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# PURE FILTERS AND STABLE TOPOLOGY ON BL–ALGEBRAS

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In this paper we introduce stable topology and $F$-topology on the set of all prime filters of a BL-algebra $A$ and show that the set of all prime filters of $A$, namely $\mathrm{Spec}(A)$ with the stable topology is a compact space but not $T_0$. Then by means of stable topology, we define and study pure filters of a BL-algebra $A$ and obtain a one to one correspondence between pure filters of $A$ and closed subsets of $\mathrm{Max}(A)$, the set of all maximal filters of $A$, as a subspace of $\mathrm{Spec}(A)$. We also show that for any filter $F$ of BL-algebra $A$ if $\sigma(F) = F$ then $U(F)$ is stable and $F$ is a pure filter of $A$, where $\sigma(F) = \{a \in A \mid y \wedge z = 0$ for some $z \in F$ and $y \in a^{\perp}\}$ and $U(F) = \{P \in \mathrm{Spec}(A) \mid F \nsubseteq P\}$.

*Keywords:* BL-algebra, prime filters, maximal filters, pure filters, stable topology, $F$-topology

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## 1. INTRODUCTION

L. P. Belluce and S. Sessa studied in [2] stable topology and pure ideals in the framework of MV-algebras. They defined the stable topology for MV-algebras as follows: let $A$ be an MV-algebra. The set of all prime ideals of $A$ is denoted by $\mathrm{Spec}(A)$. The open sets in $\mathrm{Spec}(A)$ are of the form $U(I) = \{P \in \mathrm{Spec}(A) \mid I \nsubseteq P\}$ where $I$ is an ideal of $A$. The set $U(I)$ is stable under ascent if $P \in U(I)$ and $Q \in \mathrm{Spec}(A)$ with $P \subseteq Q$, then $Q \in U(I)$. The set $U(I)$ is stable under descent if $P \in U(I)$ and $Q \in \mathrm{Spec}(A)$ with $Q \subseteq P$, then $Q \in U(I)$.

$U(I)$ is said to be stable if it is stable under ascent and under descent. The stable topology for $A$ is the collection of stable open subsets of $\mathrm{Spec}(A)$.

In 1998 Petr Hájek introduced in [4] the variety of BL-algebras and showed that the variety of MV-algebras actually is a subvariety of the variety of BL-algebras. In other words, any MV-algebra can be easily viewed as a special BL-algebra. Thus it makes sense to generalize the notion of stable topology to BL-algebras. But in fact since the multiplication ($\odot$) is a fundamental operation and filters are basic notions in BL-algebras defined in terms of $\odot$ (see the Definitions 2.1, 2.2 below) as well as a dual notion of ideals, we prefer to present stable topology based on filters. Therefore the generalization does not work easily and we face some related difficulties towards this approach. Although we get similar results as in [2], we also prove some more theorems regarding different properties of this topology on BL-algebras.

This paper consists of four sections. In the second section we recall the definition of a BL-algebra $A$, a filter $F$ and Spec($A$) with more preliminary facts that we need in the sequel. In the third section we define $F$-topology which is actually the same as spectral topology but in terms of filters (letter $F$ comes from the word filter) and introduce the stable topology on Spec($A$). We show that the topological space Spec($A$) with the stable topology is compact but not $T_0$ and hence neither $T_1$ nor $T_2$.

In the fourth section, we define pure filters of $A$ and prove some important results. In fact let Max($A$) be the set of all maximal filters of $A$. Since Max($A$)$\subseteq$ Spec($A$), we consider the topology induced by $F$-topology on Max($A$) and show that $F$-topology and stable topology coincide on subspace Max($A$). We show that pure filters of $A$ are in one to one correspondence with closed subsets of Max($A$). We also investigate some conditions for purity of a filter $F$ by considering $\sigma(F) = \{a \in A \,|\, y \wedge z = 0$ for some $z \in F$ and $y \in a^{\perp}\}$ and stability of $U(F)$ where $U(F)$ is an open set in Spec($A$) with $F$-topology.

## 2. PRELIMINARIES

**Definition 2.1.** (Hájek [6]) A BL-algebra is an algebra $A = (A, \vee, \wedge, \odot, \longrightarrow, 0, 1)$ of type (2,2,2,2,0,0) satisfying the following properties:

1. $(A, \vee, \wedge, 0, 1)$ is a lattice with 0 as the least element and 1 as the greatest element.
2. $(A, \odot, 1)$ is a commutative monoid.
3. The following statements hold for every $a, b, c \in A$:

   (i) $c \leq a \longrightarrow b$ iff $a \odot c \leq b$ (Residuation);

   (ii) $a \wedge b = a \odot (a \longrightarrow b)$ (Divisibility);

   (iii) $(a \longrightarrow b) \vee (b \longrightarrow a) = 1$ (Prelinearity).

A BL-algebra $A$ is nontrivial iff $0 \neq 1$. We also define a unary operation "$-$" on $A$ by $a \longrightarrow 0 = \overline{a}$

**Definition 2.2.** (Hájek [6]) A filter of a BL-algebra $A$ is a nonempty subset $F$ of $A$ such that:

(i) $a, b \in F$ implies $a \odot b \in F$;

(ii) $a \in F$ and $a \leq b$ imply $b \in F$.

**Definition 2.3.** (Hájek [6]) A filter $F$ of a BL-algebra $A$ is proper if $F \neq A$. A proper filter $P$ of $A$ is called prime provided that $a \vee b \in P$ implies that $a \in P$ or $b \in P$, for every $a, b \in A$.
A proper filter $M$ of $A$ is called maximal, if it is not contained in any other proper filter, that is for any filter $E$ such that $M \subseteq E \subseteq A$, either $E = M$ or $E = A$.

A BL-algebra $A$ is local if it has a unique maximal filter.

It is easy to see that $F$ is a proper filter iff $0 \notin F$.

**Remark 2.1.** The usual notions of morphisms can be defined on BL-algebras (see for example [4, 11]).

**Proposition 2.1.** (Georgescu and Leustean [5]) Let $h : A \longrightarrow B$ be a BL-morphism. Then

(i) If $G$ is a (proper, prime, maximal) filter of $B$, then $h^{-1}(G)$ is a (proper, prime, maximal) filter of $A$.

(ii) If $h$ is surjective and $F$ is a filter of $A$, then $h(F)$ is a filter of $B$.

