}$ be a graph with exactly one vertex and more than η loops. Then it is trivial that $\mathbf{G}$ cannot be directed in the type η.

It easily follows from the above example that for a unary type K there are lattices $\mathbf{L}$ which satisfy (c.1) – (c.4) of Theorem 1.1, but there is no unary partial algebra of type K such that its weak subalgebra lattice is isomorphic to $\mathbf{L}$. For instance, let $L := \{\emptyset\} \cup \{C \subseteq B \cup \{b\} : \ b \in C\}$, where B is a set such that $|B| > |K|$ and $b \notin B$. It is trivial that L with set-inclusion $\subseteq$ forms a complete lattice $\mathbf{L}$ which is a complete sublattice of the lattice of all subsets of $B \cup \{b\}$. Obviously $\mathrm{At}(\mathbf{L}) = \{\{b\}\}$ and $\mathrm{Ir}(\mathbf{L}) = \{\{b, c\} : \ c \in B\}$. These facts imply first that $\mathbf{L}$ satisfies (c.1) – (c.4) of Theorem 1.1, and secondly, $\mathbf{G}(\mathbf{L})$ contains exactly one vertex and $|\mathrm{Ir}(\mathbf{L})|$ $(= |B| > |K|)$ loops. Thus $\mathbf{G}(\mathbf{L})$ cannot be directed in the type $|K|$. Hence and by Theorem 1.2, there is no unary partial algebra $\mathbf{A}$ of type K such that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$.

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In the previous section we have reduced our algebraic problem for infinite unary types to the following graph question: Let $\mathbf{G}$ be a graph and let η be an infinite cardinal number (i.e. $\eta \geq \aleph_0$). When (necessary and sufficient conditions) can $\mathbf{G}$ be directed in the infinite type η?

In this section we first show that if $\mathbf{G}$ contains at most η_1 vertices and between every two vertices there are at most η edges, then $\mathbf{G}$ can be directed in the infinite type η.

Next we translate this result into the lattice language to obtain an analogous result for arbitrary lattices and infinite unary types. More formally, for every infinite unary type K and a lattice $\mathbf{L}$, if $\mathbf{L}$ satisfies (c.1) – (c.4) of Theorem 1.1 and contains at most $|K|_1$ atoms and for each two atoms there are at most $|K|$ join-irreducible elements which contain these two atoms, then $\mathbf{L}$ is isomorphic to the weak subalgebra lattice of some unary partial algebra of type K.

In the second part of this section we prove that if a graph $\mathbf{G}$ can be directed in an infinite type η, then there exists an algebraic closure operator on the vertex set V^G which satisfies also one special condition concerning its edges. In the subsequent section we prove that this is also a necessary condition, but the proof of this result will be more complicated.

At the beginning we formulate a few simple facts. We need the following notation: Let $\mathbf{G}$ be a graph and $v, w \in V^G$, then

$$E_s^G(v, w) := \{e \in E^G : \ I^G(e) = \{v, w\}\} \quad \text{and} \qquad s^G(v, w) := |E_s^G(v, w)|.$$

PROPOSITION 2.1. *Let η be an infinite cardinal number and let* $\mathbf{G}$ *be a graph which can be directed in the infinite type η. Then for every $v, w \in V^G$,*

$$s^G(v,w) \le \eta .$$

P r o o f. Let $\mathbf{D}$ be a graph of the type η such that $\mathbf{D}^* = \mathbf{G}$ and take $v, w \in V^G$. Then it is easy to see that $E_s^G(v,w) = E_s^{D^*}(v,w) \subseteq E_s^D(v) \cup E_s^D(w)$. Hence, $s^G(v,w) \le |E_s^D(v) \cup E_s^D(w)| \le |E_s^D(v)| + |E_s^D(w)| = s^D(v) + s^D(w) \le \eta + \eta = \eta$, because $\mathbf{D}$ is of type η and $\eta \ge \aleph_0$. □

Now we show that our graph problem "when can a graph be directed in an infinite type?" can be reduced to simple graphs. Recall that a graph $\mathbf{G}$ is *simple* if and only if it does not contain loops and between every two vertices there is at most one edge. The set of all regular edges (loops) of a graph $\mathbf{G}$ will be denoted by E^G_{reg} (E^G_{lo}); recall that $e \in E^G$ is a *regular edge* (*loop*) if and only if $|I^G(e)| = 2$ ($|I^G(e)| = 1$).

PROPOSITION 2.2. *Let η be an infinite cardinal number and let a graph* $\mathbf{G}$ *and a simple graph* $\mathbf{H}$ *satisfy the following conditions:*

($*$) $\mathbf{H} \le_w \mathbf{G}$ *and* $V^H = V^G$.
($**$) *For every* $v, w \in V^G$, $s^G(v,w) \le \eta$.
($***$) *For all* $v, w \in V^G$ *with* $v \ne w$, $s^H(v,w) = 0$ *if and only if* $s^G(v,w) = 0$.

Then the following two conditions are equivalent:

(a) $\mathbf{G}$ *can be directed in the infinite type* η.
(b) $\mathbf{H}$ *can be directed in the infinite type* η.

Remark. Note that by virtue of the axiom of choice for every graph $\mathbf{G}$ there is a simple graph $\mathbf{H}$ which satisfies ($*$) and ($***$) of Proposition 2.2.

P r o o f. By Proposition 1.4 and ($*$) we have the implication (a) $\Longrightarrow$ (b).

(b) $\Longrightarrow$ (a): Let $\mathbf{M}$ be a digraph of type η such that $\mathbf{M}^* = \mathbf{H}$. Since $\mathbf{H}$ is a simple graph, we have from ($***$) that for every $f \in E^G_{\mathrm{reg}}$ there is exactly one $e_f \in E^H_{\mathrm{reg}}$ such that $I^G(f) = I^H(e_f) = I^{M^*}(e_f) = \{I_1^M(e_f), I_2^M(e_f)\}$.

Now we must only direct each $f \in E^G_{\mathrm{reg}}$ according to e_f in the digraph $\mathbf{M}$, because every loop $f \in E^G_{\mathrm{lo}}$ can be of course considered as a directed edge. More formally, let $\mathbf{D}$ be the digraph such that $V^D := V^G$, $E^D := E^G$ and $I^D(f) := I^M(e_f)$ for $f \in E^G_{\mathrm{reg}}$ and $\{I_1^D(f)\} = \{I_2^D(f)\} = I^G(f)$ for $f \in E^G_{\mathrm{lo}}$.

It is trivial that $\mathbf{D}^* = \mathbf{G}$, so we must only show that $\mathbf{D}$ is of type η. Let $v \in V^D = V^G$. Then it is easily shown that

$$E_s^D(v) = \bigcup_{e \in E_s^M(v)} \{f \in E^G_{\mathrm{reg}} : I^G(f) = I^G(e)\} \cup \{f \in E^G_{\mathrm{lo}} : I^G(f) = \{v\}\} .$$

Moreover, by $(**)$

$$\left|\{f \in E^G_{\mathrm{reg}} :\ I^G(f) = I^G(e)\}\right| = s^G\big(I^G(e)\big) \le \eta \qquad \text{for all} \quad e \in E^M .$$

Since $\mathbf{M}$ is a digraph of type η, we obtain from the above two facts that

$$\begin{aligned} s^D(v) := |E^D_s(v)| &= \Big| \bigcup_{e \in E^M_s(v)} \{f \in E^G_{\mathrm{reg}} :\ I^G(f) = I^G(e)\} \Big| + s^G(v,v) \\ &\le |E^M_s(v)| \cdot \eta + \eta \le \eta \cdot \eta + \eta = \eta . \end{aligned}$$

Hence, $\mathbf{D}$ is of type η.

