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COMPANION d-ALGEBRAS

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ABSTRACT. In this paper we develop a theory of companion d-algebras in sufficient detail to demonstrate considerable parallelism with the theory of BCK-algebras as well as obtaining a collection of results of a novel type. Included among the latter are results on certain natural posets associated with companion d-algebras as well as constructions on $\mathrm{Bin}(X)$, the collection of binary operations on the set X, which permit construction of new companion d-algebras from companion d-algebras X also in natural ways.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([Is], [IsTa]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [HL1], [HL2] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. BCK-algebras also have some connections with other areas: A. Dvurečenskij and M. G. Graziano [DvGr], C. S. Hoo [Hoo], J. M. Font, A. J. Rodríguez and A. Torrens [FRT], D. Mundici [Mun] proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras, and J. Meng [Me] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. J. Neggers and H. S. Kim introduced the notion of d-algebras which is another useful

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generalization of BCK-algebras, and then investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs ([NK3]). After that some further aspects were studied ([LK], [NJK], [JNK]). As a generalization of BCK-algebras d-algebras are obtained by deleting identities. Given one of these deleted identities a related identities are constructed by replacing one of the terms involving the original operation by an identical term involving a second (companion) operation, thus producing the notion of companion d-algebra which (precisely) generalizes the notion of BCK-algebra and is such that not every d-algebra is one of these. In this paper we develop a theory of companion d-algebras in sufficient detail to demonstrate considerable parallelism with the theory of BCK-algebras as well as obtaining a collection of results of a novel type. Included among the latter are results on certain natural posets associated with companion d-algebras as well as constructions on $\mathrm{Bin}(X)$, the collection of binary operations on the set X, which permit construction of new companion d-algebras from companion d-algebras X also in natural ways.

2. Companion d-algebras

A *d-algebra* ([NK3]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all x, y in X.

A BCK-algebra is a d-algebra $(X; *, 0)$ satisfying the following additional axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,
(V) $(x * (x * y)) * y = 0$

for all x, y, z in X.

A BCK-algebra $(X; *, 0)$ is said to have a *condition* (S) ([MeJu]) if

$A(a, b) := \{x \in X : \ x * a \leq b\}$ has a greatest element for any $a, b \in X$.

Definition 2.1. Let $(X; *, 0)$ be a d-algebra. Define a binary operation $\cdot$ on X by

(VI) $((x \odot y) * x) * y = 0$

for any $x, y \in X$, which is called a *subcompanion operation* of X. A subcompanion operation $\odot$ is said to be a *companion operation* of X if

(VII) if $(z * x) * y = 0$, then $z * (x \odot y) = 0$ for any $x, y, z \in X$.

Example 2.2. Let $X := \{0, 1, 2, 3\}$ be a set with the following tables:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	2	2	2	0

$\odot$	0	1	2	3
0	0	1	3	3
1	1	1	3	3
2	2	2	3	3
3	3	3	3	3

Then $(X; *, 0)$ is a d-algebra, which is not a BCK/BCI-algebra, and the binary operation $\odot$ defined above is a companion operation on X.

A d-algebra X is said to be a *companion d-algebra* if it has a companion operation.

PROPOSITION 2.3. *Let $(X; *, 0)$ be a d-algebra. If X has a companion operation $\odot$, then it is unique.*

Proof. Assume the binary operations $\odot_1$ and $\odot_2$ are companion operations on X. Then $((x \odot_i y) * x) * y = 0$ for any $x, y \in X$ $(i = 1, 2)$. By (VII) we obtain

$$(x \odot_1 y) * (x \odot_2 y) = 0. \tag{1}$$

Interchange $\odot_1$ with $\odot_2$. Then

$$(x \odot_2 y) * (x \odot_1 y) = 0. \tag{2}$$

By (III) we obtain $\odot_1 = \odot_2$. Hence the operation $\odot$ is unique. $\square$

Example 2.4. Every BCK-algebra with condition (S) is a companion d-algebra.

Example 2.2 is a companion d-algebra which is not a BCK/BCI-algebra. This means that *a companion d-algebra is a generalization of a BCK/BCI-algebra with condition* (S).