(iii) If $h$ is surjective and $M$ is a maximal filter of $A$ such that $h(M)$ is proper, then $h(M)$ is a maximal filter of $B$.

We denote the lattice reduct of a BL-algebra $A$ by $L(A)$, and it is easy to see that any (prime) filter of $A$ is a (prime) filter of $L(A)$.

From now on, in this paper we consider $\mathrm{Spec}(A)$, $\mathrm{Max}(A)$ and $\mathrm{Min}(A)$ as the set of all prime filters, maximal filters and minimal prime filters of a BL-algebra $A$, respectively.

**Proposition 2.2.** (Di Nola et al. [4], Leustean [8], Turunen [10]) Let $A$ be a BL-algebra. Then the followings hold.

(i) If $F$ is a filter of $A$ and S is a nonempty $\vee$-*closed* subset of $A$, (i. e. if $a, b \in S$ then $a \vee b \in S$) such that $F \cap S = \emptyset$, then there exists a prime filter $P$ of $A$ such that $F \subseteq P$ and $P \cap S = \emptyset$.

(ii) Any maximal filter of $A$ is a prime filter.

(iii) If $A$ is nontrivial, then any proper filter $F$ of $A$ is the intersection of all prime filters containing $F$.

(iv) If $A$ is nontrivial, then any prime filter of $A$ is contained in a unique maximal filter.

(v) If $A$ is nontrivial, then any proper filter $A$ can be extended to a prime, maximal filter.

**Proposition 2.3.** (Di Nola et al. [4]) If $A$ is a nontrivial BL-algebra and $M$ a proper filter of $A$, then the following are equivalent:

(i) $M$ is maximal,

(ii) For any $x \in A$, $x \notin M$ implies that $\overline{x^n} \in M$ for some $n \in \omega$, where $\omega$ is the set of natural numbers.

**Definition 2.4.** (Leustean [8]) Let $X \subseteq A$. The filter generated by $X$ will be denoted by $\langle X \rangle$. If $X = \emptyset$ then $\langle \emptyset \rangle = \{1\}$ and if $X \neq \emptyset$ then we have $\langle X \rangle = \{y \in A \,|\, x_1 \odot x_2 \odot \cdots \odot x_n \leq y$ for some $n \in \omega$ and some $x_1, x_2, \cdots, x_n \in X\}$.

It is easy to see that if $a \in A$ then we have $\langle a \rangle = \{b \in A |\ a^n \leq b$ for some $n \in \omega\}$.

**Proposition 2.4.** (Leustean [8]) If $\mathcal{F}(A)$ is the set of all filters of $A$, then $(\mathcal{F}(A), \subseteq)$ is a complete lattice and for every family $\{F_i\}$ of filters of $A$, we have $\bigvee_{i \in I} F_i = \langle \bigcup_{i \in I} F_i \rangle$ and $\bigwedge_{i \in I} F_i = \bigcap_{i \in I} F_i$.

From [6] it follows that to every filter $F$ of a BL-algebra $A$ we can associate a congruence relation $\sim_F$ on $A$ by defining $a \sim_F b$ iff $a \longrightarrow b \in F$ and $b \longrightarrow a \in F$ iff $(a \longrightarrow b) \odot (b \longrightarrow a) \in F$. For element $a \in A$, let $\frac{a}{F}$ be the congruence class $\frac{a}{\sim_F}$. If we denote by $\frac{A}{F}$ the quotient set $\frac{A}{\sim_F}$, then $\frac{A}{F}$ becomes a BL-algebra with the natural operations induced by those of $A$.

We recall that a BL-chain is a totally ordered BL-algebra, i. e. a BL-algebra whose lattice order is total [6].

**Proposition 2.5.** (Hájek [6]) Let $F$ be a filter of $A$ and $a, b \in A$. Then

(i) $\frac{a}{F} = \frac{1}{F}$ iff $a \in F$.

(ii) $\frac{a}{F} = \frac{0}{F}$ iff $\overline{a} \in F$.

(iii) $\frac{a}{F} \leq \frac{b}{F}$ iff $a \longrightarrow b \in F$.

(iv) $\frac{A}{F}$ is a BL-chain iff $F$ is a prime filter of $A$.

**Definition 2.5.** (Busneage and Piciu [3]) Let $A$ be a BL-algebra. An element $a \in A$ is called archimedean if there is $n \in \omega,\ n \geq 1$ such that $a \vee \overline{a^n} = 1$. A BL-algebra $A$ is called hyperarchimedean if all its elements are archimedean.

**Proposition 2.6.** (Busneage and Piciu [3]) A BL-algebra $A$ is hyperarchemedian iff Spec($A$)=Max($A$).

Based on the definitions and propositions in this section, we define our main notion of stable topology on BL-algebras.

## 3. STABLE TOPOLOGY

Let $A$ be a non trivial BL-algebra. Denote by Spec($A$) the set of all its prime filters. Consider the spectral topology (Zariski topology) on Spec($A$), i. e. the topology in which its closed sets are exactly the sets of the form $V(X) = \{P \in \mathrm{Spec}(A) | X \subseteq P\}$ for each subset $X$ of $A$. Then Spec($A$) equipped with this topology is called the prime spectrum of $A$.

Now we are planning to introduce $F$-topology on Spec($A$).

**Proposition 3.1.** Let $A$ be a nontrivial BL-algebra and $F$ be a filter of $A$. Define $V(F) = \{P \in \mathrm{Spec}(A) | F \subseteq P\}$. Then the following hold:

(i) $V(\{1\}) = \mathrm{Spec}(A),\ V(A) = \emptyset$.

(ii) If $\{F_i\}_{i\in I}$ is a family of filters of $A$, then $\bigcap_{i\in I} V(F_i) = V\left(\left\langle\bigcup_{i\in I} F_i\right\rangle\right)$.

(iii) If $F_1, F_2$ are filters of $A$ then $V(F_1) \cup V(F_2) = V(F_1 \cap F_2)$.

Proof.

(i) Follows from the fact that 1 belongs to any filter $F$ and every prime filter $P$ is proper.

