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DUALITY FOR GENERALIZED EVENTS

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ABSTRACT. We study the category of effect algebras (equivalently D-posets) carrying the initial sequential convergence with respect to a set of order determining probabilities. We define distinguished subcategories and describe their mutual relationships. In particular, we deal with duality, products, and coproducts. Such structures are suitable for modelling events having fuzzy and quantum nature.

0. Introduction

Undoubtedly ([3], [19], [4], [5], [6], [15], [29], [30], [16]), D-posets and effect algebras are suitable structures in the framework of which the generalized probability can be developed. We continue (cf. [7], [8], [9], [10], [11], [12], [13], [14]) in our effort to study generalized probability using sequential convergence and categorical methods.

Given a category, $\hom(X, Y)$ denotes the set of all morphisms from the object X into the object Y; it will be clear from the context which category $\hom(X, Y)$ refers to.

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1. Convergence

There is an extensive list of papers devoted to sequential convergence on various algebraic structures. Interesting results about sequential convergence on MV-algebras can be found in [17].

By a sequential convergence on a set X we understand a subset $\mathbb{L} \subseteq X^{\mathbb{N}} \times X$ satisfying the following conditions:

(S) If $\langle x_n \rangle$ is a constant sequence and $x_n = x$, $n \in \mathbb{N}$, then $(\langle x_n \rangle, x) \in \mathbb{L}$.
(F) If $(\langle x_n \rangle, x) \in \mathbb{L}$ and $\langle x'_n \rangle$ is a subsequence of $\langle x_n \rangle$, then $(\langle x'_n \rangle, x) \in \mathbb{L}$.

In addition, we mention the following conditions:

(H) If $(\langle x_n \rangle, x) \in \mathbb{L}$ and $(\langle x_n \rangle, y) \in \mathbb{L}$, then $x = y$ (the uniqueness of limits).
(U) If $\langle x_n \rangle$ is a sequence and $x \in X$ is a point such that for each subsequence $\langle x'_n \rangle$ of $\langle x_n \rangle$ there exists a subsequence $\langle x''_n \rangle$ of $\langle x'_n \rangle$ such that $(\langle x''_n \rangle, x) \in \mathbb{L}$, then $(\langle x_n \rangle, x) \in \mathbb{L}$ (Fréchet-Urysohn condition).
(B) If $(\langle a_n \rangle, x), (\langle b_n \rangle, x) \in \mathbb{L}$ and $a_n \le x_n \le b_n$, $n \in \mathbb{N}$, then $(\langle x_n \rangle, x) \in \mathbb{L}$.

Let $X = (|X|, \le, \ominus, 0, 1)$ be a D-poset; $|X|$ always denotes the underlying set and if no confusion can arise, then the same symbol X denotes the D-poset and its underlying set. Let $\mathbb{L} \subseteq X^{\mathbb{N}} \times X$ be a sequential convergence on X such that:

(DC) If $(\langle x_n \rangle, x) \in \mathbb{L}$, $(\langle y_n \rangle, y) \in \mathbb{L}$ and $y_n \le x_n$, $n \in \mathbb{N}$, then $y \le x$ and $(\langle x_n \ominus y_n \rangle, x \ominus y) \in \mathbb{L}$.

Then (cf. [25]) X carrying $\mathbb{L}$ is said to be a *convergence D-poset.*

Let $(X, \le)$ be a partially ordered set. By the *monotone sequential convergence* on X we understand $\mathbb{M}_X \subseteq X^{\mathbb{N}} \times X$ defined as follows. Let $\langle x_n \rangle$ be a sequence in X and let $x \in X$. We say that $\langle x_n \rangle$ *converges upwards* to x in X, in symbols $x_n \uparrow x$, whenever

(i) $x_n \le x_{n+1}$, $x_n \le x$, for all $n \in \mathbb{N}$.
(ii) If $x' \in X$ and $x_n \le x'$ for all $n \in \mathbb{N}$, then $x \le x'$.

Similarly, we say that $\langle x_n \rangle$ *converges downwards* to x in X, in symbols $x_n \downarrow x$, whenever

(iii) $x_{n+1} \le x_n$, $x \le x_n$, for all $n \in \mathbb{N}$;
(iv) If $x' \in X$ and $x' \le x_n$ for all $n \in \mathbb{N}$, then $x' \le x$.

