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QBCC-ALGEBRAS INHERITED FROM QOSETS

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ABSTRACT. A new class of algebras derived from BCC-algebras, the so-called quasi-BCC-algebras (briefly QBCC-algebras), are introduced and studied. These algebras model properties of the logical connective implication " $\Longrightarrow$ ", for which the validity of formulas $x \Longrightarrow y$ and $y \Longrightarrow x$ does not mean the equivalence of x and y. A natural construction of QBCC-algebras from quasiordered sets (qosets) is then given and properties of such QBCC-algebras are studied.

1. Preliminaries

The notion of a BCK-algebra was introduced in 60's by Y. Imai and K. Iséki [7] as an algebraic formulation of Meredith's BCK-implicational calculus. When solving the problem whether the class of all BCK-algebras form a variety, Y. Komori [10] introduced the class of BCC-algebras and proved that this class is not a variety. A. Wroński [13] characterized BCC-algebras as algebras isomorphic with a subalgebra of the left-residuation reduct of some integral monoid with left-residuation.

There are several axiomatizations of BCC-algebras. We use that of [2], multiplication in which models some properties of the logical connective implication and the constant 1 means the logical value "true". For more details we refer also to [3] and [11].

DEFINITION 1. An algebra $(A, \bullet, 1)$ of type $(2,0)$ is a *BCC-algebra* if it satisfies the following identities:

(BCC1) $(x \bullet y) \bullet \big[(z \bullet x) \bullet (z \bullet y)\big] = 1$,
(BCC2) $x \bullet x = 1$,
(BCC3) $x \bullet 1 = 1$,

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(BCC4) $1 \bullet x = x$,
(BCC5) $(x \bullet y = 1 \ \& \ y \bullet x = 1) \implies x = y$.

It was shown by W. A. Dudek [2] that BCC-algebras satisfying the axiom

(C) $x \bullet (y \bullet z) = y \bullet (x \bullet z)$

are just BCK-algebras.

BCK-algebras satisfying the left-distributivity axiom

(D) $x \bullet (y \bullet z) = (x \bullet y) \bullet (x \bullet z)$

are known as Hilbert algebras, an algebraic counterpart of the logical connective implication in intuitionistic logic. Hilbert algebras were recently generalized in [5] as follows:

DEFINITION 2. A *pre-logic* is an algebra $\mathcal{A} = (A, \bullet, 1)$ of type $(2,0)$ satisfying the axioms:

(PL1) $x \bullet x = 1$,
(PL2) $1 \bullet x = x$,
(PL3) $x \bullet (y \bullet z) = (x \bullet y) \bullet (x \bullet z)$,
(PL4) $x \bullet (y \bullet z) = y \bullet (x \bullet z)$.

In other words, pre-logics, contrary to Hilbert algebras, need not satisfy the axiom (BCC5).

The axioms of a BCC-algebra $(A, \bullet, 1)$ allow us to define a *natural ordering* on A as follows:

$$x \leq y \iff x \bullet y = 1. \tag{1}$$

Indeed, reflexivity is a conclusion of (BCC2), antisymetry of (BCC5) and transitivity can be derived from (BCC1). Henceforth, from this point of view BCC-algebras are special cases of ordered sets. When extracting the axiom (BCC5) from the axiomatic system of BCC-algebras we see that $\leq$ defined in (1) is a quasiorder relation similarly as in the case of pre-logics. This leads us to a common generalization of both the classes of BCC-algebras and pre-logics:

DEFINITION 3. A *quasi-BCC-algebra* (briefly *QBCC-algebra*) is any algebra $\mathcal{A} = (A, \bullet, 1)$ satisfying the axioms (BCC1)–(BCC4). A quasiorder relation defined on A by (1) is called a *natural quasiordering* on A.

Remark. If $(A, \leq)$ is any quasiordered set, $a, b \in A$, we adopt the following terminology:

We write $a \sim b$ whenever $a \leq b$ and $b \leq a$ hold and call the pair (a,b) *indistinguishable*; the set $C(a) = \{x \in A : \ x \sim a\}$ is called the *cell* of a. We write $a < b$ if $a \leq b$ and $a \not\sim b$. If A is finite, then $(A, \leq)$ can be viewed as a poset in which elements can be substituted by cells.