Thus we have shown that $\mathbf{G}$ can be directed in the infinite type η, which completes the proof of the implication (b) $\Longrightarrow$ (a). □

PROPOSITION 2.3. *Let a graph* $\mathbf{G}$ *and an infinite cardinal number* η *satisfy the following conditions:*

$$|V^G| \le \eta \qquad \textit{and} \qquad s^G(v,w) \le \eta \quad \textit{for every } v, w \in V^G .$$

Then $\mathbf{G}$ *can be directed in the infinite type* η.

Proof. First, the cardinality of $\big\{\{v,w\} :\ v, w \in V^G\big\}$ is not greater than η, since $\eta \ge \aleph_0$. Secondly, we have obviously that $E^G = \bigcup_{v,w \in V^G} E^G_s(v,w)$. Thus E^G is the sum of at most η sets and each of them has a cardinality not greater than η. Hence, $|E^G| \le \eta$, because $\eta \ge \aleph_0$.

Now we must only take an arbitrary digraph $\mathbf{D}$ such that $\mathbf{D}^* = \mathbf{G}$ (it is enough to apply the axiom of choice and arbitrarily direct all edges of $\mathbf{G}$). It is trivial that $\mathbf{D}$ is of type η, since $s^D(v) \le |E^D| = |E^G| \le \eta$ for $v \in V^G$. □

Now we prove a stronger version of the above result. More precisely, it is enough to assume in Proposition 2.3 that $|V^G| \le \eta_1$ instead of $|V^G| \le \eta$. At the end of this paper we will show that for every infinite cardinal number there are simple graphs which cannot be directed in this type. Thus a result stronger than the theorem below does not hold.

THEOREM 2.4. *Let a graph* $\mathbf{G}$ *and an infinite cardinal number* η *satisfy the following conditions:*

$(*)$ $|V^G| \le \eta_1$.

$(**)$ *For every* $v, w \in V^G$, $s^G(v,w) \le \eta$.

Then $\mathbf{G}$ *can be directed in the infinite type* η.

Proof. By virtue of Proposition 2.3 we can of course assume that $|V^G| = \eta_1$. Thus, applying Zermelo's Theorem and the definitions of ordinal and cardinal numbers, we can put all vertices of $\mathbf{G}$ in a transfinite and injective sequence

$(v_\alpha)_{\alpha<\eta_1}$ of order type η_1, i.e. $V^G = \{v_\alpha : \alpha < \eta_1\}$ and $v_\alpha \neq v_\beta$ for all $\alpha < \beta < \eta_1$.

Now let us define the following sets of edges:

$$E_\alpha := \bigcup_{\gamma<\alpha} E_s^G(v_\gamma, v_\alpha) \qquad \text{for every} \quad \alpha < \eta_1 .$$

Take $\alpha < \eta_1$ and observe first that from $(**)$ we have the following inequalities: $|E_\alpha| \leq \sum\limits_{\gamma<\alpha} s^G(v_\gamma, v_\alpha) \leq |\{\gamma : \gamma < \alpha\}| \cdot \eta$. Secondly, from the well-known results of Set Theory we have $|\{\gamma : \gamma < \alpha\}| \leq \alpha < \eta_1$, so $|\{\gamma : \gamma < \alpha\}| \leq \eta$ by the definition of η_1. These two facts imply $|E_\alpha| \leq \eta \cdot \eta = \eta$, because $\eta \geq \aleph_0$. Thus we have shown the inequality

$$|E_\alpha| \leq \eta \qquad \text{for each} \quad \alpha < \eta_1 . \tag{1}$$

Now let $\mathbf{D}$ be the digraph such that $V^D := V^G = \{v_\alpha : \alpha < \eta_1\}$, $E^D := E^G$ and for every $e \in E^D$, $I^D(e) := (v_\alpha, v_\beta)$, where $\{v_\alpha, v_\beta\} = I^G(e)$ and $\beta \leq \alpha$.

Then it is obvious that $\mathbf{D}^* = \mathbf{G}$. Moreover, it is easy to see that

$$E_s^D(v_\alpha) = E_\alpha \cup \{e \in E^G : I^G(e) = \{v_\alpha\}\} \qquad \text{for each} \quad \alpha < \eta_1 ,$$

so

$$s^D(v_\alpha) \leq |E_\alpha| + s^G(v_\alpha, v_\alpha) .$$

This and (1) and $(**)$ imply $s^D(v_\alpha) \leq \eta + \eta = \eta$ for every $\alpha < \eta_1$, because $\eta \geq \aleph_0$. Hence, $s^D(v) \leq \eta$ for all $v \in V^D$. Thus the digraph $\mathbf{D}$ is of the type η, which completes our proof. □

Now we translate the results obtained for graphs into the lattice language to get some results about the weak subalgebra lattice of a unary partial algebra of an infinite unary type.

PROPOSITION 2.5. *Let K be an infinite unary algebraic type and let a lattice $\mathbf{L}$ be isomorphic to the weak subalgebra lattice of some unary partial algebra of the unary type K. Then for every $a, b \in \mathrm{At}(\mathbf{L})$,*

$$\big|\{i \in \mathrm{Ir}(\mathbf{L}) : \mathrm{At}(i) = \{a, b\}\}\big| \leq |K| .$$

P r o o f. By Theorem 1.2, $\mathbf{G}(\mathbf{L})$ can be directed in the type $|K|$. Moreover. it is obvious that $s^{G(L)}(a,b) = \big|\{i \in \mathrm{Ir}(\mathbf{L}) : \mathrm{At}(i) = \{a,b\}\}\big|$ for every $a, b \in V^{G(L)} = \mathrm{At}(\mathbf{L})$. These two facts and Proposition 2.1 imply our proposition. □

PROPOSITION 2.6. *Let K be an infinite unary algebraic type and let lattices $\mathbf{L}_1$, $\mathbf{L}_2$ satisfy* (c.1) – (c.4) *of Theorem* 1.1 *with $\mathbf{L}_2$ a complete sublattice of $\mathbf{L}_1$. Moreover, let K, $\mathbf{L}_1$ and $\mathbf{L}_2$ satisfy the following conditions:*

$(*)$ $\mathrm{At}(\mathbf{L}_2) = \mathrm{At}(\mathbf{L}_1)$ *and* $\mathrm{Ir}(\mathbf{L}_2) \subseteq \mathrm{Ir}(\mathbf{L}_1)$.

(**) *For every* $a, b \in \mathrm{At}(\mathbf{L}_1)$,
$\left|\{i \in \mathrm{Ir}(\mathbf{L}_1) : \ \mathrm{At}(i) = \{a,b\}\}\right| \le |K|$,
$\left|\{i \in \mathrm{Ir}(\mathbf{L}_2) : \ \mathrm{At}(i) = \{a,b\}\}\right| \le 1$,
$\left|\{i \in \mathrm{Ir}(\mathbf{L}_2) : \ \mathrm{At}(i) = \{a\}\}\right| = 0$.