Put $\mathbb{M}_X = \{(\langle x_n \rangle, x) \in X^{\mathbb{N}} \times X : x_n \uparrow x \text{ or } x_n \downarrow x\}$. It is easy to verify that $\mathbb{M}_X$ is a sequential convergence on X satisfying axioms (F), (S), (H).

Observe that if X is a D-poset, then X carrying $\mathbb{M}_X$ is a convergence D-poset.

1.1. DEFINITION. Let X be an effect algebra and let $\mathbb{L} \subseteq X^{\mathbb{N}} \times X$ be a sequential convergence on X such that

(EC) If $(\langle x_n \rangle, x) \in \mathbb{L}$, $(\langle y_n \rangle, y) \in \mathbb{L}$ and $y_n \oplus x_n$ exists for all $n \in \mathbb{N}$, then $y \oplus x$ exists and $(\langle x_n \oplus y_n \rangle, x \oplus y) \in \mathbb{L}$.

Then X carrying $\mathbb{L}$ is said to be a *convergence effect algebra.*

It is known (cf. [4; Theorem 1.3.4]) that the category of D-posets and D-poset morphisms and the category of effect algebras and effect algebra morphisms are equivalent. The result can be generalized as follows.

Denote $\mathcal{CD}$ the category whose objects are convergence D-posets and whose morphisms are sequentially continuous D-poset morphisms. Denote $\mathcal{CE}$ the category whose objects are convergence effect algebras and whose morphisms are sequentially continuous effect algebra morphisms. Since each D-poset can be considered as a convergence D-poset carrying the trivial convergence (only constant sequences converge) and each D-poset morphism is sequentially continuous with respect to the trivial convergence, D-posets (as trivial convergence D-posets) form a full subcategory of $\mathcal{CD}$. Similarly, effect algebras (as trivial convergence effect algebras) form a full subcategory of $\mathcal{CE}$.

Let $(X, \leq, \ominus, 0, 1, \mathbb{L})$ be a convergence D-poset. The dual partial operation $\oplus$ defined by the condition

(S) $a \oplus b$ exists and equals c if and only if $c \ominus b$ exists and equals a

yields an effect structure on X and $\mathbb{L}$ satisfies the axiom (EC). Analogously, let $(X, \boxplus, 0, 1, \mathbb{M})$ be a convergence effect algebra. The dual partial operation $\boxminus$ defined by the condition

(D) $a \boxminus b$ exists and equals c if and only if $b \boxplus c$ exists and equals a

yields a D-poset structure on X and $\mathbb{M}$ satisfies the axiom (DC).

The conditions (S) and (D) lead to two functors $F_1 \colon \mathcal{CD} \to \mathcal{CE}$ and $F_2 \colon \mathcal{CE} \to \mathcal{CD}$, the compositions of which are the identity functors on $\mathcal{CD}$ and $\mathcal{CE}$. Hence the two categories are isomorphic and isomorphic are also the subcategories consisting of objects carrying the trivial convergence.

In what follows, I denotes the closed unit interval of real numbers carrying the usual algebraic (also the D-poset and the effect algebra) structures and the sequential convergence.

1.2. OBSERVATION. *The categories $\mathcal{CD}$ and $\mathcal{CE}$ are isomorphic.*

Now let us turn to the initial convergence with respect to sets of morphisms into I (in fact, probability measures). In view of the isomorphism between D-posets and effect algebras, in what follows we assume that the partial operations $\ominus$ and $\oplus$ on a set X are dual, the partial order and the constants 0, 1 are compatible with $\ominus$ and $\oplus$.

Let X be a D-poset. A subset $H \subseteq \hom(X,I)$ is said to be *order determining* (cf. [5]), or an *ordering* (cf. [4]), whenever $h(b) \leq h(a)$ in I for all $h \in H$ implies $b \leq a$ in X. Denote $\mathrm{Ods}(X)$ the set of all order determining subsets of $\hom(X,I)$. Assume that $\hom(X,I) \in \mathrm{Ods}(X)$ (equivalently, $\mathrm{Ods}(X) \neq \emptyset$). For each $H \in \mathrm{Ods}(X)$ define the evaluation map $\mathrm{ev}_H : X \to I^H$ by $\mathrm{ev}_H(x) = \big(h(x);\ h \in H\big)$, $x \in X$. Let $|\mathrm{ev}_H(X)| = \big\{\mathrm{ev}_H(x) :\ x \in X\big\}$. The following is a folklore.