For example, the diagram given in Fig. 1

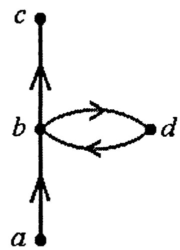

FIGURE 1.

represents a qoset in which, excluding reflexivity, the relations $a \le b$, $a \le c$, $b \le c$, $b \le d$, $d \le b$ hold.

One can easily derive that the natural quasiordering $\le$ on any QBCC-algebra $\mathcal{A} = (A, \bullet, 1)$ has the following properties:

$$1 \le x \iff x = 1\,, \tag{2}$$

$$y \le x \bullet y \qquad \text{for each} \quad x, y \in A\,. \tag{3}$$

Indeed, $1 \le x$ yields by (BCC4) $1 = 1 \bullet x = x$. Substituting $x = 1$ into (BCC1) we get the property (3). The property (2) exactly means that $C(1) = \{1\}$.

EXAMPLE 1. Let us consider an algebra $\mathcal{A} = (A, \bullet, 1)$ given by the table:

$\bullet$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	1	d
c	1	1	1	1	d
d	1	1	a	a	1

One can show that $\mathcal{A}$ is a QBCC-algebra, but $1 = b \bullet c = c \bullet b$ for $c \neq b$ and

$$1 = a \bullet a = a \bullet (d \bullet b) \neq (a \bullet d) \bullet (a \bullet b) = d \bullet b = a\,,$$

verifying that $\mathcal{A}$ is neither a BCC-algebra nor a pre-logic.

In [5] it was shown that there are pre-logics $\mathcal{A} = (A, \bullet, 1)$ having the property that every subset containing the element 1 is a subalgebra of $\mathcal{A}$. In other words, a subalgebra lattice $\operatorname{Sub} \mathcal{A} \cong \mathbf{2}^{|A \setminus \{1\}|}$.

EXAMPLE 2. Let $(Q, \leq, 1)$ be a qoset with a greatest element 1, $C(1) = \{1\}$. Let us define for $x, y \in Q$

$$x \bullet y = \begin{cases} 1 & \text{if } x \leq y\,, \\ y & \text{otherwise.} \end{cases}$$

One can verify that $\mathcal{Q} = (Q, \bullet, 1)$ is a QBCC-algebra (even a pre-logic).

The aim of the paper is to describe all the QBCC-algebras in which every subset containing 1 is a subalgebra. In the following paragraph we will present a new construction of QBCC-algebras derived from qosets.

2. Standard QBCC-algebras

To simplify expressions, a QBCC-algebra $\mathcal{A} = (A, \bullet, 1)$ in which every subset containing 1 is a subalgebra will be called *standard*.

It is clear that for $\mathcal{A}$ to be standard it is enough every subset having three elements $\{x, y, 1\}$ to be a subalgebra of $\mathcal{A}$, hence $x \bullet y \in \{1, x, y\}$. By the natural quasiorder $\leq$, the case $x \bullet y = 1$ holds if and only if $x \leq y$. The second case $x \bullet y = x$ is possible by the property (3) only when $y < x$. In other words, we have necessarily $x \bullet y = y$ whenever $x \parallel y$ (i.e. $x \not\leq y$ and $y \not\leq x$).

A pair (x, y) of (distinct) elements $x, y \in A$, $x > y$, is called *normal* if $x \bullet y = y$.

Summarizing all the cases above, it is enough to describe which pairs of elements (x, y) can be non-normal. At first we will describe a local behavior of such couples.

THEOREM 1. *Let $\mathcal{A} = (A, \bullet, 1)$ be a standard QBCC-algebra, let (x, y) be a non-normal pair of elements, i.e. $x \bullet y = x$, $x > y$. Then the following conditions hold:*

(a) *for each $z > y$ we have either $z \sim x$ and $z \bullet y = z$ or $z > x$ and the pairs (z, x), (z, y) are both normal;*

(b) *for each $z < x$ we have either $z \sim y$ and $x \bullet z = x$ or $z < y$ and the pairs (x, z), (y, z) are both normal.*

Proof.

