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ZERO INVARIANT AND IDEMPOTENT FUZZY BCC-SUBALGEBRAS

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ABSTRACT. We introduce the notion of zero invariant and idempotent fuzzy BCC-subalgebras in BCC-algebras with respect to an s-norm, and investigate some of their properties.

1. Introduction

BCK-algebras form an important class of algebras introduced by K. Iséki and were extensively investigated by several researchers. The class of all BCK-algebras is a quasivariety. K. Iséki posed an interesting problem (solved in [17]) whether the class of all BCK-algebras is a variety. In connection with this problem, Y. Komori introduced in [15] a notion of BCC-algebras, and W. A. Dudek (cf. [2], [3]) redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori.

L. A. Zadeh [20] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches, such as groups, functional analysis, probability theory, topology, and so on. In 1991, O. G. Xi [18] applied this concept to BCK-algebras, and he introduced the notion of fuzzy subalgebras (ideals) of the BCK-algebras with respect to minimum, and since then Y. B. Jun at al. studied fuzzy subalgebras and fuzzy ideals (cf. [11], [12], [13]), and moreover several fuzzy structures in BCC-algebras are considered (cf. [4], [5], [7], [8]).

In this paper, we introduce the notion of zero invariant and idempotent fuzzy BCC-subalgebras in BCC-algebras with respect to an s-norm, and investigate some of their properties. We consider also the direct product and s-normed product of zero invariant fuzzy BCC-subalgebras of BCC-algebras with respect to an s-norm.

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2. Preliminaries

In the present paper a binary multiplication will be denoted by juxtaposition. Dots we use only to avoid repetitions of brackets. For example, the formula $\big((xy)(zy)\big)(xz) = 0$ will be written as $(xy \cdot zy) \cdot xz = 0$.

A non-empty set G with a constant 0 and a binary operation denoted by juxtaposition is called a *BCC-algebra* if for all $x, y, z \in G$ the following axioms hold:

(i) $(xy \cdot zy) \cdot xz = 0$,
(ii) $xx = 0$,
(iii) $0x = 0$,
(iv) $x0 = x$,
(v) $xy = 0$ and $yx = 0$ imply $x = y$.

Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebras (cf. [3]). Note that a BCC-algebra is a BCK-algebra if and only if it satisfies:

$$xy \cdot z = xz \cdot y\,. \tag{1}$$

On any BCC-algebra (similarly as in the case of BCK-algebras) one can define the natural ordering $\leq$ by putting

$$x \leq y \iff xy = 0\,. \tag{2}$$

It is not difficult to verify that this order is partial and 0 is its smallest element. Moreover, in any BCC-algebra G, the following are true:

$$xy \leq x\,, \tag{3}$$

$$xy \cdot zy \leq xz\,, \tag{4}$$

$$x \leq y \implies xz \leq yz \;\&\; zy \leq zx\,. \tag{5}$$

A *fuzzy set* in a set G is any function $\mu\colon G \to [0,1]$. For $\alpha \in [0,1]$, the set $L(\mu;\alpha) := \big\{x \in G :\ \mu(x) \leq \alpha\big\}$ is called a *lower level set* of μ. A fuzzy set μ in a BCC-algebra G is called a *fuzzy BCC-subalgebra* of G if it satisfies the inequality:

$$\mu(xy) \geq \min\big\{\mu(x), \mu(y)\big\}$$

for all $x, y \in G$.

3. Idempotent fuzzy BCC-subalgebras

In the following, let G denote a BCC-algebra unless otherwise specified.

DEFINITION 3.1. ([19]) A binary operation S on the interval $[0,1]$ is called an *s-norm*, if

(S_1) $S(\alpha,0)=\alpha$,
(S_2) $S(\alpha,\beta)\le S(\alpha,\gamma)$ whenever $\beta\le\gamma$,
(S_3) $S(\alpha,\beta)=S(\beta,\alpha)$,
(S_4) $S(\alpha,S(\beta,\gamma))=S(S(\alpha,\beta),\gamma)$

for all $\alpha,\beta,\gamma\in[0,1]$.