1.3. LEMMA. $\mathrm{ev}_H(X)$ *is a sub-D-poset of the power D-poset I^H and X and* $\mathrm{ev}_H(X)$ *are isomorphic.*

Further, define $\mathbb{L}_H \subseteq X^{\mathbb{N}} \times X$ as follows: $\big(\langle x_n\rangle, x\big) \in \mathbb{L}_H$ whenever for each $h \in H$ the sequence $\big\langle h(x_n)\big\rangle$ converges in I to $h(x)$.

1.4. LEMMA. $\mathbb{L}_H$ *is a sequential convergence on X satisfying the axioms* (DC), (EC), (H), (U), (B).

Proof. If we identify X and $\mathrm{ev}_H(X)$, then $\mathbb{L}_H$ becomes the pointwise convergence in I^H restricted to $\mathrm{ev}_H(X)$. A straightforward calculation shows that the axioms in question are satisfied. □

For $H, G \in \mathrm{Ods}(X)$, $H \subseteq G$ implies $\mathbb{L}_G \subseteq \mathbb{L}_H$ and $\mathbb{L}_{\hom(X,I)}$ is the finest of all convergences $\mathbb{L}_H, H \in \mathrm{Ods}(X)$. For $H, G \in \mathrm{Ods}(X)$ define $H \sim G$ whenever $\mathbb{L}_H = \mathbb{L}_G$. It is an equivalence relation and each equivalence class $[H]$ contains the maximal element $H^* = \bigcup_{G \in [H]} G$. Denote $\hom_H(X,I) = \big\{h \in \hom(X,I) :$ h is sequentially continuous with respect to $\mathbb{L}_H\big\}$. Then $H^* = \hom_H(X,I)$. Clearly, $\hom_{\hom(X,I)}(X,I) = \hom(X,I)$.

1.5. DEFINITION. Let X be a D-poset and let $H \in \mathrm{Ods}(X)$. Then $\mathbb{L}_H$ is said to be an *I-convergence*. If $H = \hom(X,I)$, then $\mathbb{L}_H$ is said to be the *fine convergence.*

Sub-D-posets of the power D-posets I^H carrying the pointwise convergence, called $D(I)$-posets, have been studied in [25]. In Section 3 we shall show that such D-posets and D-posets carrying I-convergences are naturally equivalent.

Since D-posets and effect algebras are equivalent we propose a simple compromising terminology.

The idea is similar as in topology where, in fact, we have four isomorphic structures defined via open sets, closed sets, a closure operator, neighborhoods. To define a topological space, we can start with any of the four (e.g. open sets) and then to define any of the remaining (e.g. neighborhoods of points), so that going back (defining an open set as a set containing a neighborhood of each of its points) we get the original one (open sets). Further, the morphisms defined via the preimages of the open or closed sets, via preserving the closure (if a point is in the closure of a set, then the image of the point is in the closure of the image of the set), or via the continuity at each point (using the neighbourhoods), yield the identical notion of a continuous map. And, instead of four different isomorphic categories (of open or closed sets spaces, closure spaces, neighborhood spaces), we work with topological spaces and continuous maps.

1.6. Definition. Let $(X, \leq, \ominus, 0, 1)$ be a D-poset, let $H \in \mathrm{Ods}(X)$, let $\mathbb{L}_H$ be the corresponding I-convergence, and let $\oplus$ be the dual effect algebra partial operation on X. Then $(X, \leq, \ominus, \oplus, 0, 1, \mathbb{L}_H)$, or simply X, is said to be a *prob*. A sequentially continuous D-poset morphism from X into I is said to be a *probability* on X.

Denote by PB the category whose objects are probs and whose morphisms are the sequentially continuous maps preserving the prob structure. Clearly, a map preserves the prob structure if and only if it preserves the induced D-poset structure if and only if it preserves the induced effect algebra structure. This way I and each sub-D-poset of the power D-poset I^X can be considered as a prob.