(***) *For every* $a, b \in \mathrm{At}(\mathbf{L}_1)$ *with* $a \neq b$,

$$\{i \in \mathrm{Ir}(\mathbf{L}_1) : \ \mathrm{At}(i) = \{a,b\}\} = \emptyset \iff \{i \in \mathrm{Ir}(\mathbf{L}_2) : \ \mathrm{At}(i) = \{a,b\}\} = \emptyset .$$

Then the following conditions are equivalent:

(a) *There is a unary partial algebra* $\mathbf{A}$ *of type* K *such that* $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}_1$.
(b) *There is a unary partial algebra* $\mathbf{A}$ *of type* K *such that* $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}_2$.

Proof. It is easy to see that $\mathbf{G}(\mathbf{L}_2)$ is a simple graph and that the graphs $\mathbf{G}(\mathbf{L}_1)$ and $\mathbf{G}(\mathbf{L}_2)$ satisfy (*) — (***) of Proposition 2.2. Thus $\mathbf{G}(\mathbf{L}_1)$ can be directed in the type $|K|$ if and only if $\mathbf{G}(\mathbf{L}_2)$ can be directed in the type $|K|$. Hence and from Theorem 1.2 we get the desired equivalence (a) $\iff$ (b). □

Now we translate Theorem 2.4 into the lattice language to obtain that for each infinite unary type K, if a lattice $\mathbf{L}$, which satisfies (c.1) - (c.4) of Theorem 1.1, has a relatively small (with respect to the cardinal number $|K|$) set of atoms $\mathrm{At}(\mathbf{L})$, then there is a unary partial algebra $\mathbf{A}$ of type K such that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$.

THEOREM 2.7. *Let* K *be an infinite unary algebraic type and let* $\mathbf{L}$ *be a lattice which satisfies* (c.1) - (c.4) *of Theorem* 1.1. *Moreover, let* K *and* $\mathbf{L}$ *satisfy the following conditions:*

(*) $|\mathrm{At}(\mathbf{L})| \le |K|_1$.
(**) *For every* $a, b \in \mathrm{At}(\mathbf{L})$, $\left|\{i \in \mathrm{Ir}(\mathbf{L}) : \ \mathrm{At}(i) = \{a,b\}\}\right| \le |K|$.

Then there is a unary partial algebra $\mathbf{A}$ *of type* K *such that* $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$.

Proof. By the definition of $\mathbf{G}(\mathbf{L})$, $\mathbf{G}(\mathbf{L})$ and $|K|$ satisfy (*) — (***) of Proposition 2.4. Thus $\mathbf{G}(\mathbf{L})$ can be directed in the type $|K|$. Hence and by Theorem 1.2 there is a desired unary partial algebra of type K. □

Now we give a necessary condition for graphs which can be directed in an infinite type. To this purpose we first recall a few useful notations. For an arbitrary set A, $P(A)$ is the family of all subsets of A. A system $\mathcal{B} \subseteq P(A)$ is a *directed family of sets* if and only if for each $B_1, B_2 \in \mathcal{B}$ there is $B_3 \in \mathcal{B}$ such that $B_1 \cup B_2 \subseteq B_3$. Recall that a function $\mathcal{C}_A \colon P(A) \to P(A)$ is a *closure operator* if and only if for all $B \in P(A)$, $B \subseteq \mathcal{C}_A(B)$, $\mathcal{C}_A(\mathcal{C}_A(B)) = \mathcal{C}_A(B)$ and for each $B_1, B_2 \in P(A)$, $B_1 \subseteq B_2$ implies $\mathcal{C}_A(B_1) \subseteq \mathcal{C}_A(B_2)$. A closure operator $\mathcal{C}_A$ is *algebraic* if and only if for each directed family $\mathcal{B} \subseteq P(A)$, $\mathcal{C}_A(\bigcup \mathcal{B}) = \bigcup\{\mathcal{C}_A(B) : \ B \in \mathcal{B}\}$.

Let $\mathbf{D}$ be a digraph and let $r = (f_1, \dots, f_n)$ be a sequence of edges. Then r is a chain in $\mathbf{D}$ if and only if $I_2^D(f_i) = I_1^D(f_{i+1})$ for $i = 1, \dots, n-1$. Further,

r is said to be a chain from a subset $W \subseteq V^D$ to a vertex $v \in V^D$ if and only if $I_1^D(f_1) \in W$ and $I_2^D(f_n) = v$.

DEFINITION 2.8. For each digraph $\mathbf{D}$, let $\mathcal{C}_D \colon P(V^D) \to P(V^D)$ be the following function:

$$\mathcal{C}_D(W) = W \cup \{v \in V^D : \text{ there is a chain from } W \text{ to } v\} \quad \text{for each } W \subseteq V^D.$$

PROPOSITION 2.9. *Let $\mathbf{D}$ be a digraph. Then $\mathcal{C}_D$ is a closure operator such that*

$$\mathcal{C}_D\Big(\bigcup \mathcal{W}\Big) = \bigcup \{\mathcal{C}_D(W) : W \in \mathcal{W}\} \qquad \textit{for every} \quad \mathcal{W} \subseteq P(V^D);$$

hence, $\mathcal{C}_D$ is an algebraic closure operator.

P r o o f. It is trivial that $\mathcal{C}_D$ satisfies the required equality and two conditions from the definition of closure operators. Thus we must only show the equality $\mathcal{C}_D(W) = \mathcal{C}_D(\mathcal{C}_D(W))$ for $W \subseteq V^D$. The inclusion $\subseteq$ is trivial. Now let $v \in \mathcal{C}_D(\mathcal{C}_D(W))$. We can of course assume that there is a chain $(f_1, \dots, f_n)$ from $\mathcal{C}_D(W)$ to v, because otherwise $v \in \mathcal{C}_D(W)$. Then $I_1^D(f_1) \in W$ or there is a chain $(e_1, \dots, e_m)$ from W to $I_1^D(f_1)$, since $I_1^D(f_1) \in \mathcal{C}_D(W)$. Hence, $v \in \mathcal{C}_D(W)$ (in the first case $(f_1, \dots, f_n)$ connects W and v, in the second $(e_1, \dots, e_m, f_1, \dots, f_n)$ is a chain from W to v). Thus we have shown the inclusion $\supseteq$. □

LEMMA 2.10. *Let $\mathbf{D}$ be a digraph of a type η and $W \subseteq V^D$. Then*

$$|\mathcal{C}_D(W)| \le \max\{\aleph_0, \eta, |W|\}.$$

P r o o f. We first show that

$$\mathcal{C}_D(W) = \bigcup_{n \in N} X_n, \tag{1}$$

where $X_0 := W$ and $X_{n+1} := \{I_2^D(e) : e \in E^D, \ I_1^D(e) \in X_n\}$ for all $n \in \mathbb{N}$.

$\subseteq$: Let $v \in \mathcal{C}_D(W)$. We can of course assume that $v \notin W$. Then there is a chain $(f_1, \dots, f_m)$ from W to v. By simple induction on $1 \le i \le m$ we infer that $I_1^D(f_i) \in X_{i-1}$ for $i = 1, \dots, m$, because $I_1^D(f_1) \in W = X_0$. Hence in particular, $I_1^D(f_m) \in X_{m-1}$, so $v = I_2^D(f_m) \in X_m \subseteq \bigcup_{n \in N} X_n$.