Note that $\max\{\alpha,\beta\}\le S(\alpha,\beta)$ for all $\alpha,\beta\in[0,1]$. Moreover, $([0,1];S)$ is a commutative semigroup with 0 as the neutral element. In particular

$$S(S(\alpha,\beta),S(\gamma,\delta))=S(S(\alpha,\gamma),S(\beta,\delta))$$

holds for all $\alpha,\beta,\gamma,\delta\in[0,1]$.

The set of all idempotents with respect to S, i.e. the set

$$E_S=\{\alpha\in[0,1]:\ S(\alpha,\alpha)=\alpha\},$$

is closed with respect to the operation S. Hence (E_S,S) is a subsemigroup of $([0,1],S)$. If $\mathrm{Im}(\mu)\subseteq E_S$, then a fuzzy set μ is called an *idempotent with respect to an s-norm S* (briefly, an *S-idempotent*).

DEFINITION 3.2. Let S be an s-norm. A function $\mu\colon G\to[0,1]$ is called a *fuzzy BCC-subalgebra of G with respect to S* (briefly, an *S-fuzzy BCC-subalgebra*) if $\mu(xy)\le S(\mu(x),\mu(y))$ for all $x,y\in G$. If an S-fuzzy BCC-algebra μ of G is idempotent, we say that μ is an *idempotent S-fuzzy BCC-subalgebra* of G.

EXAMPLE 3.3. Let S be an s-norm defined by $S_0(\alpha,0)=\alpha=S_0(0,\alpha)$ and $S_0(\alpha,\beta)=1$ if $\alpha\ne0\ne\beta$, where $\alpha,\beta\in[0,1]$. Let $G=\{0,a,b,c\}$ be a BCC-algebra with the following Cayley table:

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Define a fuzzy set $\mu\colon G \to [0,1]$ by $\mu(0) = \alpha_0$, $\mu(a) = \mu(b) = \alpha_1$ and $\mu(c) = \alpha_2$, where $\alpha_0, \alpha_1, \alpha_2 \in [0,1]$ with $\alpha_0 < \alpha_1 < \alpha_2$. Routine calculations give that μ is an S_0-fuzzy subalgebra of G, which is not idempotent.

EXAMPLE 3.4. Let $G = \{0,1,2,3,4\}$ be a BCC-algebra with the Cayley table:

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Let S_m be an s-norm defined by $S_m(\alpha,\beta) = \min\{\alpha+\beta, 1\}$ for all $\alpha,\beta \in [0,1]$. Define a fuzzy set μ in G by

$$\mu(x) := \begin{cases} 1 & \text{if } x \in \{0,1,2\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that μ satisfies the inequality

$$\mu(xy) \le S_m\big(\mu(x), \mu(y)\big)$$

for all $x, y \in G$, and $\operatorname{Im}(\mu) \subseteq E_{S_m}$. Hence μ is an idempotent fuzzy BCC-subalgebra of G with respect to S_m.

PROPOSITION 3.5. *Let S_m be the s-norm in Example 3.4 and let A be a BCC-subalgebra of G. Then a fuzzy set μ in G defined by*

$$\mu(x) := \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{otherwise} \end{cases}$$

is an idempotent fuzzy BCC-subalgebra of G with respect to S_m.

Proof. Let $x, y \in G$. If $x \notin A$ or $y \notin A$, then $\mu(x) = 1$ or $\mu(y) = 1$ and so $S_m(\mu(x), \mu(y)) = 1 \ge \mu(xy)$. Suppose that $x \in A$ and $y \in A$. Then $xy \in A$ and thus $\mu(xy) = 0 \le S_m\big(\mu(x), \mu(y)\big)$. Obviously, $\operatorname{Im}(\mu) \subseteq E_{S_m}$, completing the proof. □

PROPOSITION 3.6. *If μ is an idempotent S-fuzzy BCC-subalgebra of G, then $\mu(0) \le \mu(x)$ and $\mu(xy) \le \max\{\mu(x), \mu(y)\}$ for all $x, y \in G$.*

Proof. Indeed, $\mu(0) = \mu(xx) \leq S(\mu(x), \mu(x)) = \mu(x)$ for all $x \in G$. Similarly

$$\max\{\mu(x), \mu(y)\} = S\Big(\max\{\mu(x), \mu(y)\}, \max\{\mu(x), \mu(y)\}\Big)$$
$$\geq S(\mu(x), \mu(y)) \geq \max\{\mu(x), \mu(y)\}$$

for all $x, y \in G$. Hence $\mu(xy) \leq S(\mu(x), \mu(y)) = \max\{\mu(x), \mu(y)\}$ for all $x, y \in G$. □