PROPOSITION 2.5. *Let $(X; *, \odot, 0)$ be a companion d-algebra. Then for any $x, y, z \in X$, we have*

(i) *if $x * z = 0$, then $x * (z \odot y) = 0$,*
(ii) $x * (x \odot y) = 0$,
(iii) $x \odot 0 = x$.

Proof.

(i) Since $(x * z) * y = 0 * y = 0$, by (VII), $x * (z \odot y) = 0$.

(ii) Put $z := x$ in (i).

(iii) We claim that if $x * 0 = 0$, then $x = 0$. In fact, since $0 * x = 0$, by (III) we have $x = 0$. Since X is a companion d-algebra, $((x \odot 0) * x) * 0 = 0$ and so $(x \odot 0) * x = 0$. If we put $y := 0$ in (ii), then $x * (x \odot 0) = 0$. By (III) we have $x \odot 0 = x$. □

THEOREM 2.6. *Let $(X; *, \odot, 0)$ be a companion d-algebra. Let $\diamond$ be a binary operation on X such that*

$$(x * y) * z = x * (y \diamond z). \tag{3}$$

Then X is a companion d-algebra and $\diamond$ is exactly the operation $\odot$.

Proof. By applying (3) and (I), we have

$$\begin{aligned} ((x \diamond y) * x) * y &= (x \diamond y) * (x \diamond y) && \text{[by (3)]} \\ &= 0, && \text{[by (I)]} \end{aligned}$$

proving the condition (VI). Let $z \in X$ with $(z * x) * y = 0$. Then by (3), $z * (x \diamond y) = (z * x) * y = 0$, proving the condition (VII). Hence $\diamond$ is a companion operation, which is unique by Proposition 2.3. □

Given a d-algebra $(X; *, 0)$, we define a partial binary relation $\leq$ by $x \leq y \iff x * y = 0$, where $x, y \in X$.

PROPOSITION 2.7. *If $(X; *, \odot, 0)$ is a bounded companion d-algebra, i.e., there is an element $1 \in X$ such that $x * 1 = 0$ for any $x \in X$, then $x \odot 1 = 1$ for any $x \in X$.*

Proof. Since $u * x \leq 1$ for any $u \in X$, $(u * x) * 1 = 0$. By applying (VII) we have $u \leq x \odot 1$, for any $u \in X$, which implies $1 = x \odot 1$. □

A d-algebra $(X; *, 0)$ is said to be *positive implicative* if $(x * y) * z = (x * z) * (y * z)$ for any $x, y, z \in X$.

PROPOSITION 2.8. *Let $(X; *, \odot, 0)$ be a companion d-algebra.*

(i) *$0 \leq x \odot y$, $x \leq x \odot y$, for any $x, y \in X$,*

(ii) *if X is positive implicative, then $y \leq x \odot y$ for any $x, y \in X$.*

Proof.

(i) Since $(0 * x) * y = 0$, $0 \leq x \odot y$. From $(x * x) * y = 0 * y = 0$, we obtain $x \leq x \odot y$.

(ii) Since X is positive implicative, $(y*x)*y = (y*y)*(x*y) = 0*(x*y) = 0$ and hence $y \leq x \odot y$. □

THEOREM 2.9. *Let $(X; *, \odot, 0)$ be a companion d-algebra. Assume that $x * 0 = x$ for any $x \in X$.*

(i) *X is positive implicative,*
(ii) *if $x \leq y$, then $x \odot y = y$,*
(iii) *$x \odot x = x$ for any $x, y \in X$.*

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

Proof.
(i) $\Longrightarrow$ (ii). If $x \leq y$, then

$$\begin{aligned} 0 &= ((x \odot y) * x) * y \\ &= [(x \odot y) * y] * (x * y) && [X\text{: positive implicative}] \\ &= [(x \odot y) * y] * 0 && [x * y = 0] \\ &= (x \odot y) * y, && [x * 0 = x] \end{aligned}$$

which means that $x \odot y \leq y$. By applying Proposition 2.8-(ii), we have $x \odot y = y$.
(ii) $\Longrightarrow$ (iii). Let $y := x$ in (ii). □

DEFINITION 2.10. ([NJK]) Let $(X; *, 0)$ be a d-algebra and $\emptyset \neq I \subseteq X$. I is called a *d-subalgebra* of X if $x * y \in I$ whenever $x \in I$ and $y \in I$. I is called a *BCK-ideal* of X if it satisfies:

(D_0) $0 \in I$,
(D_1) $x * y \in I$ and $y \in I$ imply $x \in I$.