(a) Suppose that for $z > y$, $x \sim z$ or $x < z$ does not hold. Then one of the following cases occurs:

α) $y < z < x$,
β) $z \parallel x$.

We will show that both the cases α) and β) lead to a contradiction.

The case α):

We have $y \bullet x = z \bullet x = y \bullet z = 1$, hence

$$1 = (x \bullet y) \bullet \big[(z \bullet x) \bullet (z \bullet y)\big] = x \bullet \big[1 \bullet (z \bullet y)\big] = x \bullet (z \bullet y)\,.$$

Now we have either $z \bullet y = y$ and hence $x \bullet y = 1$, a contradiction with $x > y$, or $z \bullet y = z$ and $x \bullet z = 1$, contradicting $x > z$.

The case β):

In this case it holds that $y \bullet x = y \bullet z = 1$, $x \bullet z = z$, $z \bullet x = x$ and, again by (BCC1),

$$1 = (y \bullet z) \bullet \big[(x \bullet y) \bullet (x \bullet z)\big] = 1 \bullet (x \bullet z) = x \bullet z\,,$$

which is a contradiction with $x \parallel z$.

We have shown that either $x < z$ or $x \sim z$ whenever $y < z$.

Suppose further that $z \sim x$, i.e. $z \bullet x = x \bullet z = 1$. Then (BCC1) yields

$$1 = (x \bullet y) \bullet \big[(z \bullet x) \bullet (z \bullet y)\big] = x \bullet (z \bullet y)\,.$$

Since $z > y$, we have $z \bullet y \in \{z, y\}$. The case $z \bullet y = y$ leads to

$$1 = x \bullet (z \bullet y) = x \bullet y\,,$$

which is a contradiction with $x > y$. Henceforth $z \bullet y = z$ holds and the pair (z, y) is not normal.

Consider the case $x < z$ and let us prove that then the pairs (z, x), (z, y) are normal. Assume on the contrary, that $z \bullet y = z$. Then

$$1 = (y \bullet x) \bullet \big[(z \bullet y) \bullet (z \bullet x)\big] = 1 \bullet \big[z \bullet (z \bullet x)\big] = z \bullet (z \bullet x)\,.$$

Further, if $z \bullet x = x$, then $1 = z \bullet (z \bullet x) = z \bullet x$ contradicting $z > x$. The case $z \bullet x = z$ gives us

$$1 = (z \bullet y) \bullet \big[(x \bullet z) \bullet (x \bullet y)\big] = z \bullet (1 \bullet x) = z\,,$$

hence also $1 = z = z \bullet x = 1 \bullet x = x$, a contradiction with $z > x$. Henceforth we have necessarily $z \bullet y = y$ and the pair (z, y) is normal.

Let us show the normality of (z, x). It holds that

$$1 = (x \bullet y) \bullet \big[(z \bullet x) \bullet (z \bullet y)\big] = x \bullet \big[(z \bullet x) \bullet y\big]\,,$$

and $z \bullet x \in \{x, z\}$. Supposing $z \bullet x = z$ we obtain

$$1 = x \bullet \big[(z \bullet x) \bullet y\big] = x \bullet (z \bullet y) = x \bullet y\,,$$

which does not hold. This proves $z \bullet x = x$, the normality of (z, x).

(b) Consider the dual case $z < x$ and assume that $z \parallel y$ (the case $y < z < x$ cannot occur by (a)). Then $y \bullet x = z \bullet x = 1$, $z \bullet y = y$, $y \bullet z = z$, and

$$1 = (x \bullet y) \bullet \big[(z \bullet x) \bullet (z \bullet y)\big] = x \bullet (1 \bullet y) = x \bullet y\,,$$

contradicting $x > y$. Hence we have necessarily $z \leq y$.

Supposing $z \sim y$, we get $y \bullet z = 1$ and

$$1 = (y \bullet z) \bullet \big[(x \bullet y) \bullet (x \bullet z)\big] = 1 \bullet \big[x \bullet (x \bullet z)\big] = x \bullet (x \bullet z)\,.$$

Then

$$1 = x \bullet (x \bullet z) = x \bullet z\,,$$

which does not hold. Hence, the pair (x, z) is not normal and $x \bullet z = x$.