(ii) Since each $F_i \subseteq \bigcup_{i\in I} F_i \subseteq \langle\bigcup_i F_i\rangle, V\left(\left\langle\bigcup_{i\in I} F_i\right\rangle\right) \subseteq V(F_i)$ for each $i \in I$. Then $V\left(\left\langle\bigcup_{i\in I} F_i\right\rangle\right) \subseteq \bigcap_{i\in I} V(F_i)$. Now let $P \in \bigcap_{i\in I} V(F_i)$ then $P \in V(F_i)$ and $F_i \subseteq P$ for each $i \in I$. We claim that $\langle\bigcup_i F_i\rangle \subseteq P$. Let $t \in \left\langle\bigcup_{i\in I} F_i\right\rangle$. Then $t \geq f_1 \odot f_2 \odot \cdots \odot f_k$ for some $k \in \omega$ and $f_1, f_2, \ldots, f_k \in \bigcup_{i\in I} F_i$. But for each $f_i$ there exists $F_{k_i}$ such that $f_i \in F_{k_i}$. Therefore, $f_1, f_2, \ldots, f_k \in P$ and since $P$ is a filter, $t \in P$. Thus, we conclude that $P \in V\left(\left\langle\bigcup_{i\in I} F_i\right\rangle\right)$.

(iii) Since $F_1 \cap F_2 \subseteq F_1, F_2$ we have $V(F_1), V(F_2) \subseteq V(F_1 \cap F_2)$. Thus $V(F_1) \cup V(F_2) \subseteq V(F_1 \cap F_2)$. Now let $P \in V(F_1 \cap F_2)$ but $P \notin V(F_1) \cup V(F_2)$. Then $P \notin V(F_1),\ P \notin V(F_2)$, i.e. $F_1 \nsubseteq P,\ F_2 \nsubseteq P$. There exist $x \in F_1,\ y \in F_2$ such that $x, y \notin P$. Since $x, y \leq x \vee y,\ x \vee y \in F_1,\ F_2$ and hence $x \vee y\ \in F_1 \cap F_2$. But since $F_1 \cap F_2 \subseteq P,\ x \vee y \in P$. This implies $x \in P$ or $y \in P$ which contradicts the assumption. □

Based on Proposition 3.1, we have

**Corollary 3.1.** The collection $\{V(F) \,|\, F$ is a filter of $A\}$ defines a topology on $\mathrm{Spec}(A)$ whose closed sets are of the form $V(F)$ for some filter $F$ in $A$.

We call the resulting topology in Corollary 3.1, *F-topology*.

**Remark 3.1.** From [8] since the family $\{U(a)\}_{a\in A}$ where $U(a) = \{P \in \mathrm{Spec}(A) \,|\, a \notin P\}$, is a basis for the spectral topology on $\mathrm{Spec}(A)$, this family is also a basis for $F$-topology. For let $F$ be a filter of $A$. Then by [8, Proposition 2.2], we have $U(F) = U(\bigcup_{f\in F} \{f\}) = \bigcup_{f\in F} U(f)$ and hence any open subset of $\mathrm{Spec}(A)$ with $F$-topology is the union of subsets from the family $\{U(a)\}_{a\in A}$. Thus by [8, Theorem 2.7], $\mathrm{Spec}(A)$ with $F$-topology is also a compact $T_0$ topological space.

It is obvious to see that if $F$ is a filter of $A$, then $V(F)$ is *stable under ascent*, that is if $P \in V(F),\ Q \in \mathrm{Spec}(A)$ and $P \subseteq Q$, then $Q \in V(F)$.

We also know that $U(F) = \mathrm{Spec}(A) - V(F) = \{P \in \mathrm{Spec}(A) |\ F \nsubseteq P\}$ is *stable under descent*, i.e. if $P \in U(F)$, $Q \in \mathrm{Spec}(A)$ and $Q \subseteq P$, then $Q \in U(F)$.

Let $A$ be a BL-algebra and $F$ be a filter of $A$. We say that $U(F)$ is *stable* if $U(F)$ is stable under ascent and descent. Since $U(F)$ is always stable under descent, being stable it is enough that $U(F)$ is stable under ascent.

**Remark 3.2.** If $F, G$ are filters of a nontrivial BL-algebra $A$. Then $F = G$ iff $U(F) = U(G)$.

P r o o f. If $F = G$, then obviously $U(F) = U(G)$. Now let $U(F) = U(G)$. Thus $V(F) = V(G)$ and $\bigcap_{P\in \mathrm{Spec}(A),\ F\subseteq P} P = \bigcap_{P\in \mathrm{Spec}(A),\ G\subseteq P} P$. Therefore by Proposition 2.2 (iii), $F = G$. □

In the following proposition we introduce the stable topology on Spec$(A)$.

**Proposition 3.2.** The collection of all stable open subsets of Spec$(A)$ satisfies the axioms for open sets in a topological space. The resulting topology is called stable topology on Spec$(A)$. In other words, $\{U \mid U$ is open with $F$-topology and stable$\}$ is the collection of open sets for stable topology.

P r o o f. Let $T$ be the set of all stable open subsets of Spec$(A)$. It is obvious that $\emptyset$ and Spec$(A) \in T$. Now let $T_1, T_2$ be in $T$. Then $T_1 = U(F_1)$ and $T_2 = U(F_2)$ for some $F_1, F_2 \in \mathcal{F}(A)$. Since $U(F_1) \cap U(F_2) = U(\langle F_1 \cup F_2\rangle)$, $T_1 \cap T_2$ is open. For stability, It is enough to show that $T_1 \cap T_2$ is stable under ascent. Let $P \in U(\langle F_1 \cup F_2\rangle)$, $Q \in$ Spec$(A)$ and $P \subseteq Q$. Then $P \in U(F_1)$, $U(F_2)$ and hence by stability of $U(F_1)$, $U(F_2)$ we have, $Q \in U(F_1), U(F_2)$. Thus $Q \in U(F_1) \cap U(F_2)$.

Let $\{T_i\}_{i\in I}$be a family of stable open subsets of Spec$(A)$. Then for each $i \in I$, there exists filter $F_i$ of $A$ such that $T_i = U(F_i)$. Thus $\bigcup_i T_i = \bigcup_i U(F_i) = U(\bigcap_i F_i)$. For stability, let $P \in U(\bigcap_i F_i)$, $Q \in$ Spec$(A)$ and $P \subseteq Q$, then $P \in U(F_i)$ for some $i \in I$ and by stability of $U(F_i)$ we have $Q \subseteq U(F_i) \subseteq \bigcup_i U(F_i)$ and hence $Q \in \bigcup_i U(F_i)$. □

In the next corollary, we see that there is a distinction between topological property of Spec$(A)$ with stable topology and $F$-topology. In fact, Spec$(A)$ with stable topology is a $T_0$ topological space but with $F$-topology is not.