$\supseteq$: By simple induction on $n \in \mathbb{N}$ and the definition of $\mathcal{C}_D$ we easily get the inclusion $X_n \subseteq \mathcal{C}_D(W)$ for $n \in \mathbb{N}$, because $X_0 = W \subseteq \mathcal{C}_D(W)$. Hence, $\bigcup_{n \in N} X_n \subseteq \mathcal{C}_D(W)$.

Secondly, we prove the following inequality (where $\tau := \max\{\aleph_0, \eta, |W|\}$):

$$|X_n| \le \tau \qquad \text{for all} \quad n \in \mathbb{N}. \tag{2}$$

Induction on n: If $n = 0$, then $|X_0| = |W| \leq \tau$. Now let $n \geq 0$ and let us assume that $|X_n| \leq \tau$. Observe first that $|X_{n+1}| = \left|\{I_2^D(e) :\ e \in E^D\,,\ I_1^D(e) \in X_n\}\right| \leq \left|\{e \in E^D :\ I_1^D(e) \in X_n\}\right| = \left|\bigcup_{v \in X_n} E_s^D(v)\right|$. Secondly, $|E_s^D(v)| = s^D(v) \leq \eta \leq \tau$ for $v \in V^D$, because $\mathbf{D}$ is of the type η. Hence and by induction hypothesis, $|X_{n+1}| = \left|\bigcup_{v \in X_n} E_s^D(v)\right| \leq \tau \cdot \tau = \tau$, since $\tau \geq \aleph_0$. This completes the proof of the induction step.

By (1) and (2), $|\mathcal{C}_D(W)| = \left|\bigcup_{n \in N} X_n\right| \leq \aleph_0 \cdot \tau = \tau = \max\{\aleph_0, \eta, |W|\}$. □

In the next result and also in the subsequent section we will need the following notation:

DEFINITION 2.11. Let $\mathbf{G}$ be a graph and $v \in V^G$. Then

$$V^G(v) := \left\{w \in V^G :\ (w \neq v)\ \ \&\ \ (\exists\, e \in E^G)\big(I^G(e) = \{v, w\}\big)\right\}.$$

LEMMA 2.11. *Let η be a cardinal number and let $\mathbf{D}$ be a digraph of type η and let $W \subseteq V^D$. Then*

$$|V^{D^*}(v) \cap \mathcal{C}_D(W)| \leq \eta \qquad \textit{for all} \quad v \in V^D \setminus \mathcal{C}_D(W).$$

Proof. Let $v \in V^D \setminus \mathcal{C}_D(W)$. It is trivial that

$$\begin{aligned} V^{D^*}(v) &= \{I_2^D(e) :\ e \in E^D_{\mathrm{reg}}\,,\ I_1^D(e) = v\} \cup \{I_1^D(e) :\ e \in E^D_{\mathrm{reg}}\,,\ I_2^D(e) = v\} \\ &\subseteq \{I_2^D(e) :\ e \in E_s^D(v)\} \cup \{I_1^D(e) :\ e \in E^D\,,\ I_2^D(e) = v\}. \end{aligned}$$

Now let us assume that there is an $f \in E^D$ such that $I_1^D(f) \in \mathcal{C}_D(W)$ and $I_2^D(f) = v$. Then the one-element sequence (f) is a chain from $\mathcal{C}_D(W)$ to v. Hence, $v \in \mathcal{C}_D\big(\mathcal{C}_D(W)\big) = \mathcal{C}_D(W)$ by Proposition 2.9. But this is impossible, so

$$\{I_1^D(e) :\ e \in E^D\,,\ I_2^D(e) = v\} \cap \mathcal{C}_D(W) = \emptyset.$$

The above two facts imply

$$V^{D^*}(v) \cap \mathcal{C}_D(W) \subseteq \{I_2^D(e) :\ e \in E_s^D(v)\} \cap \mathcal{C}_D(W).$$

Hence,

$$\left|V^{D^*}(v) \cap \mathcal{C}_D(W)\right| \leq \left|\{I_2^D(e) :\ e \in E_s^D(v)\}\right| \leq |E_s^D(v)| = s^D(v) \leq \eta,$$

because $\mathbf{D}$ is of type η. □

Now we can give a necessary condition for graphs which can be directed in a given infinite type η.

PROPOSITION 2.13. *Let η be an infinite cardinal number and let* $\mathbf{G}$ *be a graph which can be directed in the infinite type η. Then there exists a closure operator $\mathcal{C}_G\colon P(V^G) \to P(V^G)$ such that*

(a) $\mathcal{C}_G(\bigcup \mathcal{W}) = \bigcup\{\mathcal{C}_G(W) :\ W \in \mathcal{W}\}$ *for every family* $\mathcal{W} \subseteq P(V^G)$ *(in particular, $\mathcal{C}_G$ is algebraic).*

(b) *For every $W \in P(V^G)$,*

$$|\mathcal{C}_G(W)| \leq \max\{\eta, |W|\},$$

$$|V^G(v) \cap \mathcal{C}_G(W)| \leq \eta \qquad \textit{for all} \quad v \in V^G \setminus \mathcal{C}_G(W).$$

P r o o f. Let $\mathbf{D}$ be a digraph of type η such that $\mathbf{D}^* = \mathbf{G}$ and take the function $\mathcal{C}_G\colon P(V^G) \to P(V^G)$ such that $\mathcal{C}_G := \mathcal{C}_D$, i.e. for each $W \in P(V^G)$,

$$\mathcal{C}_G(W) := W \cup \{v \in V^G :\ \text{there is a chain in } \mathbf{D} \text{ from } W \text{ to } v\}.$$

Then by Proposition 2.9, Lemma 2.10 and Lemma 2.12, $\mathcal{C}_G$ is the desired closure operator. □

3

In this section we completely characterize all graphs which can be directed in a given infinite type. More precisely, we prove that the necessary conditions in Proposition 2.13 are also sufficient, and moreover, we show that the condition (b) of Proposition 2.13 can be replaced by a weaker one. In other words, we prove that a graph $\mathbf{G}$ can be directed in an infinite type η if and only if there exists an algebraic closure operator $\mathcal{C}_G$ on the set V^G such that $|\mathcal{C}_G(W)| \leq \max\{\eta_1, |W|\}$ and $|V^G(v) \cap \mathcal{C}_G(W)| \leq \eta$ for all $W \subseteq V^G$ and $v \in V^G \setminus \mathcal{C}_G(W)$.

Next we apply the obtained results on graphs and Theorem 1.2 to formulate an algebraic characterization of the weak subalgebra lattice of an arbitrary unary partial algebra of a given infinite unary type.

The proof of our main result for graphs is divided into several steps, i.e. we first prove two technical lemmas (in which we use notations from Section 1).

LEMMA 3.1. *Let an infinite cardinal number η, a graph* $\mathbf{G}$ *and a transfinite sequence $(\mathbf{G}_\alpha)_{\alpha<\xi}$ of relative subgraphs of* $\mathbf{G}$ *(of order type ξ) satisfy the following conditions:*

(∗) *For all $v, w \in V^G$, $s^G(v,w) \leq \eta$.*

(∗∗) *For each $\alpha < \xi$,*

(1) $\mathbf{G}_\alpha$ *can be directed in the infinite type η,*

(2) $\left|V^G(v) \cap \left(\bigcup_{\gamma<\alpha} V^{G_\gamma}\right)\right| \leq \eta$ *for every $v \in V^{G_\alpha}$.*

(***) *For every* $\alpha < \beta < \xi$, $V^{G_\alpha} \cap V^{G_\beta} = \emptyset$.