Consider the second possible case, i.e. $z < y < x$ and let us prove the normality of (x, z) and (y, z). The pair (x, z) has to be normal because of (a) (non-normal pair is a covering pair).

If the pair (y, z) is not normal, then by (a) again, the pair (x, y) is normal, a contradiction. Hence the pair (y, z) is normal. □

Theorem 1 motivates us to introduce the following concept:

DEFINITION 4. Let $(Q, \leq, 1)$ be a qoset with a greatest element 1, and $C(1) = \{1\}$. A pair $(x,y) \in Q \times Q$, $x > y$, is called a *bridge* if for each $z \in Q$ the following (dual) conditions hold:

(b1) $z > y$ implies $z \geq x$,
(b2) $z < x$ implies $z \leq y$.

Remark. It is clear that if (x, y) is a bridge in Q, then x covers y, i.e. there is no $z \in Q$ with $y < z < x$. The notion of "bridge" is motivated by the diagram of Q around the pair (x, y), which looks like a bridge between x and y. In account of Theorem 1 we have seen that bridges are the only candidates for pairs of elements which need not be normal. Next we will describe all the standard QBCC-algebras.

THEOREM 2. *Let $(Q, \leq, 1)$ be a qoset with a greatest element 1 and $C(1) = \{1\}$. Let us define the operation $\bullet$ on Q as follows:*

(q1) *$x \bullet y = 1$ if $x \leq y$,*
(q2) *$1 \bullet x = x$,*
(q3) *$x \bullet y = y$ if $x \parallel y$,*
(q4) *$x \bullet y = y$ if $x > y$ and (x,y) is not a bridge,*
(q5) *if (x,y) is a bridge in Q and $x \neq 1$, one can set $x \bullet y = y$ or $x \bullet y = x$; in the latter case for each $z \geq x$ we have either $z \sim x$ and $z \bullet y = z$ or $z > x$ and $z \bullet x = x$, $z \bullet y = y$; for each $z \leq y$ we have either $z \sim y$ and $x \bullet z = x$ or $z < y$ and $x \bullet z = y \bullet z = z$.*

Then $(Q, \bullet, 1)$ *is a standard QBCC-algebra and each standard QBCC-algebra is of this form.*

Proof. It is sufficient to show the validity of the axiom

$$(x \bullet y) \bullet \big[(z \bullet x) \bullet (z \bullet y)\big] = 1\,. \tag{$*$}$$

One can easily show that $(*)$ holds whenever there are identical elements among x, y, z, so we can suppose that they are pairwise distinct. The same holds if one of the elements x, y, z is equal to 1. Comparing the elements y, z we distinguish several cases.

Case 1.

Suppose $z \leq y$. Then by (q1), $z \bullet y = 1$ and $(*)$ is valid.

Case 2.

Suppose further $z \parallel y$. Then due to (q3), $z \bullet y = y$ and the left hand side of $(*)$ has the form

$$(x \bullet y) \bullet \big[(z \bullet x) \bullet (z \bullet y)\big] = (x \bullet y) \bullet \big[(z \bullet x) \bullet y\big]\,.$$

With respect to elements x, z the following cases can occur:

Subcase 2.1. Let $z \parallel x$. Then by (q3) again $z \bullet x = x$ and applying (q1) we get $(x \bullet y) \bullet \big[(z \bullet x) \bullet y\big] = (x \bullet y) \bullet (x \bullet y) = 1$.

Subcase 2.2. Let us have $z \leq x$, i.e. $z \bullet x = 1$ and $(x \bullet y) \bullet \big[(z \bullet x) \bullet y\big] = (x \bullet y) \bullet y$. We have $x \not\leq y$, otherwise we would have $z \leq x \leq y$, a contradiction with $z \parallel y$. The case $x \bullet y = y$ gives us $(x \bullet y) \bullet y = y \bullet y = 1$. If $x \bullet y = x$, then $x > y$ and the pair (x, y) is a bridge. But it is impossible because of $z \leq x$ and $z \parallel y$.