I is called a *d-ideal* of X if it satisfies (D_1) and

(D_2) $x \in I$ and $y \in X$ imply $x * y \in I$, i.e., $I * X \subseteq I$.

DEFINITION 2.11. Let $(X; *, \odot, 0)$ be a companion d-algebra and $\emptyset \neq I \subseteq X$. I is called a *$\odot$-subalgebra* if $x \odot y \in I$ for any $x, y \in I$.

In Example 2.2, the set $I_1 := \{0, 1\}$ is a $\odot$-subalgebra of X, while $I_2 := \{0, 1, 2\}$ is not a $\odot$-subalgebra of X.

THEOREM 2.12. *Let $(X; *, \odot, 0)$ be a companion d-algebra. If I is a BCK-ideal of X, then I is a $\odot$-subalgebra of X.*

Proof. If X is a companion d-algebra, then $((x \odot y) * x) * y = 0 \in I$ for any $x, y \in I$. Since I is a BCK-ideal of X and $y \in I$, $(x \odot y) * x \in I$. Moreover, since $x \in I$, we obtain $x \odot y \in I$, proving the theorem. □

The converse of Theorem 2.12 need not be true in general. For example, $J := \{0, 1, 3\}$ is a $\odot$-subalgebra of X, but not a BCK-ideal of X, since $2 * 3 = 0 \in J$, $3 \in J$, but $2 \notin J$ in Example 2.2.

PROPOSITION 2.13. *Let $(X; *, \odot, 0)$ be a companion d-algebra and let I be a BCK-ideal of X. If $x \odot y \in I$, then $x \in I$ where $x, y \in X$.*

Proof. By Proposition 2.5-(ii), $x * (x \odot y) = 0 \in I$. Since $x \cdot y \in I$ and I is a BCK-ideal of X, we have $x \in I$. □

COROLLARY 2.14. *Let $(X; *, \odot, 0)$ be a companion d-algebra and let I be a BCK-ideal of X. If $x \odot y = y \odot x \in I$, then $x, y \in I$ where $x, y \in X$.*

COROLLARY 2.15. *Let $(X; *, \odot, 0)$ be a companion d-algebra and let I be a BCK-ideal of X. Then $x \in I \iff x \odot x \in I$.*

Proof. It follows immediately from Theorem 2.12 and Proposition 2.13. □

3. *dsu* condition

In a d-algebra X, the identity $(x * y) * x = 0$ does not hold in general.

DEFINITION 3.1. ([NJK]) A d-algebra X is called a *d^*-algebra* if it satisfies the identity $(x * y) * x = 0$ for all $x, y \in X$.

Clearly, a BCK-algebra is a d^*-algebra, but the converse need not be true.

Example 3.2. Let $X := \{0, 1, 2, \dots\}$ and let the binary operation $*$ be defined as follows:

$$x * y := \begin{cases} 0 & \text{if } x \leq y, \\ 1 & \text{otherwise.} \end{cases}$$

Then $(X, *, 0)$ is a d-algebra which is not a BCK-algebra (see [NK3, Example 2.8]). We can easily see that $(X, *, 0)$ is a d^*-algebra.

THEOREM 3.3. ([NJK]) *In a d^*-algebra, every BCK-ideal is a d-ideal.*

The following corollary is obvious.

COROLLARY 3.4. ([NJK]) *In a d^*-algebra, every BCK-ideal is a d-subalgebra.*

For companion d-algebras the condition $(x * y) * (x \cdot y) = 0$ is also one which is not unusual, since in 'usual' circumstances we expect the difference to be smaller than the usual (*dsu* condition).

DEFINITION 3.5. Let $(X; *, \odot, 0)$ be a companion d-algebra. X is said to have a *dsu condition* if

$$(x * y) * (x \odot y) = 0 \tag{4}$$

for any $x, y \in X$.