**Corollary 3.2.** With the stable topology, Spec$(A)$ is a compact topological space but not $T_0$ and hence neither $T_1$ nor $T_2$.

P r o o f. We know that every stable open set is also open in $F$-topology. Therefore, since by Remark 3.1, Spec$(A)$ is compact in $F$-topology, it is also compact in stable topology. Now let $P$, $Q \in$ Spec$(A)$ such that $P \subsetneqq Q$. Since all open sets ($U(F)$ for some filter $F$ of $A$) are stable under descent, every $U(F)$ that contains $Q$, will contain $P$. Now suppose that $U(F)$ is stable, and $P \in U(F)$. Then since $Q$ contains $P$, and $U(F)$ is stable, we have $Q \in U(F)$. Hence we see that $P$ and $Q$ can not be separated by stable open sets, so the stable topology is not $T_0$ and therefore is neither $T_1$ nor $T_2$. □

**Lemma 3.1.** Let $A$ be a nontrivial BL-algebra, $M \in \mathrm{Max}(A)$ and $F$ be a proper filter of $A$. Suppose that $\widehat{W}_M = \{P \in \mathrm{Spec}(A) | P \subseteq M\}$ and $W_M = \bigcap \widehat{W}_M = \bigcap_P P$ for $P \in \widehat{W}_M$ and $W_M \subseteq F$. Then $F \subseteq M$.

Proof. Let $W_M \subseteq F$ but $F \nsubseteq M$. Therefore by Proposition 2.3, there exists $x \in F$ such that $x \notin M$ and $\overline{x^n} \in M$ for some $n \in \omega$. Let $P \in \widehat{W}_M$ be arbitrary. Then $\frac{M}{P} \in \mathrm{Max}\left(\frac{A}{P}\right)$. Since $P$ is a prime filter, by Proposition 2.5, $\frac{A}{P}$ is linear ordered and we have $\frac{x^n}{P} < \frac{\overline{x^n}}{P}$ or $\frac{\overline{x^n}}{P} < \frac{x^n}{P}$. If $\frac{x^n}{P} < \frac{\overline{x^n}}{P}$, then $\frac{x^n}{P} \odot \frac{x^n}{P} < \frac{\overline{x^n}}{P} \odot \frac{x^n}{P} = \frac{0}{P}$, i.e. $\frac{x^{2n}}{P} = \frac{0}{P}$. Thus $(x^{2n} \longrightarrow 0) \in P$, i.e. $\overline{x^{2n}} \in P$. But since $P$ is arbitrary in $\widehat{W}_M$, $\overline{x^{2n}} \in \bigcap_P P = W_M$ for $P \in \widehat{W}_M$. Since $W_M \subseteq F$, $\overline{x^{2n}} \in F$. But by assumption $x \in F$. Therefore $x^{2n} \in F$ and hence $0 \in F$ which is a contradiction. Now if $\frac{\overline{x^n}}{P} < \frac{x^n}{P}$, since $\overline{x^n} \in M$, $\frac{\overline{x^n}}{P} \in \frac{M}{P} \in \mathrm{Max}\left(\frac{A}{P}\right)$ and hence $\frac{x^n}{P} \in \frac{M}{P}$, i.e. $x^n \in M$. Thus $x^n \odot \overline{x^n} = 0 \in M$ which is a contradiction. □

**Proposition 3.3.** With the notation of Lemma 3.1 the following hold:

(i) $V(W_M) = \widehat{W}_M$ and hence $\widehat{W}_M$ is closed with respect to $F$-topology.

(ii) $\mathrm{Spec}(A)$ is the disjoint union of subspaces $\widehat{W}_M,\ M \in \mathrm{Max}(A)$.

Proof.

(i) Let $P \in V(W_M)$. Then $W_M \subseteq P$ and by Lemma 3.1, $P \subseteq M$, i.e. $P \in \widehat{W}_M$. Obviously $\widehat{W}_M \subseteq V(W_M)$ and hence $\widehat{W}_M$ is a closed set.

(ii) The proof follows from [1, p. 333]. □

## 4. PURE FILTERS AND SOME RESULTS

In this section we introduce pure filters and study some of their properties.

**Definition 4.1.** Let $F$ be a filter of $A$. We say that $F$ is *pure* if $U(F)$ is stable.

It is easy to see that if $A$ is a BL-algebra, then $A$ and $\{1\}$ are pure filters.

**Lemma 4.1.** Let $F$ be a pure filter of $A$, $P \in \mathrm{Spec}(A),\ M \in \mathrm{Max}(A)$ and $P, F \subseteq M$. Then $F \subseteq P$.

Proof. Assume on the contrary that $F \nsubseteq P$. Then $P \in U(F)$. Since $P \subseteq M,\ M \in \mathrm{Max}(A)$ and $\mathrm{Max}(A) \subseteq \mathrm{Spec}(A)$, by stability of $U(F)$, we conclude that $M \in U(F)$, i.e. $F \nsubseteq M$ and this is a contradiction. □

**Corollary 4.1.** Let $F$ be a pure filter of $A$, $M \in \mathrm{Max}(A)$ and $F \subseteq M$. Then $F \subseteq W_M$.

Proof. Let $P$ be an arbitrary prime filter such that $P \subseteq M$. Since $F \subseteq M$ and $U(F)$ is stable, by Lemma 4.1, $F \subseteq P$. Thus $F \subseteq \bigcap_P P$ for each $P \subseteq M$ and hence $F \subseteq W_M$. □

**Theorem 4.1.** Let $F$ be a pure filter of $A$. Then $F = \bigcap_M \{W_M | F \subseteq M \in \mathrm{Max}(A)\}$.

Proof. Let $U(F)$ be stable. Then by Corollary 4.1, for each $M \in \mathrm{Max}(A)$ such that $F \subseteq M$, we have $F \subseteq W_M$ and hence $F \subseteq \bigcap_{M \supseteq F} W_M$. Now let $x \in \bigcap_M \{W_M | F \subseteq M \in \mathrm{Max}(A)\}$. Then we have $x \in W_M$ for each $M \in \mathrm{Max}(A)$ such that $F \subseteq M$. Let $P$ be any prime filter of $A$ such that $F \subseteq P$. By Proposition 2.2 there exists a unique maximal filter $M_P$ over $P$, i.e. $P \subseteq M_P$. Therefore $F \subseteq M_P$ and by assumption $x \in W_{M_P}$. Since $V(W_M) = \widehat{W}_M$ and $P \in \widehat{W}_{M_P}$, $P \in V(W_M)$, i.e. $W_{M_P} \subseteq P$ and hence $x \in P$. Since $P$ is an arbitrary prime filter such that $F \subseteq P$, $x \in \bigcap_{F \subseteq P} P$ and by Proposition 2.2 since $F = \bigcap_{F \subseteq P} P$, $x \in F$. □

We are planning to obtain a relation between subsets of $\mathrm{Max}(A)$ and stable open sets.