Subcase 2.3. Suppose finally that $z > x$. Then $z \bullet x \in \{x, z\}$. If $z \bullet x = z$, then (z, x) is a bridge and since $z \parallel y$, necessarily also $x \parallel y$, and due to (q3) $x \bullet y = y$. From this we can derive $(x \bullet y) \bullet \big[(z \bullet x) \bullet y\big] = y \bullet (z \bullet y) = y \bullet y = 1$.

The case $z \bullet x = x$ leads to $(x \bullet y) \bullet \big[(z \bullet x) \bullet y\big] = (x \bullet y) \bullet (x \bullet y) = 1$.

Case 3.

Suppose that $z > y$. The following two subcases can occur:

Subcase 3.1. Let the pair (z, y) be normal, i.e. $z \bullet y = y$, and

$$(x \bullet y) \bullet \big[(z \bullet x) \bullet (z \bullet y)\big] = (x \bullet y) \bullet \big[(z \bullet x) \bullet y\big]\,.$$

Further if $z \bullet x = 1$, i.e. $z \leq x$, then $x \bullet y \in \{x, y\}$ (otherwise we would have $z \leq y$). The case $x \bullet y = y$ gives $(x \bullet y) \bullet \big[(z \bullet x) \bullet y\big] = y \bullet (1 \bullet y) = y \bullet y = 1$. For $x \bullet y = x$ we have $x > y$ and the pair (x, y) is a bridge with $z > y$. This leads to $z \geq x$ and $z \not> x$, otherwise we would have, by (q5), $z \bullet y = z$, which does not hold. Hence $z > x$, a contradiction.

Suppose further that $z \bullet x = x$. Then $(x \bullet y) \bullet \big[(z \bullet x) \bullet y\big] = (x \bullet y) \bullet (x \bullet y) = 1$.

Let us consider the last case $z \bullet x = z$. We have $z > x$, and the pair (z, x) is a bridge. Then $(x \bullet y) \bullet \left[(z \bullet x) \bullet y\right] = (x \bullet y) \bullet (z \bullet y) = (x \bullet y) \bullet y$. Since $y < z$ and (z, x) is a bridge, we have also $y \le x$ due to (b2). The case $y \sim x$ is impossible, since by (q5) the pair (z, y) would not be normal. Hence $y < x$ and by (q5) again, the pair (x, y) is normal, i.e. $x \bullet y = y$ and $(x \bullet y) \bullet y = y \bullet y = 1$, finishing the Subcase 3.1.

Subcase 3.2. Let us consider that the pair (z, y) is not normal, hence $z \bullet y = z$, and

$$(x \bullet y) \bullet \left[(z \bullet x) \bullet (z \bullet y)\right] = (x \bullet y) \bullet \left[(z \bullet x) \bullet z\right].$$

Suppose further $z \bullet x = 1$, i.e. $z \le x$. If $z \sim x$, then by (q5) the pair (x, y) is also not normal, hence $x \bullet y = x$ and $(x \bullet y) \bullet \left[(z \bullet x) \bullet z\right] = x \bullet (1 \bullet z) = x \bullet z = 1$. For $z < x$ we have again $x \bullet y = y$ by (q4), and so $(x \bullet y) \bullet \left[(z \bullet x) \bullet z\right] = y \bullet (1 \bullet z) = y \bullet z = 1$.

Consider the case when $z \bullet x = x$, hence $(x \bullet y) \bullet \left[(z \bullet x) \bullet z\right] = (x \bullet y) \bullet (x \bullet z)$. Suppose further $x \parallel z$, i.e. $x \bullet z = z$ and so $(x \bullet y) \bullet (x \bullet z) = (x \bullet y) \bullet z$. Since (y, z) forms a bridge and $z \parallel x$, also $x \parallel y$ holds and $x \bullet y = y$ yields $(x \bullet y) \bullet z = y \bullet z = 1$. The case $z > x$ leads to $x \bullet z = 1$ and $(x \bullet y) \bullet (x \bullet z) = (x \bullet y) \bullet 1 = 1$.

The last possible case with respect to x and z is $z \bullet x = z$. But then $(x \bullet y) \bullet \left[(z \bullet x) \bullet (z \bullet y)\right] = (x \bullet y) \bullet (z \bullet z) = (x \bullet y) \bullet 1 = 1$, finishing the proof of Subcase 3.2. □

Theorem 2 allows us to construct a standard QBCC-algebra from a given qoset Q. It shows that one can have non-normal pairs of elements only when Q contains bridges. Hence, for a qoset without bridges the only possibility to get standard QBCC-algebras is as shown in Example 2.