THEOREM 3.7. *Let μ be a fuzzy BCC-subalgebra of G with respect to an s-norm S and let $\alpha \in [0,1]$. Then*

(i) *if $S(\alpha, \alpha) = \alpha$, then the lower level set $L(\mu; \alpha)$ of μ is either empty or a BCC-subalgebra of G;*

(ii) *if $S(\alpha, \beta) = \max\{\alpha, \beta\}$, then the lower level set $L(\mu; \alpha)$ of μ is either empty or a BCC-subalgebra of G, and moreover $\mu(0) \leq \mu(x)$ for all $x \in G$.*

Proof.

(i) Assume that $S(\alpha, \alpha) = \alpha$ and $L(\mu; \alpha)$ is non-empty. If $x, y \in L(\mu; \alpha)$, then

$$\mu(xy) \leq S(\mu(x), \mu(y)) \leq S(\mu(x), \alpha) = S(\alpha, \mu(x)) \leq S(\alpha, \alpha) = \alpha,$$

which means that $xy \in L(\mu; \alpha)$. Hence $L(\mu; \alpha)$ is a BCC-subalgebra of G.

(ii) If $S(\alpha, \beta) = \max\{\alpha, \beta\}$, then S is an idempotent s-norm and by (i) every non-empty $L(\mu; \alpha)$ is a BCC-subalgebra of G. Moreover, for all $x \in G$, we get

$$\mu(0) = \mu(xx) \leq S(\mu(x), \mu(x)) = \max\{\mu(x), \mu(x)\} = \mu(x),$$

which completes the proof. □

COROLLARY 3.8. *If μ in G is a fuzzy BCC-subalgebra of G with respect to an idempotent s-norm S, then every non-empty lower level set $L(\mu; \alpha)$ is a BCC-subalgebra of G.*

THEOREM 3.9. *Let S be an s-norm and let μ be a fuzzy set in G. If every non-empty lower level set $L(\mu; \alpha)$ is a BCC-subalgebra of G, then μ is an S-fuzzy BCC-subalgebra of G.*

Proof. Assume that $\mu(x_0 y_0) > S(\mu(x_0), \mu(y_0))$ for some $x_0, y_0 \in G$. Then

$$\max\{\mu(x_0), \mu(y_0)\} \leq S(\mu(x_0), \mu(y_0)) < \alpha_0 < \mu(x_0 y_0)$$

by taking $\alpha_0 := \frac{1}{2}[\mu(x_0 y_0) + S(\mu(x_0), \mu(y_0))]$. It follows that $x_0, y_0 \in L(\mu; \alpha_0)$ and $x_0 y_0 \notin L(\mu; \alpha_0)$. This is a contradiction, and hence μ satisfies the inequality $\mu(xy) \leq S(\mu(x), \mu(y))$ for all $x, y \in G$. □

4. Zero invariant fuzzy BCC-subalgebras

DEFINITION 4.1. A fuzzy set μ in G is said to be a *zero invariant* (0-*invariant*) if $\mu(0) = 0$.

Note that the fuzzy BCC-subalgebra of G with respect to S_m in Proposition 3.5 is zero invariant.

In the following theorem, we give a condition for a fuzzy BCC-subalgebra of G with respect to an s-norm to be zero invariant.

THEOREM 4.2. *Let S be an s-norm and let μ be an S-fuzzy BCC-subalgebra of G. If there is a sequence $\{x_n\}$ in G such that*

$$\liminf_{n\to\infty} S\big(\mu(x_n), \mu(x_n)\big) = 0\,,$$

then μ is a zero invariant.

Proof. For any $x \in G$, we have $\mu(0) = \mu(xx) \le S\big(\mu(x), \mu(x)\big)$. Therefore $\mu(0) \le S\big(\mu(x_n), \mu(x_n)\big)$ for each $n \in \mathbb{N}$, and so

$$0 \le \mu(0) \le \liminf_{n\to\infty} S\big(\mu(x_n), \mu(x_n)\big) = 0\,.$$

It follows that $\mu(0) = 0$, i.e. μ is zero invariant. □

If μ is a fuzzy set in G and Θ is a mapping from G into itself, we define a mapping $\mu_\Theta \colon G \to [0,1]$ by $\mu_\Theta(x) = \mu\big(\Theta(x)\big)$ for all $x \in G$.