**Theorem 4.2.** Let $F$ be a filter of $A$ such that $U(F)$ is stable. Let $T = \{M \in \mathrm{Max}(A) | F \subseteq M\}$. Then $U(F) = \mathrm{Spec}(A) - \bigcup_{M \in T} \widehat{W}_M$.

Proof. Let $P \in U(F)$. Then $F \nsubseteq P$ and by Lemma 4.1, $F \nsubseteq M$. Let $M_P$ be the unique maximal filter such that $P \subseteq M_P$. Then $F \nsubseteq M_P$, i.e. $M_P \notin T$ and hence $P \in \widehat{W}_{M_P} \subseteq \bigcup_{M \in \mathrm{Max}(A) - T} \widehat{W}_M$. Since $\mathrm{Spec}(A) - \bigcup_{M \in T} \widehat{W}_M = \bigcup_{M \in \mathrm{Max}(A)} \widehat{W}_M - \bigcup_{M \in T} \widehat{W}_M = \bigcup_{M \in \mathrm{Max}(A) - T} \widehat{W}_M$, we have $U(F) \subseteq \mathrm{Spec}(A) - \bigcup_{M \in T} \widehat{W}_M$. Now let $P \in \mathrm{Spec}(A) - \bigcup_{M \in T} \widehat{W}_M$ but $P \notin U(F)$. Then $P \notin \bigcup_{M \in T} \widehat{W}_M$, i.e. $P \notin \widehat{W}_M$ and $P \nsubseteq M$ for each $M \in T$. Since $P \notin U(F)$, $F \subseteq P$ and hence $F \subseteq M_P$, i.e. $M_P \in T$ but by above $P \nsubseteq M_P$ therefore, this is impossible. □

**Remark 4.1.** For any $T \subseteq \mathrm{Max}(A)$, $X_T = \mathrm{Spec}(A) - \bigcup_{M \in T} \widehat{W}_M$ is stable.

Proof. It is enough to show that $X_T$ is stable under ascent. Let $P \in X_T$ , $Q \in \mathrm{Spec}(A)$ , $P \subseteq Q$. Then, $P \notin \bigcup_{M \in T} \widehat{W}_M$, i.e. $P \notin \widehat{W}_M$ for each $M \in T$. This means that $P \nsubseteq M$ for each $M \in T$. Thus $Q \nsubseteq M$ for each $M \in T$. That is equivalent to $Q \notin \widehat{W}_M$ for each $M \in T$. Hence $Q \in X_T$. □

**Theorem 4.3.** Let $T$ be a finite subset of $\mathrm{Max}(A)$ and $F$ be a filter of $A$ such that $F = \bigcap_{M \in T} W_M$. Then $U(F)$ is stable, i.e. $F$ is a pure filter.

Proof. Let $X = \mathrm{Spec}(A) - \bigcup_{M \in T} \widehat{W}_M$. Since each $\widehat{W}_M$ is closed and $T$ is finite, the finite union of $\widehat{W}_M$ is closed and hence $X$ is an open set. By Remark 4.1, it is enough to show that $X = U(F)$. Let $P \in X$ then $P \notin \widehat{W}_M$ for each $M \in T$. Since $V(W_M) = \widehat{W}_M$, $P \notin V(W_M)$, i.e. $W_M \nsubseteq P$ for each $M \in T$ and hence $\bigcap_{M \in T} W_M \nsubseteq P$, which implies that $P \in U(\bigcap_{M \in T} W_M) = U(F)$. Now let $P \in U(F)$. Then $F \nsubseteq P$, i.e. $\bigcap_{M \in T} W_M \nsubseteq P$. This implies that $W_M \nsubseteq P$ for each $M \in T$. But since $P \notin V(W_M)$, $P \notin \widehat{W}_M$ for each $M \in T$ and hence $P \in \mathrm{Spec}(A) - \bigcup_{M \in T} \widehat{W}_M$. □

We recall that a BL-algebra $A$ is semilocal iff $\mathrm{Max}(A)$ is a finite set [5], then:

**Corollary 4.2.** A BL-algebra $A$ is semilocal iff the stable topology on $\mathrm{Spec}(A)$ is finite.

P r o o f. By Theorems 4.2 and 4.3, there is a relation between subsets of $\mathrm{Max}(A)$ and stable open sets. Let $A$ be semilocal. Then, there are only finitely many maximal filters and hence only finitely many stable open sets. Thus the stable topology is finite. Now let the stable topology on $\mathrm{Spec}(A)$ be finite. Then, there can be only finitely many maximal filters and hence $A$ is semilocal. $\square$

In the following theorems we provide needed facts to obtain a one to one correspondence between pure filters and closed subsets of $\mathrm{Max}(A)$.

**Theorem 4.4.** The map $\varphi : \mathrm{Spec}(A) \longrightarrow \mathrm{Max}(A)$ by $P \longmapsto M_P$ is a continuous retraction.

P r o o f. From [8], it is known that associated to each BL-algebra $A$, there is a bounded distributive lattice $\beta(A)$ such that the topological space $\mathrm{Spec}(A)$ and $\mathrm{Spec}(\beta(A))$ are homeomorphic. On the other hand, by ([7] p. 68), if $A$ is a normal distributive lattice, then the map $\varphi : \mathrm{Spec}(A) \longrightarrow \mathrm{Max}(A)$ is a continuous retraction. Thus the Theorem holds for BL-algebras. $\square$