(****) $V^G = \bigcup_{\alpha<\xi} V^{G_\alpha}$.

Then $\mathbf{G}$ *can be directed in the infinite type* η.

P r o o f . First, by (**1) there is a transfinite sequence of digraphs $(\mathbf{D}_\alpha)_{\alpha<\xi}$ (of order type ξ) such that for every $\alpha < \xi$, $\mathbf{D}^*_\alpha = \mathbf{G}_\alpha$ and $\mathbf{D}_\alpha$ is of the type η.

Secondly, the conditions (***) and (****) imply that for every $e \in E^G$ there is exactly one (directed) pair (α_e, β_e) such that $\beta_e \leq \alpha_e < \xi$, and one endpoint vertex of e belongs to $V^{G_{\alpha_e}}$ and the second endpoint vertex of e belongs to $V^{G_{\beta_e}}$, i.e. $I^G(e) \subseteq V^{G_{\alpha_e}} \cup V^{G_{\beta_e}}$ and $I^G(e) \cap V^{G_{\alpha_e}} \neq \emptyset$, $I^G(e) \cap V^{G_{\beta_e}} \neq \emptyset$.

Observe that $\alpha_e \neq \beta_e$ for each $e \in E^G \setminus \bigcup_{\alpha<\xi} E^{G_\alpha}$. More precisely, let $e \in E^G$ be an edge such that $\alpha_e = \beta_e$. Then $I^G(e) \subseteq V^{G_{\alpha_e}}$, so $e \in E^{G_{\alpha_e}} \subseteq \bigcup_{\alpha<\xi} E^{G_\alpha}$, because $\mathbf{G}_{\alpha_e}$ is a relative subgraph of $\mathbf{G}$.

Thus we can introduce the following definition: for every $e \in E^G \setminus \bigcup_{\alpha<\xi} E^{G_\alpha}$ let v_1^e and v_2^e be vertices such that $I^G(e) = \{v_1^e, v_2^e\}$, $v_1^e \in V^{G_{\alpha_e}}$, $v_2^e \in V^{G_{\beta_e}}$ and $\beta_e < \alpha_e$.

Moreover, let $\mathbf{D}$ be the digraph such that $V^D := V^G$, $E^D := E^G$ and $I^D|_{\bigcup_{\alpha<\xi} E^{G_\alpha}} := \bigcup_{\alpha<\xi} I^{D_\alpha}$ and $I^D(e) := (v_1^e, v_2^e)$ for all $e \in E^G \setminus \bigcup_{\alpha<\xi} E^{G_\alpha}$.

Note that $\mathbf{D}$ is indeed a digraph (i.e. I^D is a well-defined function), because by (***), $E^{G_\alpha} \cap E^{G_\beta} = \emptyset$ for all $\alpha < \beta < \xi$.

Since $\mathbf{D}^*_\alpha = \mathbf{G}^*_\alpha$ for all $\alpha < \xi$, we easily obtain that

$$\mathbf{D}^* = \mathbf{G}.$$

Thus we must only prove now that $\mathbf{D}$ is of type η. To this purpose let us take an arbitrary $v \in V^D = V^G$. Then by (***) and (****) there is exactly one $\gamma < \xi$ such that $v \in V^{G_\gamma}$.

First, we show that

$$E_s^D(v) = E_s^{D_\gamma}(v) \cup A_\gamma(v), \tag{1}$$

where $A_\gamma(v) := \left\{e \in E^G \setminus \bigcup_{\alpha<\xi} E^{G_\alpha} : \ v = v_1^e, \ v_2^e \in \bigcup_{\alpha<\gamma} V^{G_\alpha}\right\}$.

By the definition of $\mathbf{D}$, $E_s^{G_\gamma}(v) \subseteq E_s^G(v)$ and $A_\gamma(v) \subseteq E_s^G(v)$. Hence, the inclusion $\supseteq$ holds.

Now let us take an arbitrary $e \in E_s^D(v)$. If $e \in \bigcup_{\alpha<\xi} E^{G_\alpha}$, then $e \in E^{G_\beta}$ for some $\beta < \xi$. Hence, $v = I_1^D(e) = I_1^{D_\beta}(e) \in V^{D_\beta} = V^{G_\beta}$, so $\beta = \gamma$ by (***). Thus $e \in E_s^{D_\gamma}(v)$. If $e \notin \bigcup_{\alpha<\xi} E^{G_\alpha}$, then $v = v_1^e$. Hence, $\alpha_e = \gamma$ by (***), so

$v_2^e \in V^{G_{\beta_e}} \subseteq \bigcup_{\alpha<\gamma} V^{G_\alpha}$, because $\beta_e < \alpha_e$. Thus $e \in A_\gamma(v)$. These two cases complete the proof of the inclusion $\subseteq$.

Secondly, we prove that

$$|A_\gamma(v)| \leq \eta. \tag{2}$$

It is easy to see that $A_\gamma(v) = \bigcup_{w\in B} E_s^G(v,w)$, where $B := \bigcup_{\alpha<\gamma} V^{G_\alpha}$. By the definition of $V^G(v)$ we have also $E_s^G(v,w) = \emptyset$ for all $w \notin V^G(v) \cup \{v\}$. Moreover, by (***), $B \cap V^{G_\gamma} = \emptyset$, so in particular $v \notin B$. Thus

$$A_\gamma(v) = \bigcup_{w\in V^G(v)\cap B} E_s^G(v,w).$$

This with (*) and (**2) imply

$$|A_\gamma(v)| = |V^G(v) \cap B| \cdot \eta = \left|V^G(v) \cap \Big(\bigcup_{\alpha<\gamma} V^{G_\alpha}\Big)\right| \cdot \eta \leq \eta \cdot \eta = \eta.$$

The above two facts (1) and (2) imply

$$s^D(v) = |E_s^D(v)| = |E_s^{G_\gamma}(v) \cup A_\gamma(v)| \leq s^{G_\gamma}(v) + |A_\gamma(v)| \leq \eta + \eta = \eta.$$

Thus we have shown that $\mathbf{D}$ is a digraph of type η. This completes our proof.

□

LEMMA 3.2. *Let an infinite cardinal number η, a graph $\mathbf{G}$ and a transfinite sequence $(\mathbf{G}_\alpha)_{\alpha<\xi}$ of relative subgraphs of $\mathbf{G}$ (of order type ξ) satisfy the following conditions:*

(*) *For all $v, w \in V^G$, $s^G(v,w) \leq \eta$.*

(**) *For each $\alpha < \xi$,*

(1) *$\mathbf{G}_\alpha$ can be directed in the infinite type η,*

(2) $\left|V^G(v) \cap \Big(\bigcup_{\gamma<\alpha} V^{G_\gamma}\Big)\right| \leq \eta$ *for every $v \in V^{G_\alpha} \setminus \bigcup_{\gamma<\alpha} V^{G_\gamma}$.*

(***) $V^G = \bigcup_{\alpha<\xi} V^{G_\alpha}$.

Then $\mathbf{G}$ can be directed in the infinite type η.

Proof. For every $\alpha < \xi$ let $V_\alpha := V^{G_\alpha} \setminus \bigcup_{\gamma<\alpha} V^{G_\gamma}$ (of course $\bigcup_{\gamma<0} V^{G_\gamma} = \emptyset$, so $V_0 := V^{G_0}$) and let $\mathbf{H}_\alpha$ be the relative subgraph of $\mathbf{G}$ on V^{G_α}.