1.7. Proposition. *Let X and Y be D-posets admitting order determining sets of D-morphisms into I. Let h be a D-poset morphism from X into Y. Then h is sequentially continuous with respect to the fine convergences on X and Y (hence it is a prob-morphism).*

Proof. The easy proof is omitted. □

In view of 1.7. Proposition, the subcategory of D-posets (effect algebras) admitting an order determining set of D-poset morphisms into I becomes a full subcategory of the category PB. Indeed, each D-poset X admitting an ordering can be considered as a prob carrying the fine convergence. Each state on X, i.e. a D-poset morphism from X into I, is sequentially continuous and hence a probability.

The categorical product in PB is the usual product of D-posets (and effect algebras) carrying the pointwise convergence. A subprob of a prob is the usual sub-D-poset (and subeffect algebra) carrying the restricted convergence. If X is a prob and Y is a nonempty subset of X, then the intersection of all subprobs of X containing Y is a subprob.

2. Epireflection

It is known (cf. [12], [13]) that passing from a field of sets $\mathbb{A}$ to the generated σ-algebra $\sigma(\mathbb{A})$, or from a bold algebra of fuzzy sets to the generated tribe, has a distinguished categorical aspect and it is intimately related to the "absolute" sequential closedness. In this section we generalize such extension process to probs. The idea of the construction goes back to [23], [24].

2.1. Definition. Let E be a prob. If E is a subprob of a prob $\overline{E}$ and each morphism from E into I can be extended to a morphism from $\overline{E}$ into I, then E is said to be *I-embedded* in $\overline{E}$. Moreover, if each extension is uniquely determined, then E is said to be *hom-dense* in $\overline{E}$. If E is sequentially closed in each prob in which it is I-embedded, then E is said to be *absolutely sequentially closed*, or simply *absolute*.

Denote APB the full subcategory of PB consisting of all absolute probs.

2.2. Lemma. *Let E be a prob. The following are equivalent:*

(i) *E is absolute.*

(ii) *In E the following implication holds true: if a sequence $\langle x_n \rangle$ of elements of E does not converge in E, then there exists $\varphi \in \mathrm{hom}(E, I)$ such that the sequence $\langle \varphi(x_n) \rangle$ does not converge in I.*

Proof.

(i) implies (ii). Since E is a prob, the evaluation map ev is an isomorphism from E onto a subprob of $I^{\mathrm{hom}(E,I)}$. If we identify E with its image $\mathrm{ev}(E)$, then E becomes an I-embedded subprob of $I^{\mathrm{hom}(E,I)}$. Assume (i) and let $\langle x_n \rangle$ be a sequence in E which does not converge in E. Contrariwise, suppose that for each $\varphi \in \mathrm{hom}(E, I)$ the sequence $\langle \varphi(x_n) \rangle$ converges in I. The sequence $\langle \mathrm{ev}(x_n) \rangle$ converges in $I^{\mathrm{hom}(E,I)}$ and it follows from (i) that the limit belongs to $\mathrm{ev}(E)$. This is a contradiction.

(ii) implies (i). Assume (ii). Contrariwise, suppose that (i) does not hold. Then E can be I-embedded in a prob $\overline{E}$ such that there exists a sequence $\langle x_n \rangle$ of elements of E converging in $\overline{E}$ to a point $x \in \overline{E} \setminus E$. According to (ii) there exists $\varphi \in \mathrm{hom}(E, I)$ such that the sequence $\langle \varphi(x_n) \rangle$ does not converge in I and hence φ does not have a continuous extension over $\overline{E}$. This is a contradiction. □

2.3. Proposition. *The category APB is closed with respect to the formation of products and closed subprobs.*

Proof. Both assertions easily follow from 2.2. Lemma. We leave out the details. □

Recall that a subcategory $\mathcal{B}$ of $\mathcal{A}$ is *epireflective* if for each object X of $\mathcal{A}$ there exists an object X_r of $\mathcal{B}$ and an epimorphism r of X into X_r such that for each morphism f of X into an object Y of $\mathcal{B}$ there exists a unique morphism f_r of X_r into Y such that $f_r \circ r = f$. Sending X to X_r yields a functor from $\mathcal{A}$ to $\mathcal{B}$, called *epireflector*.

2.4. THEOREM. *APB is an epireflective subcategory of PB.*

P r o o f. Let E be a prob. First, we shall construct a prob $\overline{E}$ such that:

(e1) There is no proper subprob F of $\overline{E}$ such that $|E| \subseteq |F|$ and $|F|$ is sequentially closed in $\overline{E}$;
(e2) E is I-embedded in $\overline{E}$;
(e3) $\overline{E}$ is absolute.