**Corollary 4.3.** The stable topology and $F$-topology coincide on subspace $\mathrm{Max}(A)$.

P r o o f. We know that if $T_1 = \{U_i \,|\, i \in I\}$ is stable topology for $\mathrm{Spec}(A)$ then $T_2 = \{U_i \cap \mathrm{Max}(A) \,|\, i \in I\}$ is stable topology for $\mathrm{Max}(A)$. Let $G$ be an open set of $\mathrm{Max}(A)$ and $Q = \{P \in \mathrm{Spec}(A) \,|\, M_P \in G\}$ where $M_P$ is the unique maximal filter of $A$ such that $P \subseteq M_P$. Since $\varphi : \mathrm{Spec}(A) \longrightarrow \mathrm{Max}(A)$ by $P \longmapsto M_P$ is a continuous retraction, $\varphi^{-1}(G)$ is open in $\mathrm{Spec}(A)$. But $\varphi^{-1}(G) = Q$. Hence $Q$ is open in $\mathrm{Spec}(A)$. Now we claim that $Q$ is stable. Let $P_1 \in Q$, $P_2 \in \mathrm{Spec}(A)$ such that $P_1 \subseteq P_2$. Let $M_{P_2}$ be the unique maximal filter such that $P_2 \subseteq M_{P_2}$. Since $P_1 \subseteq M_{P_1}$ and $P_2 \subseteq M_{P_2}$, $M_{P_1} = M_{P_2}$. But $M_{P_1} \in G$ implies $M_{P_2} \in G$, i. e. $P_2 \in Q$. Therefore $Q$ is stable and clearly $G = Q \cap \mathrm{Max}(A)$. $\square$

**Corollary 4.4.** Let $A$ be a nontrivial BL-algebra. Then every stable open subset $G$ has the form $G = \bigcup_{M \in Y} \widehat{W}_M$ for some open subset $Y \subseteq \mathrm{Max}(A)$.

P r o o f. Let $G$ be a stable open set. We take $Y = G \cap \mathrm{Max}(A)$. Then if we consider the map $\varphi : \mathrm{Spec}(A) \longrightarrow \mathrm{Max}(A)$ by $P \longmapsto M_P$, then it is trivial that $\varphi^{-1}(Y) = G$. But $\mathrm{Spec}(A) = \bigcup_{M \in \mathrm{Max}(A)} \widehat{W}_M$. Therefore, $G = \bigcup_{M \in Y} \widehat{W}_M$. $\square$

**Remark 4.2.** If $G$ is open in $\text{Spec}(A)$ and $G$ is a union of closed sets in $\text{Spec}(A)$, then $G$ is stable.

Proof. Let $G = \bigcup_{M\in T\subseteq \text{Max}(A)} \widehat{W}_M$, $P \in G$, $Q \in \text{Spec}(A)$ and $P \subseteq Q$. Since $Q \subseteq M_Q$ (unique maximal filter over Q), $Q \in \widehat{W}_{M_Q} \subseteq \bigcup_{M\in\text{Max}(A)} \widehat{W}_M$. Thus $G$ is stable. □

**Theorem 4.5.** Let $A$ be a nontrivial BL-algebra, $T \subseteq \text{Max}(A)$, $Y = \bigcup_{M\in T} \widehat{W}_M$ be closed in $\text{Spec}(A)$. Then $E = \bigcap_{M\in T} W_M$ is a pure filter.

Proof. Let $E = \bigcap_{M\in T} W_M$. We know that $Y \cap \text{Max}(A) = \bigcup_{M\in T} \widehat{W}_M \cap \text{Max}(A) = \bigcup_{M\in T}\{P | P \in \text{Max}(A),\ P \subseteq M\} = \bigcup_{M\in T} M = T$. Since Y is closed, $\text{Spec}(A) - Y$ is open and stable. Thus there exists a pure filter $F$ of $A$ such that $\text{Spec}(A) - Y = U(F)$. It is enough to show that $E = F$. Since $U(F)$ is stable, by Theorem 4.1, $F = \bigcap_{M\supseteq F,\ M\in\text{Max}(A)} W_M = \bigcap_{M\supseteq F} (\bigcap_{P\subseteq M} P) \subseteq \bigcap_{P\subseteq M} P = W_M$ for each $M \in Y \cap \text{Max}(A) = T$. Then $F \subseteq \bigcap_{M\in F}$, i.e. $F \subseteq E$. On the other hand, since $\text{Spec}(A) - Y = U(F)$, $\text{Spec}(A) = U(F) \cup V(F)$ and $U(F) \cap V(F) = \emptyset$ this implies that $Y = V(F)$. Thus $M \in V(F) = Y$ iff $F \subseteq M$, i.e. $M \in Y \cap \text{Max}(A) = T$ iff $F \subseteq M$. Therefore $E = \bigcap_{M\in T} W_M \subseteq W_M$ for each $M \in T$, that is for each $M \supseteq F$. Hence $E \subseteq \bigcap_{F\subseteq M} W_M = F$. □

**Corollary 4.5.** There is a one to one correspondence between pure filters and subsets $T \subseteq \text{Max}(A)$ *where* $\bigcup_{M\in T} \widehat{W}_M$ is closed in $\text{Spec}(A)$.

In the next theorem we prove a good relation between closed subsets of $\text{Max}(A)$ and closed subsets of $\text{Spec}(A)$.

**Theorem 4.6.** Let $T$ be a subset of $\text{Max}(A)$. Then $T$ is closed in $\text{Max}(A)$ iff $\bigcup_{M\in T} \widehat{W}_M$ is closed in $\text{Spec}(A)$.

Proof. Let $T$ be closed in $\text{Max}(A)$. Then $G = \text{Max}(A) - T$ is open in $\text{Max}(A)$. By Corollary 4.3, there is a stable open subset $U(I)$ such that $U(I) \cap \text{Max}(A) = G$. We claim that $\text{Spec}(A) - Y = U(I)$. Since $U(I)$ is stable, by Theorem 4.2, we have $U(I) = \text{Spec}(A) - \bigcup_{I\subseteq M} \widehat{W}_M$. It is enough to show that $M \in T$ iff $I \subseteq M$ (since $Y = \bigcup_{I\subseteq M} \widehat{W}_M$ iff $\bigcup_{I\subseteq M} \widehat{W}_M = \bigcup_{M\in T} \widehat{W}_M$). Now let $I \subseteq M$. Then $M \notin U(I)$, hence $M \notin G$, i.e. $M \in T$. Conversely, let $M \in T$. Since $M \subseteq M$ and $M \in \text{Max}(A) \subseteq \text{Spec}(A)$, $M \in \widehat{W}_M \subseteq \bigcup_{M\in T} W_M = Y$. Then $M \in Y$. It is enough to show that $M \in V(I)$. Suppose that $M \notin V(I)$, i.e. $M \in U(I)$. Then $M \in G$. Thus $M \in \text{Max}(A) - T$ and hence $M \notin T$ which is a contradiction.