PROPOSITION 4.3. *Let S be an s-norm. If μ is an S-fuzzy BCC-subalgebra of G and Θ is an endomorphism of G, then μ_Θ is an S-fuzzy BCC-subalgebra of G. Moreover, if μ is a zero invariant, then so is μ_Θ.*

Proof. For any $x, y \in G$, we have

$$\begin{aligned} \mu_\Theta(xy) &= \mu\big(\Theta(xy)\big) = \mu\big(\Theta(x)\Theta(y)\big) \\ &\le S\Big(\mu\big(\Theta(x)\big), \mu\big(\Theta(y)\big)\Big) = S\big(\mu_\Theta(x), \mu_\Theta(y)\big)\,. \end{aligned}$$

Hence μ_Θ is an S-fuzzy BCC-subalgebra of G.

Assume that μ is a zero invariant. Since $\Theta(0) = 0$, we get $\mu_\Theta(0) = \mu\big(\Theta(0)\big)$ $\mu(0) = 0$. This completes the proof.

Let f be a mapping defined on G. If ν is a fuzzy set in $f(G)$, then the fuzzy set $f^{-1}(\nu)$ in G defined by $\big[f^{-1}(\nu)\big](x) = \nu\big(f(x)\big)$ for all $x \in G$ is called the *preimage* of ν under f.

THEOREM 4.4. *Let S be an s-norm. An onto homomorphic preimage of a zero invariant S-fuzzy BCC-subalgebra is also a zero invariant S-fuzzy BCC-subalgebra.*

Proof. Let $f\colon G \to H$ be a homomorphism from a BCC-algebra G onto a BCC-algebra H and let ν be a zero invariant S-fuzzy BCC-subalgebra of H. Then

$$\begin{aligned}[f^{-1}(\nu)](xy) &= \nu(f(xy)) = \nu(f(x)f(y)) \\ &\le S\Big(\nu(f(x)), \nu(f(y))\Big) = S\Big([f^{-1}(\nu)](x), [f^{-1}(\nu)](y)\Big)\end{aligned}$$

for all $x, y \in G$, and

$$[f^{-1}(\nu)](0) = \nu(f(0)) = \nu(0) = 0\,.$$

Hence $f^{-1}(\nu)$ is a zero invariant S-fuzzy BCC-subalgebra of G. $\square$

DEFINITION 4.5. Let μ be a fuzzy set in G and f a mapping defined on G. The fuzzy set μ^f in $f(G)$ defined by $\mu^f(y) = \inf\limits_{x \in f^{-1}(y)} \mu(x)$ for all $y \in f(G)$ is called the *anti image* of μ under f.

An s-norm S on $[0,1]$ is said to be *continuous* if S is a continuous function from $[0,1] \times [0,1]$ to $[0,1]$ with respect to the usual topology.

THEOREM 4.6. *Let S be a continuous s-norm and let f be a homomorphism on G. If μ is a zero invariant S-fuzzy BCC-subalgebra of G, then the anti image of μ under f is a zero invariant S-fuzzy BCC-subalgebra of $f(G)$.*

Proof. Let $A_1 = f^{-1}(y_1)$, $A_2 = f^{-1}(y_2)$ and $A_{12} = f^{-1}(y_1y_2)$, where $y_1, y_2 \in f(G)$. Consider the set

$$A_1A_2 := \{x \in G :\ x = a_1a_2 \text{ for some } a_1 \in A_1,\ \ a_2 \in A_2\}\,.$$

If $x \in A_1A_2$, then $x = x_1x_2$ for some $x_1 \in A_1$, $x_2 \in A_2$, and so $f(x) = f(x_1x_2) = f(x_1)f(x_2) = y_1y_2$, i.e., $x \in f^{-1}(y_1y_2) = A_{12}$. Thus $A_1A_2 \subseteq A_{12}$. It follows that

$$\begin{aligned}\mu^f(y_1y_2) &= \inf_{x \in f^{-1}(y_1y_2)} \mu(x) = \inf_{x \in A_{12}} \mu(x) \\ &\le \inf_{x \in A_1A_2} \mu(x) \le \inf_{\substack{x_1 \in A_1,\\ x_2 \in A_2}} \mu(x_1x_2) \\ &\le \inf_{\substack{x_1 \in A_1,\\ x_2 \in A_2}} S(\mu(x_1), \mu(x_2))\,.\end{aligned}$$