PROPOSITION 3.6. *Let $(X;*,\odot,0)$ be a companion d-algebra having the dsu condition. If I is a BCK-ideal of X, then it is a d-subalgebra of X.*

Proof. By Theorem 2.12, $x\odot y\in I$ for any $x,y\in I$. Since X has the *dsu* condition, $(x*y)*(x\odot y)=0\in I$ and I is a BCK-ideal of X, we obtain $x*y\in I$. $\square$

Let $(X;*,0)$ be a d-algebra and $x\in X$. Define $x*X:=\{x*a:\ a\in X\}$. X is said to be *edge* ([NK3]) if for any x in X, $x*X=\{x,0\}$.

LEMMA 3.7. ([NJK]) *If $(X;*,0)$ is a edge d-algebra, then $(x*(x*y))*y=0$ and $x*0=x$ for any $x,y\in X$.*

THEOREM 3.8. *Let $(X;*,\odot,0)$ be a companion edge d^*-algebra. If*

$$(z*(x\odot y))*((z*x)*y)=0, \tag{5}$$

then X has a dsu condition.

Proof. Let $z:=x*y$ in (5). Then

$$\begin{aligned}0&=((x*y)*(x\odot y))*(((x*y)*x)*y)\\&=((x*y)*(x\odot y))*(0*y) &&[X\text{: } d^*\text{-algebra}]\\&=((x*y)*(x\odot y))*0\\&=((x*y)*(x\odot y)), &&[X\text{: edge}]\end{aligned}$$

proving the theorem. $\square$

PROPOSITION 3.9. *Let $(X;*,\odot,0)$ be a companion edge d-algebra. If*

$$(z*(x\odot y))*((x*z)*y)=0, \tag{6}$$

then X has a dsu condition.

Proof. Let $z:=x*y$ in (6). Then by Lemma 3.7

$$\begin{aligned}0&=((x*y)*(x\odot y))*((x*(x*y))*y)\\&=((x*y)*(x\odot y))*0\\&=((x*y)*(x\odot y)),\end{aligned}$$

proving the proposition. $\square$

4. Completeness

A companion d-algebra $(X;*,\odot,0)$ is said to be *complete* if for any $x\in X$, there exists an x^* in X such that $x\odot x^*=x$. Note that such an x^* need not be unique. For such an example, we find $2\odot 0=2\odot 1=2$, $3\odot 1=3\odot 2=3$ in Example 2.2.

Proposition 4.1. *Let $(X;*,\odot,0)$ be a companion d-algebra. If we define a partial binary relation $\ll$ by*

$$x \ll y \iff (x\odot z)*(y\odot z)=0 \qquad \text{for all} \quad z\in X, \tag{7}$$

then $\ll$ is reflexive and anti-symmetric.

Proof. Clearly, $\ll$ is reflexive. If $x \ll y$, $y \ll x$, then $(x\odot z)*(y\odot z) = 0 = (y\odot z)*(x\odot z)$ for any $z\in X$. By applying (III) we have

$$x\odot z = y\odot z \tag{8}$$

for any $z\in X$. Since X is complete, there exist $x^*, y^* \in X$ such that $x = x\odot x^*$, $y = y\odot y^*$. If we let $z := x^*$ and $z := y^*$ in (8), respectively, then $x = x\odot x^* = y\odot x^*$, $y = y\odot y^* = x\odot y^*$. Thus by Proposition 2.5-(ii), $x*y = x*(x\odot y^*) = 0$ and $y*x = y*(y\odot x^*) = 0$ and hence $x=y$, proving the proposition. □

For any BCK/BCI-algebras the following *transitivity condition* holds:

$$\text{if } x*y=0 \text{ and } y*z=0, \text{ then } x*z=0 \tag{9}$$

(see [MeJu, Theorem 1.2-(b)]). This condition does not hold in d-algebra in general.

Example 4.2. Let $X := \{0,a,b,c\}$ be a set with the following tables:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Then $(X;*,0)$ is a d-algebra, which is not a BCK/BCI-algebra (see [NJK]), and $a*b = 0 = b*c$, but $a*c = a \neq 0$.

Thus, if a d-algebra satisfies the transitivity condition, then the natural order $\le$ given by $x\le y$ if and only if $x*y=0$ is a partial order.