Let $Y = \bigcup_{M\in T} \widehat{W}_M$ be closed in $\text{Spec}(A)$. Then $Y \cap \text{Max}(A)$ is closed in $\text{Max}(A)$. But $Y \cap \text{Max}(A) = T$. Therefore $T$ is closed in $\text{Max}(A)$. □

**Theorem 4.7.** Let $P$ be a prime filter and $T$ a closed subset of $\operatorname{Max}(A)$. If $\bigcap_{M\in T} W_M \subseteq P$ then $W_M \subseteq P$ for some $M \in T$.

Proof. Suppose that $F = \bigcap_{M\in T} W_M$. Then we have $U(F) = \operatorname{Spec}(A) - \bigcup_{M\in T} \widehat{W}_M$. Let $F \subseteq P$ but $W_M \nsubseteq P$ for all $M \in T$, i.e. $P \notin V(W_M)$. In other words, $P \notin \widehat{W}_M$ for all $M \in T$. Thus $P \notin \bigcup_{M\in T} \widehat{W}_M$, i.e. $P \in U(F)$ and hence $F \nsubseteq P$ which is a contradiction.

The converse of the above Theorem is also true.

**Theorem 4.8.** Let $T \subseteq \operatorname{Max}(A)$, $F = \bigcap_{M\in T} W_M$ and suppose that each prime filter $P$, $F \subseteq P$ implies that $W_M \subseteq P$ *for some* $M \in T$. Then $F$ is a pure filter and $T$ is a closed set.

Proof. Let U be an open set in $\operatorname{Spec}(A)$. We must show that $U(F)$ is stable. Let $P \in U(F)$, $Q \in \operatorname{Spec}(A)$ and $P \subseteq Q$. Suppose that $Q \notin U(F)$, i.e. $F \subseteq Q$. Then $\bigcap_{M\in T} W_M \subseteq Q$. Thus by Theorem 4.7, $W_M \subseteq Q$ for some $M \in T$, i.e. $Q \in V(W_M)$. Therefore $Q \in \widehat{W}_M$ and $Q \subseteq M$. But $P \subseteq Q \subseteq M$ implies that $P \in \widehat{W}_M$ and hence $P \in V(W_M)$. Therefore $W_M \subseteq P$. On the other hand $F \subseteq W_M \subseteq P$. Thus we have $F \subseteq P$. This means that $P \notin U(F)$ which is a contradiction. Therefore we have $Q \in U(F)$. $\square$

**Corollary 4.6.** From Theorems 4.7 and 4.8, pure filters correspond to close subsets of $\operatorname{Max}(A)$.

In the next proposition we prove that a prime filter having a certain condition is a pure filter and vice versa.

**Proposition 4.1.** Let $P$ be a prime filter of $A$. Then $P$ is pure iff $\widehat{W}_{M_P}$ is a chain and $P = W_{M_P}$.

Proof. Suppose that $P$ is a pure filter. Then $U(P)$ is stable and by Theorem 4.2, $U(P) = \operatorname{Spec}(A) - \bigcup_{P\subseteq M} \widehat{W}_M$. Since $\operatorname{Max}(A) \cap V(P) = M_P$, $U(P) = \operatorname{Spec}(A) - \widehat{W}_{M_P}$. We show that for each $Q \in \widehat{W}_{M_P}$, $P \subseteq Q$. Let $Q \in \widehat{W}_{M_P}$. Then $Q \subseteq M_P$. Since $U(P)$ is stable and $P \subseteq M_P$ by Lemma 4.1, $P \subseteq Q$, i.e. $\widehat{W}_{M_P}$ is a chain and $P$ is a minimal prime filter of $A$. But $W_{M_P} = \bigcap_{Q\in\widehat{W}_{M_P}} Q = \bigcap_{Q\subseteq M_P} Q = P$ (since $P \subseteq M_P$ and $P$ is minimal prime). Therefore $W_{M_P} = P$.
Now let $\widehat{W}_{M_P}$ be a chain and $P = W_{M_P}$. Thus $U(P) = U(W_{M_P}) = \operatorname{Spec}(A) - V(W_{M_P}) = \operatorname{Spec}(A) - \widehat{W}_{M_P}$ and by Remark 4.1, $U(P)$ is stable and hence $P$ is a pure filter. $\square$

Now based on the above we get an equivalent statement for a BL-algebra to be hyperarchimedean, that is

**Corollary 4.7.** A BL-algebra $A$ is hyperarchimedean iff every maximal filter of $A$ is pure.

Proof. Let $A$ be hyperarchimedean. Then by Proposition 2.6, $\mathrm{Spec}(A) = \mathrm{Max}(A)$. Suppose that N is a maximal filter of $A$. Thus $M_N = N$ and $\widehat{W}_{M_N} = \widehat{W}_N = N$. Since $W_{M_N} = W_N = \bigcap_{\{P \in \mathrm{Spec}(A),\ P \in \widehat{W}_N\}} P = N$, by Proposition 4.1, we get that N is a pure filter. Conversely, we know that $\mathrm{Max}(A) \subseteq \mathrm{Spec}(A)$. Let $P \in \mathrm{Spec}(A)$ and $M \in \mathrm{Max}(A)$ such that $P \subseteq M$ but $M \nsubseteq P$. This means that $P \in U(M)$. Since $P \subseteq M$ and every maximal filter is pure, $U(M)$ is stable and hence $M \in U(M)$ which is a contradiction. Thus $M \subseteq P$ and $A$ is hyperarchimedean. □

**Proposition 4.2.** Let $F$ be a filter of $A$ such that $U(F)$ is stable and $G = U(F) \cap \mathrm{Max}(A)$. Then $U(F)$ is minimal among all $U(E)$ such that $G = U(E) \cap \mathrm{Max}(A)$.

Proof. Suppose that $F$ and $E$ are filters of $A$ such that $U(F)$ is stable and $G = U(E) \cap \mathrm{Max}(A)$. Let $P \in U(F)$ and $M_P$ be the unique maximal filter over $P$, i.e. $P \subseteq M_P$. By stability of $U(F)$ we have $M_P \in U(F)$. Thus $M_P \in G$ and $M_P \in U(E)$, i.e. $E \nsubseteq M_P$. Hence $E \nsubseteq P$ and $P \in U(E)$. Therefore $U(F) \subseteq U(E)$. □

**Proposition 4.3.** Let S be a nonempty $\vee$-closed subset of $A$ and $F$ be a filter of $A$ such that $F \cap S = \emptyset$. Then there exists a minimal prime filter $Q$ of $A$ such that $F \subseteq Q$ and $Q \cap S = \emptyset$.