Now S is continuous, and therefore if ε is any positive number, then there exists a number $\delta > 0$ such that

$$S(x_1^*, x_2^*) \le S\Big(\inf_{x_1 \in A_1} \mu(x_1), \inf_{x_2 \in A_2} \mu(x_2)\Big) + \varepsilon\,,$$

whenever $x_1^* \le \inf_{x_1 \in A_1} \mu(x_1) + \delta$ and $x_2^* \le \inf_{x_2 \in A_2} \mu(x_2) + \delta$. Choose $a_1 \in A_1$ and $a_2 \in A_2$ such that

$$\mu(a_1) \le \inf_{x_1 \in A_1} \mu(x_1) + \delta \qquad \text{and} \qquad \mu(a_2) \le \inf_{x_2 \in A_2} \mu(x_2) + \delta .$$

Then

$$S(\mu(a_1), \mu(a_2)) \le S\Big(\inf_{x_1 \in A_1} \mu(x_1), \inf_{x_2 \in A_2} \mu(x_2) \Big) + \varepsilon .$$

Consequently

$$\begin{aligned} \mu^f(y_1 y_2) &\le \inf_{\substack{x_1 \in A_1, \\ x_2 \in A_2}} S(\mu(x_1), \mu(x_2)) \\ &\le S\Big(\inf_{x_1 \in A_1} \mu(x_1), \inf_{x_2 \in A_2} \mu(x_2) \Big) \\ &= S(\mu^f(y_1), \mu^f(y_2)) . \end{aligned}$$

Moreover, $\mu^f(0) = \inf_{x \in f^{-1}(0)} \mu(x) \le \mu(0) = 0$ and hence $\mu^f(0) = 0$. Thus μ^f is a zero invariant fuzzy BCC-subalgebra of $f(G)$ with respect to S. □

5. Direct products and s-normed products

PROPOSITION 5.1. *Let S be an s-norm and let $G = G_1 \times G_2$ be the direct product of BCC-algebras G_1 and G_2. If μ_1 (resp. μ_2) is a zero invariant S-fuzzy BCC-subalgebra of G_1 (resp. G_2), then $\mu = \mu_1 \times \mu_2$ is a zero invariant S-fuzzy BCC-subalgebra of G defined by*

$$\mu(x_1, x_2) = (\mu_1 \times \mu_2)(x_1, x_2) = S(\mu_1(x_1), \mu(x_2))$$

for all $(x_1, x_2) \in G$.

P r o o f. Note that $\mu(0,0) = S(\mu_1(0), \mu_2(0)) = S(0,0) = 0$. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any elements of G. Then

$$\begin{aligned} \mu(xy) &= \mu((x_1, x_2)(y_1, y_2)) = \mu(x_1 y_1, x_2 y_2) \\ &= S(\mu_1(x_1 y_1), \mu_2(x_2 y_2)) \\ &\le S\Big(S(\mu_1(x_1), \mu_1(y_1)), S(\mu_2(x_2), \mu_2(y_2))\Big) \\ &= S\Big(S(\mu_1(x_1), \mu_2(x_2)), S(\mu_1(y_1), \mu_2(y_2))\Big) \\ &= S(\mu(x_1, x_2), \mu(y_1, y_2)) = S(\mu(x), \mu(y)) . \end{aligned}$$

Hence μ is a zero invariant S-fuzzy BCC-subalgebra of G. □

Now we will generalize the idea to the product of $n \ge 2$ fuzzy BCC-subalgebras with respect to S. We first need to generalize the domain of s-norm to $\prod_{i=1}^{n} [0,1]$ as follows:

Definition 5.2. The function $S_n \colon \prod_{i=1}^{n} [0,1] \to [0,1]$ is defined by

$$S_n(\alpha_1, \alpha_2, \dots, \alpha_n) = S\big(\alpha_i, S_{n-1}(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n)\big)$$

for all $1 \le i \le n$, where $n \ge 2$, $S_2 = S$ and $S_1 = \mathrm{id}$ (identity).