Proposition 4.3. *Let $(X;*,\odot,0)$ be a complete companion d-algebra. If X satisfies the transitivity condition, then $(X;\ll)$ is a poset.*

Proposition 4.4. *Let $(X;*,\odot,0)$ be a complete companion d-algebra. If $x\ll y$, then $x\le y$ in X.*

Proof. If $x\ll y$, then $(x\odot \alpha)*(y\odot\alpha)=0$ for any $\alpha\in X$. This implies $(x\odot 0)*(y\odot 0)=0$ and hence $x*y=0$ by Proposition 2.5-(iii). □

The converse of Proposition 4.4 need not be true in general.

Example 4.5. Let $X := \{0, a, b, c, d, 1\}$ be a set with the following table:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	a	0	0	a	0	0
b	b	a	0	b	a	0
c	c	c	b	0	0	0
d	d	c	b	a	0	0
1	1	d	b	a	a	0

$\odot$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	b	b	d	1	1
b	b	b	1	b	b	1
c	c	1	1	c	1	1
d	d	1	1	d	1	1
1	1	1	1	1	1	1

Then $(X; *, \odot, 0)$ is a companion d-algebra, which is not a BCK/BCI-algebra, since $(c * b) * d = a \neq 0 = (c * b) * b$. We know that $a \leq b$, but $a \odot c = d$ and $b \odot c = b$, and d and b are incomparable. Hence $a \ll b$ does not hold.

The converse of Proposition 4.4 holds for BCK/BCI-algebras (see [Hu, BCI-algebras, p. 98, Theorem 8]).

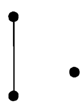
5. Pogroupoid and subcompanion operators

In [Ne], J. Neggers defined a groupoid $S(\cdot)$ to be a *pogroupoid* if

(i) $x \cdot y \in \{x, y\}$;
(ii) $x \cdot (y \cdot x) = y \cdot x$;
(iii) $(x \cdot y) \cdot (y \cdot z) = (x \cdot y) \cdot z$

for all $x, y, z \in S$. For a given pogroupoid $S(\cdot)$ he defined an associated partial order $\mathrm{po}(S)$ by $x \leq y$ iff $y \cdot x = y$ and he then demonstrated that $\mathrm{po}(S)$ is a poset. On the one hand, for a given poset $S(\leq)$ he also defined a binary operation on S by $y \cdot x = y$ if $x \leq y$, $y \cdot x = x$ otherwise, and proved that $S(\cdot)$ is a pogroupoid. Thus, denoting this pogroupoid by $\mathrm{pogr}(S)$, it may be shown that $\mathrm{pogr}(\mathrm{po}(S)) = S(\cdot)$ and $\mathrm{po}(\mathrm{pogr}(S)) = S(\leq)$ provide a natural isomorphism between the category of pogroupoids and the category of posets.

Given a poset $P(\leq)$ it is *A-free* if there is no full-subposet $X(\leq)$ of $P(\leq)$ which is order isomorphic to the poset $A(\leq)$. If C_n denotes a chain of length n and if $\underline{n}$ denotes an antichain of cardinal number n, while $+$ denotes the disjoint union of posets, then the poset $(C_2 + \underline{1})$ (or $C_2 + C_1$) has Hasse-diagram:



and may be represented as $\{p \le q, p \circ r, q \circ r\}$, where $a \circ b$ denotes the relation of not being comparable (i.e., $a \circ b$ iff $a \le b$ and $b \le a$ are both false) (see [NK2]). J. Neggers and H. S. Kim [NK1] proved that the pogroupoid $S(\cdot)$ is a semigroup if and only if $S(\cdot) = \mathrm{pogr}(P)$ where $P(\le)$ is $(C_2 + \underline{1})$-free as a poset.

Given a d-algebra $(X; *, 0)$, we define a binary operation $\star$ on X by

$$\begin{aligned} x \star y = y \star x = y \qquad & \text{if } x * y = 0, \\ x \star y = y, \quad y \star x = x \qquad & \text{otherwise.} \end{aligned}$$

The operation $\star$ described above is said to be a *pogroupoid*. Even though the derived digraph from a d-algebra may have no $(C_2 + \underline{1})$-full subposet, its derived algebra $(X, \star)$ need not be a pogroupoid.