Then $V^{H_\alpha} \cap V^{H_\beta} = \emptyset$ for every $\alpha < \beta < \xi$. It is also trivial that $\mathbf{H}_\alpha$ is a weak subgraph of $\mathbf{G}_\alpha$ for each $\alpha < \xi$. Thus by Proposition 1.4 and (**2) we obtain that for every $\alpha < \xi$, $\mathbf{H}_\alpha$ can be directed in the type η. Thus the transfinite sequence $(\mathbf{H}_\alpha)_{\alpha<\xi}$ of relative subgraphs of $\mathbf{G}$ (of order type ξ) satisfies (**1) and (***) of Lemma 3.1.

Now observe that $\bigcup_{\alpha<\gamma} V^{G_\alpha} = \bigcup_{\alpha<\gamma} V^{H_\alpha}$ for all $\gamma \leq \xi$. The inclusion $\supseteq$ is trivial, because $V^{H_\alpha} = V_\alpha \subseteq V^{G_\alpha}$ for each $\alpha < \gamma$. On the other hand, let $v \in \bigcup_{\alpha<\gamma} V^{G_\alpha}$. Then there is $\alpha_0 < \gamma$ such that $v \in V^{G_{\alpha_0}}$ and $v \notin V^{G_\beta}$ for $\beta < \alpha_0$, because $\{\alpha : \alpha < \gamma\}$ is a well-ordered set. Hence, $v \in V^{G_{\alpha_0}} \setminus \bigcup_{\beta<\alpha_0} V^{G_\beta} = V_{\alpha_0} = V^{H_{\alpha_0}}$. Thus we obtain the second inclusion $\subseteq$.

By the above fact (for $\gamma = \xi$) and (***),

$$\bigcup_{\alpha<\xi} V^{H_\alpha} = \bigcup_{\alpha<\xi} V^{G_\alpha} = V^G .$$

Moreover, this fact and (**2) imply that for every $\alpha < \xi$ and for every vertex $v \in V^{G_\alpha} \setminus \bigcup_{\gamma<\alpha} V^{G_\gamma} = V^{H_\alpha}$,

$$\Big|V^G(v) \cap \Big(\bigcup_{\gamma<\alpha} V^{H_\gamma}\Big)\Big| = \Big|V^G(v) \cap \Big(\bigcup_{\gamma<\alpha} V^{G_\gamma}\Big)\Big| \leq \eta .$$

Thus we have shown that the transfinite sequence $(\mathbf{H}_\alpha)_{\alpha<\xi}$ of relative subgraphs of $\mathbf{G}$ (of order type ξ) satisfies the conditions of Lemma 3.1, so $\mathbf{G}$ can be directed in the infinite type η. □

THEOREM 3.3. *Let η be an infinite cardinal number and let $\mathbf{G}$ be a graph. Then the following conditions are equivalent:*

(a) $\mathbf{G}$ *can be directed in the infinite type η.*

(b) $\mathbf{G}$ *is a graph such that:*

(b.1) *For all $v, w \in V^G$, $s^G(v,w) \leq \eta$.*

(b.2) *There exists an algebraic closure operator $\mathcal{C}_G \colon P(V^G) \to P(V^G)$ such that for every $W \subseteq V^G$:*

$$|\mathcal{C}_G(W)| \leq \max\{\eta, |W|\} ,$$
$$|V^G(v) \cap \mathcal{C}_G(W)| \leq \eta \qquad \textit{for each} \quad v \in V^G \setminus \mathcal{C}_G(W) .$$

(c) $\mathbf{G}$ *is a graph such that:*

(c.1) *For all $v, w \in V^G$, $s^G(v,w) \leq \eta$.*

(c.2) *There exists an algebraic closure operator $\mathcal{C}_G \colon P(V^G) \to P(V^G)$ such that for every $W \subseteq V^G$:*

$$|\mathcal{C}_G(W)| \leq \max\{\eta_1, |W|\} ,$$
$$|V^G(v) \cap \mathcal{C}_G(W)| \leq \eta \qquad \textit{for each} \quad v \in V^G \setminus \mathcal{C}_G(W) .$$

Proof.

(a) $\Longrightarrow$ (b) follows from Proposition 2.1 and Proposition 2.13, and

(b) $\Longrightarrow$ (c) is trivial.

(c) $\Longrightarrow$ (a): Let $\xi := |V^G|$ (thus ξ is an initial ordinal number). We first prove that there is a transfinite sequence of sets $(V_\alpha)_{\alpha<\xi}$ (of order type ξ) such that

(1) For every $\alpha < \xi$, $V_\alpha \subseteq V^G$, and if $\xi > \eta_1$, then $|V_\alpha| < \xi$.

(2) For every $\alpha < \xi$ and for each $v \in V^G \setminus \bigcup_{\gamma<\alpha} V_\gamma$, $\left|V^G(v) \cap \left(\bigcup_{\gamma<\alpha} V_\gamma\right)\right| \leq \eta$.

(3) $V^G = \bigcup_{\alpha<\xi} V_\alpha$.

Applying Zermelo's Theorem and the definitions of ordinal and cardinal numbers we can put all vertices of $\mathbf{G}$ in a transfinite and injective sequence $(v_\alpha)_{\alpha<\xi}$ (of order type ξ), i.e. $V^G = \{v_\alpha : \alpha < \xi\}$ and $v_\alpha \neq v_\beta$ for all $\alpha < \beta < \xi$.

Now for every $\alpha < \xi$ let

$$V'_\alpha := \{v_\gamma : \gamma < \alpha\} \qquad \text{and} \qquad V_\alpha := \mathcal{C}_G(V'_\alpha).$$

We now have $|V'_\alpha| = |\{v_\gamma : \gamma < \alpha\}| = |\{\gamma : \gamma < \alpha\}| \leq \alpha < \xi$ for each $\alpha < \xi$. Moreover, by (c.2), $|V_\alpha| = |\mathcal{C}_G(V'_\alpha)| \leq \max\{\eta_1, |V'_\alpha|\}$ for all $\alpha < \xi$. These facts imply that if $\xi > \eta_1$, then $|V_\alpha| < \xi$ for each $\alpha < \xi$. Thus we have shown (1).

Now we show (3). Since ξ is a limit ordinal number, we have by the definition of V'_α that $\bigcup_{\alpha<\xi} V'_\alpha = \bigcup_{\alpha<\xi} \{v_\gamma : \gamma < \alpha\} = \{v_\alpha : \alpha < \xi\} = V^G$. Moreover, $V'_\alpha \subseteq V_\alpha$ for all $\alpha < \xi$. Thus $\bigcup_{\alpha<\xi} V_\alpha = V^G$.