Using the induction on n, we have the following two lemmas.

Lemma 5.3. *For any s-norm S and every $\alpha_i, \beta_i \in [0,1]$, where $1 \le i \le n$ and $n \ge 2$, we have*

$$\begin{aligned} S_n\big(S(\alpha_1,\beta_1), S(\alpha_2,\beta_2), \dots, S(\alpha_n,\beta_n)\big) \\ = S\big(S_n(\alpha_1, \alpha_2, \dots, \alpha_n), S_n(\beta_1, \beta_2, \dots, \beta_n)\big). \end{aligned}$$

Lemma 5.4. *For an s-norm S and every $\alpha_1, \dots, \alpha_n \in [0,1]$, where $n \ge 2$, we have*

$$\begin{aligned} S_n(\alpha_1, \dots, \alpha_n) &= S\big(\dots S\big(S\big(S(\alpha_1, \alpha_2), \alpha_3\big) \alpha_4\big), \dots, \alpha_n\big) \\ &= S\big(\alpha_1, S\big(\alpha_2, S\big(\alpha_3, \dots S(\alpha_{n-1}, \alpha_n) \dots\big)\big)\big). \end{aligned}$$

Theorem 5.5. *Let S be an s-norm and let $G = \prod_{i=1}^{n} G_i$ be the direct product of BCC-subalgebras $\{G_i\}_{i=1}^{n}$. If every μ_i is a zero invariant S-fuzzy BCC-subalgebra of G_i, where $1 \le i \le n$, then $\mu = \prod_{i=1}^{n} \mu_i$ defined by*

$$\mu(x_1, x_2, \dots, x_n) = \left(\prod_{i=1}^{n} \mu_i\right)(x_1, x_2, \dots, x_n) = S_n\big(\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n)\big)$$

is a zero invariant S-fuzzy BCC-subalgebra of G.

Proof. We first note that

$$\mu(0, 0, \dots, 0) = S_n\big(\mu_1(0), \mu_2(0), \dots, \mu_n(0)\big) = S_n(0, 0, \dots, 0) = 0.$$

Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ be any elements of G. Then, by the above Lemma 5.3,

$$\begin{aligned} \mu(xy) &= \mu(x_1y_1, x_2y_2, \dots, x_ny_n) \\ &= S_n\big(\mu_1(x_1y_1), \mu_2(x_2y_2), \dots, \mu_n(x_ny_n)\big) \\ &\le S_n\Big(S\big(\mu_1(x_1), \mu_1(y_1)\big),\ S\big(\mu_2(x_2), \mu_2(y_2)\big), \dots, S\big(\mu_n(x_n), \mu_n(y_n)\big)\Big) \\ &= S\Big(S_n\big(\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n)\big),\ S_n\big(\mu_1(y_1), \mu_2(y_2), \dots, \mu_n(y_n)\big)\Big) \\ &= S\big(\mu(x_1, x_2, \dots, x_n), \mu(y_1, y_2, \dots, y_n)\big) = S\big(\mu(x), \mu(y)\big). \end{aligned}$$

Hence μ is a zero invariant S-fuzzy BCC-subalgebra of G. □

DEFINITION 5.6. Let S be an s-norm and let μ and ν be two fuzzy sets in G. Then the *S-product of μ and ν*, denoted by $[\mu \cdot \nu]_S$, is defined by

$$[\mu \cdot \nu]_S(x) = S\big(\mu(x), \nu(x)\big)$$

for all $x \in G$.

Obviously $[\mu \cdot \nu]_S$ is a fuzzy set in G such that $[\mu \cdot \nu]_S = [\nu \cdot \mu]_S$.

THEOREM 5.7. *Let S be an s-norm and let μ and ν be zero invariant S-fuzzy BCC-subalgebras of G. Let S^* be an s-norm which dominates S, i.e.,*

$$S^*\big(S(\alpha,\gamma), S(\beta,\delta)\big) \le S\big(S^*(\alpha,\beta), S^*(\gamma,\delta)\big)$$

for all $\alpha, \beta, \gamma, \delta \in [0,1]$. Then S^*-product $[\mu \cdot \nu]_{S^*}$ is a zero invariant S-fuzzy BCC-subalgebra of G.