Second, we shall prove that the embedding of E into $\overline{E}$ is the desired epireflection:

1. For each morphism φ from E into an absolute prob F there exists a unique morphism $\overline{\varphi}$ from $\overline{E}$ into F such that the restriction of $\overline{\varphi}$ to E is equal to φ, in symbols $\overline{\varphi} \restriction E = \varphi$.
2. The embedding of E into $\overline{E}$ is an epimorphism. Hence passing from E to $\overline{E}$ yields the desired epireflector.

1. Consider the evaluation mapping ev from E into $I^{\hom(E,I)}$ defined by $\mathrm{ev}(x) = \big(\varphi(x);\ \varphi \in \hom(E,I)\big)$, $x \in E$. Since it is an isomorphism into, to avoid complicated notation, we identify E with its image under ev and denote $\overline{E}$ the intersection of all sequentially closed subprobs F of $I^{\hom(E,I)}$ such that $|E| \subseteq |F|$. The condition (e1) follows directly from the construction of $\overline{E}$. Further, for each $\varphi \in \hom(E,I)$, the restriction to $\overline{E}$ of the corresponding projection from $I^{\hom(E,I)}$ is a morphism from $\overline{E}$ into I and its restriction to E is equal to φ. Hence (e2) holds true. Now, let $\overline{E}$ be I-embedded in a prob $\overline{\overline{E}}$ and let $\langle x_n \rangle$ be a sequence in $\overline{E}$ converging in $\overline{\overline{E}}$ to some element $x \in |\overline{\overline{E}}|$. Since for each morphism $\overline{\overline{\varphi}}$ from $\overline{\overline{E}}$ into I its restriction $\overline{\overline{\varphi}} \restriction \overline{E}$ belongs to $\hom(\overline{E},I)$ and $\overline{\overline{\varphi}} \restriction E$ belongs to $\hom(E,I)$, the sequence $\langle x_n \rangle$ converges in $I^{\hom(E,I)}$. Thus $x \in |\overline{E}|$ and $\overline{E}$ is sequentially closed in $\overline{\overline{E}}$. Hence (e3) is satisfied, too.

2. Let φ be a mapping from E into an absolute prob F. Let us identify F with its isomorphic image under the evaluation from F into $I^{\hom(F,I)}$. Then φ can be considered as a morphism from the subprob E of $I^{\hom(E,I)}$ into a sequentially closed subprob F of $I^{\hom(F,I)}$. It follows from [25; Corollary 2.17] that φ can be extended to a morphism from $\overline{E}$ into F and the extension is uniquely determined. Further (cf. [25; Lemma 2.9]), the embedding of E into $\overline{E}$ is an epimorphism and hence an epireflection (defined uniquely up to a commuting isomorphism) of E into APB. □

2.5. COROLLARY. *A prob is absolute if and only if it is isomorphic to a sequentially closed subprob of a power prob I^T.*

P r o o f. The necessity follows from the proof of 2.4. Theorem. The sufficiency follows from 2.3. Proposition. □

Denote σ the epireflector from *PB* onto *APB* assigning to a prob E the prob $\overline{E} = \sigma(E)$ and to each morphism ψ from a prob E into a prob F the unique morphism $\sigma(\psi)$ from $\sigma(E)$ into $\sigma(F)$ defined as follows: let φ be the composition of ψ and the embedding map from F into $\sigma(F)$, let $\overline{\varphi}$ be the unique extension of φ over $\sigma(E) = \overline{E}$, and put $\sigma(\psi) = \overline{\varphi}$.

Recall (cf. [4]) that a σ*-effect algebra* is an effect algebra in which each nondecreasing sequence $\langle a_n \rangle$ has the supremum $\bigvee_{n=1}^{\infty} a_n$. Accordingly, we say that $(X, \leq, \ominus, \oplus, 0, 1, \mathbb{L})$ is a σ*-prob* whenever each nondecreasing sequence $\langle a_n \rangle$ has the supremum $\bigvee_{n=1}^{\infty} a_n$.