To prove the property (2), observe first that $V'_\alpha \subseteq V'_\beta$ for all $\alpha \leq \beta < \xi$. Hence and by the definition of closure operators, $V_\alpha \subseteq V_\beta$ for all $\alpha \leq \beta < \xi$. Thus the family $\{V_\alpha\}_{\alpha<\xi}$ is a chain with respect to set-inclusion, in particular it is a directed family. Hence, since $\mathcal{C}_G$ is an algebraic closure operator (i.e. $\mathcal{C}_G$ is closed under the union of an arbitrary directed family), we obtain

$$\begin{aligned}\mathcal{C}_G\left(\bigcup_{\gamma<\alpha} V_\gamma\right) &= \mathcal{C}_G\left(\bigcup_{\gamma<\alpha} \mathcal{C}_G(V'_\gamma)\right) = \bigcup_{\gamma<\alpha} \mathcal{C}_G(\mathcal{C}_G(V'_\gamma)) \\ &= \bigcup_{\gamma<\alpha} \mathcal{C}_G(V'_\gamma) = \bigcup_{\gamma<\alpha} V_\gamma \qquad \text{for all} \quad \alpha < \xi.\end{aligned}$$

Hence and from (c.2) we get (2).

Now we apply transfinite induction on the cardinal number $\xi := |V^G|$ to prove the implication (c) $\Longrightarrow$ (a).

Basis: if $\xi \leq \eta_1$, then by (c.1) and Theorem 2.4, $\mathbf{G}$ can be directed in the type η.

Induction step: Let $\xi > \eta_1$ and assume that for every graph $\mathbf{H}$, if $\mathbf{H}$ satisfies (c.1), (c.2) (with insert $\mathbf{H}$ instead of $\mathbf{G}$) and $|V^H| < \xi$, then $\mathbf{H}$ can be directed in the type η.

Now take an arbitrary $\mathbf{H} \leq_w \mathbf{G}$. It is a well-known fact that the function $\mathcal{C}_H\colon P(V^H) \to P(V^H)$ such that $\mathcal{C}_H(W) := \mathcal{C}_G(W) \cap V^H$ for all $W \subseteq V^H$ is an algebraic closure operator on V^H. Observe also that for every $W \subseteq V^H$ and $v \in V^H \setminus \mathcal{C}_H(W) = V^H \setminus \mathcal{C}_G(W)$ we have

$$|\mathcal{C}_H(W)| = |\mathcal{C}_G(W) \cap V^H| \leq |\mathcal{C}_G(W)| \leq \max\{\eta_1, |W|\}$$

and

$$\begin{aligned}|V^H(v) \cap \mathcal{C}_H(W)| &= |V^H(v) \cap \mathcal{C}_G(W) \cap V^H| \\ &\leq |V^H(v) \cap \mathcal{C}_G(W)| \\ &\leq |V^G(v) \cap \mathcal{C}_G(W)| \leq \eta,\end{aligned}$$

because $V^H(v) \subseteq V^G(v)$.

Thus we have shown that $\mathbf{H}$ satisfies (c.2). Since $\mathbf{H}$ is a weak subgraph of $\mathbf{G}$, $\mathbf{H}$ satisfies (c.1). Hence and by induction hypothesis, for every $\mathbf{H} \leq_w \mathbf{G}$, if $|V^H| < \xi$, then $\mathbf{H}$ can be directed in the type η.

Now for each $\alpha < \xi$ let $\mathbf{H}_\alpha$ be the relative subgraph of $\mathbf{G}$ such that $V^{H_\alpha} = V_\alpha$, i.e. $E^{H_\alpha} = \{e \in E^G : I^G(e) \subseteq V_\alpha\}$.

Since $\xi > \eta_1$, we have by (1) and the above fact that $\mathbf{H}_\alpha$ can be directed in the type η for each $\alpha < \xi$. Hence and by (2) and (3), and also by (*), the transfinite sequence $(\mathbf{H}_\alpha)_{\alpha<\xi}$ satisfies (*), (**), (***) of Lemma 3.2. Thus $\mathbf{G}$ can be directed in the infinite type η, which completes the proof of the induction step. This and the theorem of transfinite induction prove (c) $\Longrightarrow$ (a). □

Now we show that the necessary condition of Proposition 2.1 for infinite types is not sufficient. In other words, for a given infinite type there are a lot of simple graphs which cannot be directed in this type.

We also show that for infinite types there is no characterization as simple as in the case of finite types (see [Pió2]). More precisely, for each infinite type there is a simple graph $\mathbf{G}$ which cannot be directed in this type, even if each weak subgraph $\mathbf{H}$ of $\mathbf{G}$ with $|V^H| < |V^G|$ can be directed in this type.

EXAMPLE 1. Let η be an infinite cardinal number and let $\mathbf{G}$ be a clique, i.e. a simple graph such that between each two different vertices there is exactly one edge, and moreover, assume $|V^G| \geq \eta_2$.

Then of course the necessary condition of Proposition 2.1 for η holds. Now we show that $\mathbf{G}$ cannot be directed in the type η. Let $\mathcal{C}$ be an arbitrary algebraic closure operator on V^G such that $|\mathcal{C}(U)| \leq \max\{\eta_1, |U|\}$ for each $U \subseteq V^G$ and let $W \subseteq V^G$ be a subset such that $|W| = \eta_1$. Then $|\mathcal{C}(W)| = \eta_1$ (since $W \subseteq$

$\mathcal{C}(W)$) and $V^G(v) \cap \mathcal{C}(W) = \mathcal{C}(W)$ for $v \in V^G \setminus \mathcal{C}(W)$. Hence, $V^G \setminus \mathcal{C}(W) \neq \emptyset$ and $|V^G(v) \cap \mathcal{C}(W)| = \eta_1$ for $v \in V^G \setminus \mathcal{C}(W)$. Thus $\mathcal{C}$ does not satisfy the second part of the condition (c.2) of Theorem 3.3, so $\mathbf{G}$ cannot be directed in the type η.

Applying this construction and Proposition 1.4 we can form, for every infinite type η, a lot of different graphs (also simple) which cannot be directed in the type η. More precisely, we must only take a graph (a simple graph) which contains a graph as in the above example.

EXAMPLE 2. Let η be an infinite cardinal number and let $\mathbf{G}$ be a clique and $|V^G| = \eta_2$. Then by Example 1, $\mathbf{G}$ cannot be directed in the infinite type η. On the other hand, let $\mathbf{H}$ be an arbitrary weak subgraph of $\mathbf{G}$ such that $|V^H| < |V^G|$. Then $|V^H| \leq \eta_1$ by the definition of η_2. Hence and by Theorem 2.4, $\mathbf{H}$ can be directed in the infinite type η.

Finally, we apply the above results for graphs to completely characterize the weak subalgebra lattice of a unary partial algebra of a given infinite unary type. In other words, for a given infinite unary type K we describe all lattices $\mathbf{L}$ for which there exists a unary partial algebra of the unary type K such that its weak subalgebra lattice is isomorphic to $\mathbf{L}$. In the theorem below we use notations from Section 1.

THEOREM 3.4. *Let K be an infinite unary algebraic type and let $\mathbf{L}$ be a lattice which satisfies* (c.1) – (c.4) *of Theorem* 1.1. *Then the following conditions are equivalent:*

(a) *There is a unary partial algebra $\mathbf{A}$ of type K such that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$.*

(b) $\mathbf{L}$ *satisfies the following conditions:*

(b.1) *For all* $a, b \in \mathrm{At}(\mathbf{L})$, $\big|\{i \in \mathrm{Ir}(\mathbf{L}) :\ \mathrm{At}(i) = \{a, b\}\}\big| \leq |K|$.