Proof. Note that $[\mu \cdot \nu]_{S^*}(0) = S^*\big(\mu(0), \nu(0)\big) = S^*(0,0) = 0$. Hence $[\mu \cdot \nu]_{S^*}$ is zero invariant. For any $x, y \in G$, we have

$$\begin{aligned} [\mu \cdot \nu]_{S^*}(xy) &= S^*\big(\mu(xy), \nu(xy)\big) \\ &\le S^*\Big(S\big(\mu(x), \mu(y)\big), S\big(\nu(x), \nu(y)\big)\Big) \\ &\le S\Big(S^*\big(\mu(x), \nu(x)\big), S^*\big(\mu(y), \nu(y)\big)\Big) \\ &= S\big([\mu \cdot \nu]_{S^*}(x),\, [\mu \cdot \nu]_{S^*}(y)\big), \end{aligned}$$

which proves that $[\mu \cdot \nu]_{S^*}$ is an S-fuzzy BCC-subalgebra of G. □

Let $f\colon G \to G'$ be an onto homomorphism of BCC-algebras. If μ and ν are zero invariant S-fuzzy BCC-subalgebras of G', then the S^*-product $[\mu \cdot \nu]_{S^*}$ is a zero invariant S-fuzzy BCC-subalgebra of G' whenever S^* dominates S. Using Theorem 4.4, the preimages $f^{-1}(\mu)$, $f^{-1}(\nu)$ and $f^{-1}\big([\mu \cdot \nu]_{S^*}\big)$ are zero invariant S-fuzzy BCC-subalgebras of G.

THEOREM 5.8. *Let S and S^* be s-norms in which S^* dominates S. Let $f\colon G \to G'$ be an onto homomorphism of BCC-algebras. For any zero invariant S-fuzzy BCC-subalgebras μ and ν of G', we have*

$$f^{-1}\big([\mu \cdot \nu]_{S^*}\big) = \big[f^{-1}(\mu) \cdot f^{-1}(\nu)\big]_{S^*}.$$

Proof. Let $x \in G$. Then

$$\begin{aligned} f^{-1}\big([\mu \cdot \nu]_{S^*}\big)(x) &= [\mu \cdot \nu]_{S^*}\big(f(x)\big) \\ &= S^*\Big(\mu\big(f(x)\big), \nu\big(f(x)\big)\Big) = S^*\Big(\big[f^{-1}(\mu)\big](x), \big[f^{-1}(\nu)\big](x)\Big) \\ &= \big[f^{-1}(\mu) \cdot f^{-1}(\nu)\big]_{S^*}(x), \end{aligned}$$

completing the proof. □

THEOREM 5.9. *Let S be a continuous s-norm, f a homomorphism on G, and let μ and ν be zero invariant S-fuzzy BCC-subalgebras of G. If an s-norm S^* dominates S, then:*

$$[\mu^f \cdot \nu^f]_{S^*} \subseteq ([\mu \cdot \nu]_{S^*})^f .$$

P r o o f . Using Theorem 5.7 and Theorem 4.6 we known that the S^*-product $[\mu \cdot \nu]_{S^*}$ is a zero invariant S-fuzzy BCC-subalgebra of G, and that the S^*-product $[\mu^f \cdot \nu^f]_{S^*}$ is a zero invariant S-fuzzy BCC-subalgebra of $f(G)$. Moreover, for each $y \in f(G)$, we have

$$\begin{aligned}([\mu \cdot \nu]_{S^*})^f(y) &= \inf_{x\in f^{-1}(y)} [\mu \cdot \nu]_{S^*}(x) = \inf_{x\in f^{-1}(y)} S^*\big(\mu(x), \nu(x)\big)\\ &\geq S^*\Big(\inf_{x\in f^{-1}(y)} \mu(x), \inf_{x\in f^{-1}(y)} \nu(x)\Big)\\ &= S^*\big(\mu^f(y), \nu^f(y)\big) = [\mu^f \cdot \nu^f]_{S^*}(y) ,\end{aligned}$$

and hence $[\mu^f \cdot \nu^f]_{S^*} \subseteq ([\mu \cdot \nu]_{S^*})^f$. □

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