EXAMPLE 3. Let us consider a qoset Q with the diagram in Fig. 2.

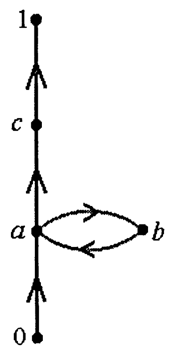

FIGURE 2.

By setting $a \bullet 0 = a$ we get $b \bullet 0 = b$, $c \bullet 0 = 0$, $c \bullet a = a$, $c \bullet b = b$ by (q5). The rest of cases is given by (q1), (q2) and (q4), hence the operation $\bullet$ is completely determined.

It is immediately seen that the algebras described in Theorem 2 need not satisfy the distributivity axiom (D), hence these need not be pre-logics in general. Indeed, if (x, y) is any non-normal pair, i.e. $x \bullet y = x$, then $1 = x \bullet x = x \bullet (x \bullet y) \neq (x \bullet x) \bullet (x \bullet y) = x$.

COROLLARY 1. *Every standard QBCC-algebra satisfies the axiom* (C).

Proof. We show that every standard QBCC-algebra $(Q, \bullet, 1)$ satisfies the axiom (C):

$$x \bullet (y \bullet z) = y \bullet (x \bullet z).$$

If there are identical elements among x, y and z, (C) holds (we use the identity $a \bullet (b \bullet a) = 1$). The same holds if one of the elements x, y, z is equal to 1. So we can suppose that x, y, z are distinct elements of Q.

Comparing y and z we obtain three possibilities:

Case 1.
Let $y \leq z$, i.e. $y \bullet z = 1$. We get $x \bullet (y \bullet z) = 1$. Considering $x \bullet z = 1$ or $x \bullet z = z$ one gets $y \bullet (x \bullet z) = 1$ and the equality holds. If $x \bullet z = x$, i.e. the pair (x, z) forms a bridge and $x > z$, we obtain, using transitivity, $y < x$. So $y \bullet (x \bullet z) = y \bullet x = 1$.

Case 2.
Suppose $y \parallel z$, i.e. $y \bullet z = z$. For $x \bullet z = 1$ both the sides of (C) are equal to 1. If $x \bullet z = z$, we obtain $x \bullet (y \bullet z) = y \bullet (x \bullet z) = z$. In the last subcase $x \bullet z = x$ we get $x \bullet (y \bullet z) = x$. Since (x, z) is a bridge and $y \parallel z$, also $y \parallel x$. It follows that $y \bullet (x \bullet z) = y \bullet x = x$ and (C) holds.

Case 3.
Suppose finally that $y > z$. Let further $y \bullet z = z$. If $x \bullet z = 1$, then both the sides of (C) are equal to 1. In the subcase $x \bullet z = z$ we get $x \bullet (y \bullet z) = y \bullet (x \bullet z) = z$. The last possible subcase is $x \bullet z = x$, i.e. (x, z) is not normal and forms a bridge. So we have either $y \sim x$ and then by (q5) the pair (z, y) is also non-normal, a contradiction with $y \bullet z = z$, or $y > x$. For $y > x$ we get, by (q5), $y \bullet x = x$ and $x \bullet (y \bullet z) = y \bullet (x \bullet z) = x$.

Now let $y \bullet z = y$ holds, i.e. (y, z) is non-normal and forms a bridge. In the subcase $x \bullet z = 1$, we have by transitivity $x < y$ and both the sides are equal to 1. For $x \bullet z = z$ only the possibility $x \bullet y = y$ can occur ($x \bullet y = 1$ or $x \bullet y = x$ lead to a contradiction) and so we get $x \bullet (y \bullet z) = y \bullet (x \bullet z) = y$. Finally let $x \bullet z = x$. Since (y, z), (z, x) are bridges, we have necessarily $x \sim y$. From this we derive $x \bullet (y \bullet z) = y \bullet (x \bullet z) = 1$. □

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