(b.2) *There exists an algebraic closure operator $\mathcal{C}_L$ on* $\mathrm{At}(\mathbf{L})$ *such that for every* $B \subseteq \mathrm{At}(\mathbf{L})$,

$$|\mathcal{C}_L(B)| \leq \max\{|K|, |B|\},$$
$$\big|\{b \in \mathcal{C}_L(B) :\ (\exists i \in \mathrm{Ir}(L))(\mathrm{At}(i) = \{a, b\})\}\big| \leq |K| \quad \textit{for each} \quad a \in \mathrm{At}(\mathbf{L}) \setminus \mathcal{C}_L(B).$$

(c) $\mathbf{L}$ *satisfies the following conditions:*

(c.1) *For all* $a, b \in \mathrm{At}(\mathbf{L})$, $\big|\{i \in \mathrm{Ir}(\mathbf{L}) :\ \mathrm{At}(i) = \{a, b\}\}\big| \leq |K|$.

(c.2) *There exists an algebraic closure operator $\mathcal{C}_L$ on* $\mathrm{At}(\mathbf{L})$ *such that for every* $B \subseteq \mathrm{At}(\mathbf{L})$,

$$|\mathcal{C}_L(B)| \leq \max\{|K|_1, |B|\},$$
$$\big|\{b \in \mathcal{C}_L(B) :\ (\exists i \in \mathrm{Ir}(L))(\mathrm{At}(i) = \{a, b\})\}\big| \leq |K| \quad \textit{for each} \quad a \in \mathrm{At}(\mathbf{L}) \setminus \mathcal{C}_L(B).$$

P r o o f . It is easily shown that $\mathbf{L}$ satisfies (b) ((c)) if and only if $\mathbf{G}(\mathbf{L})$ satisfies (b) ((c)) of Theorem 3.3. Hence and by Theorem 1.2 we get the equivalences (a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (c), which completes the proof. $\square$

Now we show that the necessary condition of Proposition 2.5 for infinite unary types is not sufficient. In other words, for a given infinite unary type K there are a lot of lattices $\mathbf{L}$ which satisfy this condition and (c.1) – (c.4) of Theorem 1.1 and for which there is no unary partial algebra $\mathbf{A}$ of this unary type K such that its weak subalgebra lattice $\mathbf{S}_w(\mathbf{A})$ is isomorphic to $\mathbf{L}$.

Moreover, we also show that for infinite unary types there is no characterization as simple as in the case of finite unary types (see [Pió2]). More precisely, for every infinite unary type there is a lattice $\mathbf{L}$ which satisfies (c.1) – (c.4) of Theorem 1.1 and for which there is no unary partial algebra of this unary type such that its weak subalgebra lattice is isomorphic to $\mathbf{L}$, even if such an algebra exists for every complete sublattice $\mathbf{M}$ of $\mathbf{L}$ generated by a set $\{0\} \cup A \cup I$ (where 0 is the least element of $\mathbf{L}$ and $A \subseteq \mathrm{At}(\mathbf{L})$, $I \subseteq \mathrm{Ir}(\mathbf{L})$ and $\mathrm{At}(i) \subseteq A$ for all $i \in I$) such that $|\mathrm{At}(\mathbf{M})| < |\mathrm{At}(\mathbf{L})|$.

EXAMPLE 3. Let K be an infinite unary type and let W be an arbitrary set such that $|W| \geq |K|_2$ and let $V_W := \big\{\{w_1, w_2\} \subseteq W : w_1 \neq w_2\big\}$. Let $\mathbf{L} = (L, \leq_L)$ be the complete sublattice of the lattice $\mathbf{P}(W \cup V_W)$ of all subsets of $W \cup V_W$ generated by $\big\{\{w\} : w \in W\big\} \cup \big\{\{w_1, w_2, \{w_1, w_2\}\} : w_1, w_2 \in W, \ w_1 \neq w_2\big\}$.

It is easy to see that $\mathbf{L}$ is just the family of all sets $U \in P(W \cup V_W)$ such that for every $w_1, w_2 \in W$, $w_1 \neq w_2$, $\{w_1, w_2\} \in U$ implies $w_1, w_2 \in U$ (in particular $\emptyset \in L$). Thus $\mathrm{At}(\mathbf{L}) := \big\{\{w\} : w \in W\big\}$ and $\mathrm{Ir}(\mathbf{L}) := \big\{\{w_1, w_2, \{w_1, w_2\}\} : w_1, w_2 \in W, \ w_1 \neq w_2\big\}$. Hence, $\mathbf{L}$ satisfies (c.1) – (c.4) of Theorem 1.1 and the condition in Proposition 2.5. Moreover, this fact implies that $\mathbf{G}(\mathbf{L})$ is simple and contains $|W|$ vertices and between any two different vertices there is exactly one edge. Thus by Example 1, $\mathbf{G}(\mathbf{L})$ cannot be directed in the infinite type $|K|$. This fact and Theorem 1.2 imply that there is no unary partial algebra of type K such that its weak subalgebra lattice is isomorphic to $\mathbf{L}$.

Let $\mathbf{L}$ be a lattice which satisfies (c.1) – (c.4) of Theorem 1.1 and let $A \subseteq \mathrm{At}(\mathbf{L})$, $I \subseteq \mathrm{Ir}(\mathbf{L})$ be sets such that $\mathrm{At}(i) \subseteq A$ for all $i \in I$ and let $\mathbf{M}$ be the complete sublattice of $\mathbf{L}$ generated by $\{0\} \cup A \cup I$ (where 0 is the least element of $\mathbf{L}$). Recall that we showed in [Pió2] that A is the set of all atoms of $\mathbf{M}$ and I is the set of all non-zero and non-atomic join-irreducible elements (i.e. $\mathrm{At}(\mathbf{M}) = A$ and $\mathrm{Ir}(\mathbf{M}) = I$). Moreover, $\mathbf{M}$ satisfies (c.1) – (c.4) of Theorem 1.1.

EXAMPLE 4. Let K be an infinite unary type and let $\mathbf{L}$ be the lattice of Example 3 for some set W such that $|W| = |K|_2$. Then by Example 3 there is no unary partial algebra of type K such that its weak subalgebra lattice is isomorphic to $\mathbf{L}$. On the other hand, take an arbitrary complete sublattice

$\mathbf{M}$ of $\mathbf{L}$ such that $|\operatorname{At}(\mathbf{M})| < |\operatorname{At}(\mathbf{L})|$ and generated by $\{0\} \cup A \cup I$, where $A \subseteq \operatorname{At}(\mathbf{L})$ and $I \subseteq \operatorname{Ir}(\mathbf{L})$ are sets such that $\operatorname{At}(i) \subseteq A$ for all $i \in I$. Then $\mathbf{M}$ satisfies (c.1) - (c.4) of Theorem 1.1 and $\operatorname{At}(\mathbf{M}) = A$, $\operatorname{Ir}(\mathbf{M}) = I$. From this and from the definition of $\mathbf{L}$ we infer that $\mathbf{M}$ satisfies (**) of Theorem 2.7. Moreover, by Example 3 we have $|\operatorname{At}(\mathbf{L})| = \big|\{\{w\} : w \in W\}\big| = |W| = |K|_2$, so $|\operatorname{At}(\mathbf{M})| < |\operatorname{At}(\mathbf{L})|$ implies $|\operatorname{At}(\mathbf{M})| \leq |K|_2$ by the definition of $|K|_2$. Thus by Theorem 2.7, $\mathbf{M}$ is isomorphic to the weak subalgebra lattice of some unary partial algebra of type K.

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