Example 5.1. Consider a d-algebra $(X; *, 0)$ with the following left table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	c	c	c

$\star$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	a	c	c

Then $(X; *, 0)$ is a d-algebra, which is not a BCK/BCI-algebra. It is easy to see that its derived digraph has no $(C_2 + \underline{1})$-full subposet, but $(X; \star)$ is not a pogroupoid, since $(c \star b) \star a = c \star a = a$, while $(c \star b) \star (b \star a) = c \star b = c$.

Proposition 5.2. *Let $(X; *, 0)$ be a d^*-algebra. Then $((x \star y) * x) * y = 0$ for any $x, y \in X$.*

Proof. It follows immediately from the definition of pogroupoid. $\square$

Proposition 5.3. *Let $(X; *, 0)$ be a d^*-algebra. Assume $(y * x) * y = 0$ provided $x * y = 0$. Then $((y \star x) * x) * y = 0$ for any $x, y \in X$.*

Proof. If $x * y = 0$, then $y \star x = y$ and hence $((y \star x) * x) * y = (y * x) * y = 0$. If $x * y \neq 0$, then $y \star x = x$ and $((y \star x) * x) * y = (x * x) * y = 0 * y = 0$, proving the proposition. $\square$

There exists an example of non-d^*-algebra satisfying $(y * x) * y = 0$ when $x * y = 0$. The d-algebra X in Example 5.1 is such an algebra, since $(a * c) * a = b * a \neq 0$. Propositions 5.2 and 5.3 hold for any $BCK/BCI/BCH$-algebras. The notion of a subcompanion operation is a generalized concept of Proposition 5.2.

PROPOSITION 5.4. *Let $(X;*,\odot,0)$ be a companion d-algebra. If $(X;\star)$ is a pogroupoid, then $(x\star y)*(x\odot y)=0$ for any $x,y\in X$.*

P r o o f . Since X is a d^*-algebra, by Proposition 5.2, $((x\star y)*x)*y=0$ for any $x,y\in X$. Since $\odot$ is a companion operation, by (VII), $(x\star y)*(x\odot y)=0$. $\square$

Let $(X;*,\odot,0)$ be a d-algebra. If we define $x\star y=0$, then $\star$ is a (trivial) subcompanion operation on X.

Let $(X;*,0)$ be a d-algebra and $\diamond_i$ be a binary operation on X $(i=1,2)$. Define a relation:

$$\diamond_1\le\diamond_2 \iff (x\diamond_1 y)*(x\diamond_2 y)=0$$

for any $x,y\in X$. Then it is reflexive and anti-symmetric. Let $\mathrm{Bin}(X):=\{\diamond : \diamond \text{ binary operation on } X\}$. Define a binary operation $\oplus$ on $\mathrm{Bin}(X)$ by

$$x(\diamond_1\oplus\diamond_2)y:=(x\diamond_1 y)*(x\diamond_2 y).$$

Denote by $\diamond_a$, $a\in X$, the binary operation $x\diamond_a y:=a$ for any $x,y\in X$.

THEOREM 5.5. *If $(X;*,0)$ is a d-algebra, then $(\mathrm{Bin}(X),\oplus,\diamond_0)$ is also a d-algebra and the mapping $a\mapsto\diamond_a$ is an injection of $(X;*,0)$ into $(\mathrm{Bin}(X);\oplus,\diamond_0)$ which is a d-morphism.*

P r o o f . Clearly, $\mathrm{Bin}(X)$ satisfies the conditions (I) and (III). For any $\diamond\in\mathrm{Bin}(X)$ and for any $x,y\in X$, $x(\diamond_0\oplus\diamond)y=(x\diamond_0 y)*(x\diamond y)=0*(x\diamond y)=0=x\diamond_0 y$, which means that $\diamond_0\oplus\diamond=\diamond_0$, proving that $(\mathrm{Bin}(X),\oplus,\diamond_0)$ is a d-algebra. We claim that $\diamond_a*\diamond_b=\diamond_{a*b}$ for any $a,b\in X$. In fact, $x(\diamond_a*\diamond_b)y=(x\diamond_a y)*(x\diamond_b y)=a*b=x\diamond_{a*b}y$ for any $x,y\in X$. If we define a map $\varphi\colon X\to\mathrm{Bin}(X)$ by $\varphi(a):=\diamond_a$, then $\varphi(a*b)=\diamond_{a*b}=\diamond_a\oplus\diamond_b=\varphi(a)\oplus\varphi(b)$ for any $a,b\in X$, proving the theorem. $\square$