2.6. Lemma. *Let X be an absolute prob. Then X is a σ-prob.*

P r o o f. Consider the evaluation map ev from X into the power prob $I^{\hom(X,I)}$ and identify X with the sequentially closed subprob $\mathrm{ev}(X)$ of $I^{\hom(X,I)}$. Let $\langle a_n \rangle$ be a nondecreasing sequence in X. Then, for each $h \in \hom(X, I)$, $\langle h(a_n) \rangle$ is a nondecreasing and hence convergent sequence in I. Thus $\langle a_n \rangle$ converges in $I^{\hom(X,I)}$ to a point $a \in X$. Clearly $a = \bigvee_{n=1}^{\infty} a_n$. □

Observe that for each prob E, its epireflection $\sigma(E)$ is a σ-prob and it is the maximal prob in which E is I-embedded and hom-dense. Further, E and $\sigma(E)$ have "the same" probabilities.

3. Duality

Recall (cf. [25]) that the objects of the category *ID* are (reduced) D-posets of fuzzy sets carrying the pointwise sequential convergence and that the morphisms of *ID* are sequentially continuous D-poset morphisms.

An *ID*-object $\mathcal{X} \subseteq I^X$ is *sober* if for each $h \in \hom(\mathcal{X}, I)$ there exists a unique $x \in X$ such that h is the evaluation at x (sending $u \in \mathcal{X}$ to $h(u) = u(x) \in I$). Denote *SID* the subcategory of *ID* consisting of sober objects.

An *ID-measurable space* is a pair $(X, \mathcal{X})$, where X is a set and $\mathcal{X} \subseteq I^X$ is an *ID*-object. If $\mathcal{X}$ is sober, then $(X, \mathcal{X})$ is said to be sober, too.

If $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are *ID*-measurable spaces, then a map f of Y into X is said to be *measurable*, or more exactly $(\mathcal{X}, \mathcal{Y})$-measurable, whenever $u \circ f \in \mathcal{Y}$ for each $u \in \mathcal{X}$.

Then $f^{\triangleleft} \colon \mathcal{X} \to \mathcal{Y}$, defined by $f^{\triangleleft}(u) = u \circ f$ is a sequentially continuous D-poset morphism. Denote *MID* the category of *ID*-measurable spaces and *ID*-measurable maps. Denote *SMID* its subcategory consisting of sober spaces.

If $(X, \mathcal{X})$ is a sober space, then for each sequentially continuous D-poset morphism h of $\mathcal{X}$ into $\mathcal{Y}$ there exists an $(\mathcal{X}, \mathcal{Y})$-measurable map f such that $h = f^{\triangleleft}$. This is the essence of the duality between *ID* and *SMID* ([25; Theorem 1.12]).

The duality can be extended to probs. Indeed, the duality means that there exist a natural equivalence functor $F\colon \mathit{PB} \to \mathit{SMID}^{\mathrm{op}}$, where the target category $\mathit{SMID}^{\mathrm{op}}$ is the opposite category associated to *SMID* (the same objects but the direction of the arrows is reversed). It is known that *SID* and $\mathit{SMID}^{\mathrm{op}}$ are isomorphic ([25; Theorem 1.11]). Hence to show that *PB* and *SMID* are dual, it suffices to construct a natural equivalence functor $G\colon \mathit{PB} \to \mathit{SID}$.

Let X be a D-poset admitting an order determining set $H \in \hom(X, I)$, let $\mathbb{L}_H$ be the corresponding I-convergence, and let $H^* = \hom_H(X, I)$. Consider the resulting prob $(X, \leq, \ominus, \oplus, 0, 1, \mathbb{L}_H)$, condensed to X. It follows from Lemma 3 that the evaluation ev_{H^*} is an isomorphism sending X to $\mathcal{H} = \mathrm{ev}_{H^*}(X) \subseteq I^{H^*}$. Since $\mathcal{H}$ is sober, it suffices to verify that such evaluations (via $H^* = \hom_H(X, I)$) yield a functor $G\colon \mathit{PB} \to \mathit{SID}$, the functor is full and faithful, and each object of *SID* is isomorphic to the evaluation of some prob (cf. [20; Theorem IV.4.1]). But this is obvious.

COROLLARY 3.1. *The categories PB and SMID are dual.*

Further, reorganizing each *ID*-object $\mathcal{X} \subseteq I^X$ into a prob, we get the category *MPB* of *PB*-measurable spaces and its subcategory *SMPB* of sober objects.