THEOREM 5.6. *Let $(X;*,\odot,0)$ be a companion d-algebra. If we define a binary operation $\square$ by*

$$x(\diamond_1\,\square\,\diamond_2)y:=(x\diamond_1 y)\odot(x\diamond_2 y)$$

for any $x,y\in X$, then $(\mathrm{Bin}(X);\oplus,\square,\diamond)$ is also a companion d-algebra containing $(X;*,\odot,0)$ via the identification $a\mapsto\diamond_a$.

P r o o f . Since X is a companion d-algebra, $x\big[((\diamond_1\,\square\,\diamond_2)\oplus\diamond_1)\diamond_2\big]y=\big[\{(x\diamond_1 y)\oplus(x\diamond_2 y)\}*(x\diamond_1 y)\big]*(x\diamond_2 y)=0$ for any $x,y\in X$. Since the proof of (VII) is similar, we omit it. $\square$

PROPOSITION 5.7. *If d-algebra $(X;*,0)$ has the transitivity condition, then $(\mathrm{Bin}(X),\oplus,\diamond_0)$ has also the transitivity condition.*

P r o o f . Straightforward. $\square$

PROPOSITION 5.8. *Let $(X; *, 0)$ be a d-algebra. If $\diamond \in \operatorname{Bin}(X)$ is commutative with $x * (x \diamond y) = 0$ for all $x, y \in X$, then $(x \star y) * (x \diamond y) - 0$.*

Proof. For any $x, y \in X$, either $x \star y = x$ or $x \star y = y$. If $x \star y = x$, then $(x \star y) * (x \diamond y) = x * (x \diamond y) = 0$. If $x \star y = y$, since $\diamond$ is commutative, $(x \star y) * (x \diamond y) = y * (x \diamond y) = y * (y \diamond x) = 0$, proving the proposition. □

A d-algebra $(X; *, 0)$ is said to be a *d-chain* if $x * y \neq 0$, then $y * x = 0$. $x, y \in X$.

Note that $\operatorname{Bin}(X)$ need not be a d-chain, even though X is a d-chain. Consider a $BCK/BCI/d$-algebra $X := \{0, a, b\}$ with the following table:

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	a	0

Then $(X; *, 0)$ is a d-chain. Define maps $f \colon X \to X$ by $f(0) = b$, $f(a) = a$, $f(b) = 0$, $g \colon X \to X$ by $g(0) = 0$, $g(a) = a$, $f(b) = b$. If we define binary operations on X by $x \diamond_f y := f(x)$, $x \diamond_g y := g(x)$, for all $x \in X$, then $(0 \diamond_f a) * (0 \diamond_g a) = f(0) * g(0) = b * 0 = b \neq 0$ and $(b \diamond_g 0) * (b \diamond_f 0) = g(b) * f(b)$ $b * 0 = b \neq 0$. Hence $\diamond_f * \diamond_g \neq \diamond_0 \neq \diamond_g * \diamond_f$, showing that $\operatorname{Bin}(X)$ is not a d-chain.

PROPOSITION 5.9. *Let $(X; *, \odot, 0)$ be a d-algebra and $\star$ be a pogroupoid operation on X. Then X is a d-chain if and only if $x \star y = y \star x$ for all $x, y \in X$.*

Proof. Let X be a d-chain. If $x * y = 0$, then $x \star y = y \star x = y$. If $x * y \neq 0$, then $y * x = 0$, since X is a d-chain, and hence $x \star y = y \star x = x$ Conversely, assume that there are $x, y \in X$ such that $x * y \neq 0 \neq y * x$. Then $y = x \star y = y \star x = x$, a contradiction. □

THEOREM 5.10. *Let $(X; *, \odot, 0)$ be a companion d-algebra. If the companion operation is the pogroupoid operation, then the algebra $(X; *, 0)$ is a d-chain and companion operation is commutative.*