COROLLARY 3.2. *The categories PB and SMPB are dual.*

An *ID*-measurable space $(X, \mathcal{X})$ is said to be *closed* if $\mathcal{X}$ is sequentially closed in I^X. The subcategory *CMID* of *MID* consisting of closed *ID*-measurable spaces is monocoreflective in *MID* (cf. [25]). Accordingly, we say that a prob $\mathcal{X}$ and the *PB*-measurable space $(X, \mathcal{X})$ are *closed* whenever $\mathcal{X}$ as an *ID*-object is closed. It follows that the subcategory *CMPB* of closed *PB*-measurable spaces is monocoreflective in the category *MPB*. For a *PB*-measurable space $(X, \mathcal{X})$ denote $\big(X, \sigma(\mathcal{X})\big)$ the monocoreflection. If $(X, \mathcal{X})$ is sober, then $\big(X, \sigma(\mathcal{X})\big)$ is sober, too.

COROLLARY 3.3. *The duality between PB and SMPB commutes with the epireflector sending a prob X to the absolute prob $\sigma(X)$ and the monocoreflector sending the sober PB-measurable space $(H, \mathcal{H})$, where $H = \hom(X, I)$ and $\mathcal{H}$ is the image of X under the evaluation map (via H), to the closed sober PB-measurable space $\big(H, \sigma(\mathcal{H})\big)$. If X is absolute, then $\mathcal{H}$ is closed.*

Observe that a 2-chain $\{0, a, 1\}$ admits an order determining family, but the diamond $\{0, a, b, 1\}$ as the coproduct in the category of D-posets of two

2-chains does not admit any order determining family. M. Papčo, at the SCAM-conference in Bratislava, April 2003 (cf. [25]), presented the construction of the product of ID-measurable spaces leading to the construction of the coproduct in the category ID. The latter yields the coproduct in the category PB.

Let $\{X_s : s \in S\}$ be a family of probs. Denote $H_s = \hom(X_s, I)$ and, as a rule, identify X_s and its image $\mathrm{ev}_s(X_s) \subseteq I^{H_s}$ under the evaluation map $\mathrm{ev}_s \colon X_s \to I^{H_s}$. Consider the product $H = \prod_{s\in S} H_s$ and the prob I^H. For $t \in S$, define a natural embedding κ_t of $\mathrm{ev}_t(X_t) = X_t$ into I^H, sending $u \in X_t \subseteq I^{H_t}$ to $u_t \in I^H$ defined as follows: for $h = (h_s;\ s \in S) \in H$ put $u_t(h) = u(h_t)$, i.e. u_t depends only on the tth coordinate. It is a sequentially continuous D-poset morphism of X_t into I^H. Let $X \subseteq I^H$ be the minimal subprob of I^H which contains $\kappa_s(X_s)$, $s \in S$. If all involved probs are considered as ID objects and all PB morphisms are considered as ID morphisms, then (as shown by M. Papčo) X, together with the coprojections $\{\kappa_s \colon X_s \to X : s \in S\}$, is the coproduct of the family $\{X_s : s \in S\}$ in ID.

THEOREM 3.4. *Let $\{X_s : s \in S\}$ be a family of probs. Then X, together with the coprojections $\{\kappa_s \colon X_s \to X : s \in S\}$, is the coproduct of the family $\{X_s : s \in S\}$ in PB.*

Proof. The assertion follows directly from the definition of a coprodut and the fact that each prob and its image under the evaluation (via the set of all morphisms) are isomorphic. □

4. Remarks

The categories PB and MPB provide a tool for studying generalized probability: events, measures, random variables, observables. In [1], [2] (see also [15]) the operational random variable is defined as a suitable map of the set of all probability measures on one measurable space into the set of all probability measures on another measurable space. If the map sends a point measure (an elementary event) to a nontrivial probability measure, then the random variable has a quantum nature. We claim that ID-measurable maps, and hence PB-measurable maps, generalize the operational random variables. Further, the duality between PB and $SMPB$ covers the duality between operational random variables and the observables (going the opposite direction) as it is described in [1], [2]. We also claim that the sequential convergence and the categorical approach shed more light on the duality between the operational random variables and the observables.

As already stated in [25], in order to develop further probability notions, it might be useful to introduce and study on D-posets and hence on probs a multiplication operation (cf. [21], [22], [18], [27], [28]).

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