Proof. Assume that $(X; *, 0)$ is not a d-chain. Then there are $x, y \in X$ such that $x * y \neq 0 \neq y * x$. This means that $x \star y = y$ and $x * (x \star y) = x * y \neq 0$. By Proposition 2.5-(ii), we have $0 = x * (x \odot y) = x * (x \star y)$, a contradiction. Hence, $(X; *, 0)$ is a d-chain. When $(X; *, 0)$ is a d-chain, at least one of $x * y$, $y * x$ is zero, and hence by definition of $\star$, the companion operation is commutative. □

COROLLARY 5.11. *Let $(X;*,\odot,0)$ be a companion d-algebra. If the companion operation is the pogroupoid operation, then $(X;*,0)$ is a d^*-algebra.*

Proof. By Theorem 5.10, the situation $x*y \neq 0, y*x \neq 0$ does not occur. If $x*y=0$, then $x\odot y = x\star y = y$ and hence $(y*x)*y = ((x\odot y)*x)*y = 0$, $(x*y)*x = 0*x = 0$. The case $y*x=0$ is the same case to the above case. □

Consider the following example. Let $X := \{0,a,b,c\}$ be a set with

$*$	0	a	b	c
0	0	0	0	0
a	a	0	b	0
b	b	0	0	a
c	c	a	0	0

Then $(X;*,0)$ is a d-chain, but $(b*c)*b = a*b = b \neq 0$, i.e., X is not a d^*-algebra. Note that X is not a companion d-algebra, since $a\odot c$ is not defined.

REFERENCES

[DvGr] DVUREČENSKIJ, A.—GRAZIANO, M. G.: *Commutative BCK-algebras and lattice ordered groups*, Math. Japon. **32** (1999), 227–246.

[FRT] FONT, J. M. RODRÍGEZ, A. J.—TORRENS, A.: *Wajsberg algebras*, Stochastica **8** (1984), 5 31.

[HL1] HU, Q. P. LI, X.: *On BCH-algebras*, Math. Seminar Notes **11** (1983), 313 320.

[HL2] HU, Q. P. LI, X.: *On proper BCH-algebras*, Math. Japon. **30** (1985), 659 661.

[Hoo] HOO, C. S.: *Unitary extensions of MV and BCK-algebras*, Math. Japon. **37** (1992), 683-686.

[Hu] HU, Q.: *BCI-algebras*, Shaanxi Scientific and Technological Press, Xian, 1984. (Chinese)

[Is] ISÉKI, K.: *On BCI-algebras*, Math. Seminar Notes **8** (1980), 125 130.

[IsTa] ISÉKI, K. TANAKA, S.: *An introduction to theory of BCK-algebras*, Math. Japon. **23** (1978), 1 26.

[JNK] JUN, Y. B.—NEGGERS, J.—KIM, H. S.: *Fuzzy d-ideals of d-algebras*, J. Fuzzy Math. **8** (2000), 123 130.

[LK] LEE, Y. C. KIM, H. S.: *On d-subalgebras of d-transitive d^*-algebras*, Math. Slovaca **49** (1999), 27 33.

[Me] MENG, J.: *Implicative commutative semigroups are equivalent to a class of BCK-algebras*, Semigroup Forum **50** (1995), 89 96.

[MeJu] MENG, J. JUN, Y. B.: *BCK-algebras*, Kyung Moon Sa, Seoul, 1994.

[Mun] MUNDICI, D.: *MV-algebras are categorically equivalent to bounded commutative BCK-algebras*, Math. Japon. **31** (1986), 889–894.

[NJK] NEGGERS, J.—JUN, Y. B.—KIM, H. S.: *On d-ideals in d-algebras*, Math. Slovaca **49** (1999), 243–251.

[Ne] NEGGERS, J.: *Partially ordered sets and pogroupoids*, Kyungpook Math. J. **16** (1976), 7–20.

[NK1] NEGGERS, J.—KIM, H. S.: *Basic Posets*, World Scientific Pub. Co., Singapore, 1998.

[NK2] NEGGERS, J.—KIM, H. S.: *Modular posets and semigroups*, Semigroup Forum **53** (1996), 57–62.

[NK3] NEGGERS, J.—KIM, H. S.: *On d-algebras*, Math. Slovaca **49** (1999), 19–26.

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