Proof. Let $F$ be a proper filter of $A$. Consider $T = \{P \in \mathrm{Spec}(A) | F \subseteq P,\ F \cap S = \emptyset\}$. By Proposition 2.2, $T$ is nonempty and by Zorn's Lemma, $T$ has a minimal element. □

**Corollary 4.8.** Let S be a nonempty $\vee$-closed subset of $A$ such that $1 \notin S$ then there exists a minimal prime filter $Q$ such that $Q \cap S = \emptyset$.

Proof. Take $F = \{1\}$ and apply Proposition 4.3. □

We recall that if $F$ is a filter of $A$ and $x \in F$ then $x^{\perp} = \{y \in A \,|\, x \vee y = 1\}$ is a filter of $A$ [9].

**Theorem 4.9.** Let $F$ be a filter of $A$ such that $U(F)$ is stable and let $x \in F$. Then we have $x^{\perp} \vee F = A$.

Proof. Let $U(F)$ be stable but $x^{\perp} \vee F \neq A$ for some $x \in F$, i.e. $x^{\perp} \vee F \subset A$. In other words, $x^{\perp} \vee F$ is a proper filter of $A$. Then by Proposition 2.2, there exists a maximal filter $M$ of $A$ such that $x^{\perp} \vee F \subseteq M$. Then we have $x^{\perp} \subseteq M$. We define $T = \{x \vee y \,|\, y \notin M\}$. Since $x = x \vee 0$, $x \in T$ and $T$ is nonempty. Now let $x \vee y$ and $x \vee z$ be two elements of $T$. Then we have $(x \vee y) \vee (x \vee z) = x \vee (y \vee z) \in T$. Since, $M \in \mathrm{Max}(A) \subseteq \mathrm{Spec}(A)$ then $M$ is a prime filter and $y \notin M$ and $z \notin M$. Thus we have $y \vee z \notin M$. Now we claim that $1 \notin T$ (otherwise, if $1 \in T$, $x \vee y = 1$ for $y \notin M$, i.e. $y \in x^{\perp}$ and since $x^{\perp} \subseteq M$, $y \in M$ which is a contradiction). Therefore by Corollary 4.8, there exists a minimal prime filter $Q$ such that $Q \cap T = \emptyset$. Then $Q \subseteq M$. On the other hand, since $x = x \vee 0$, we have $F \nsubseteq Q$ which means $Q \in U(F)$. Since $Q \in U(F)$ and $Q \subseteq M$ by stability of $U(F)$, we conclude that $M \in U(F)$, i.e. $F \nsubseteq M$. This is a contradiction since $F \subseteq x^{\perp} \vee F \subseteq M$. □

The converse of Theorem 4.9 is true, that is

**Theorem 4.10.** Let $F$ be a filter of $A$ such that for each $x \in A$, $x^{\perp} \vee F = A$. Then $U(F)$ is stable and hence $F$ is a pure filter.

Proof. Let $P \in U(F)$, $P \subseteq Q$ and $Q \in \mathrm{Spec}(A)$. We must show that $Q \in U(F)$. Assume on the contrary that $Q \notin U(F)$ which implies $F \subseteq Q$. Choose $J \in \mathrm{Min}(A)$ such that $J \subseteq P$. Then $F \nsubseteq J$. In fact if $F \subseteq J \subseteq P$, then $F \subseteq P$ and hence $P \notin U(F)$ which is a contradiction. Hence $F \nsubseteq J$. Thus there exists $x \in F - J$ such that $x^{\perp} \subseteq J$ (otherwise, if for each $x \in F - J$, $x^{\perp} \nsubseteq J$, i.e. there exists $t \in x^{\perp}$, $t \notin J$, we have $t \vee x = 1 \in J$ but $x \notin J$ and $t \notin J$ which is impossible). But $J \subseteq P \subseteq Q$. Then $x^{\perp} \subseteq Q$. Since $F \subseteq Q$, $\langle x^{\perp} \cup F\rangle \subseteq Q$, i.e. $x^{\perp} \vee F \subseteq Q$. Thus $A = Q$. Hence by this contradiction, we conclude that $Q \in U(F)$. □

**Corollary 4.9.** Let $F$ be a filter of $A$. Then $U(F)$ is stable iff $U(F) = \bigcup_{x \in F} V(x^{\perp})$.

Proof. Let $U(F) = \bigcup_{x \in F} V(x^{\perp})$, $P \in U(F)$, $Q \in \mathrm{Spec}(A)$ and $P \subseteq Q$. Then $F \nsubseteq P$. Therefore there exists $x_1 \in F$ such that $x_1^{\perp} \subseteq P \subseteq Q$ and hence $P \in V(x_1)^{\perp}$, that is $x_1^{\perp} \subseteq Q$. This implies that $Q \in V(x_1^{\perp}) \subseteq \bigcup_{x_1 \in F} V(x_1^{\perp}) = U(F)$. Thus $U(F)$ is stable.

Now let $P \in U(F)$, i.e. $F \nsubseteq P$. Therefore, there exists an element $x_1 \in F - P$ such that $x_1^{\perp} \subseteq P$, that is $P \in V(x_1^{\perp}) \subseteq \bigcup_{x_1 \in F} V(x_1^{\perp})$.

Conversely let $P \in \bigcup_{x_1 \in F} V(x_1^{\perp})$. Then there is an element $y \in F$ such that $P \in V(y^{\perp})$, i.e. $y^{\perp} \subseteq P$. We see that $F \nsubseteq P$. In fact if $F \subseteq P$, since $y^{\perp} \subseteq P$, we conclude that $y^{\perp} \vee F \subseteq P$. By the stability of $U(F)$ and Theorem 4.9, we have $A \subseteq P$ which is a contradiction and hence $P \in U(F)$. □

**Proposition 4.4.** Let $F$ be a filter of $A$ such that $U(F)$ is stable and $P_1, P_2 \in \mathrm{Min}(A)$, $M \in \mathrm{Max}(A)$ such that $P_1, P_2 \subseteq M$. Then we have $F \subseteq P_1$ iff $F \subseteq P_2$.

Proof. Let $P_1$ and $P_2$ be two minimal prime filters contained in a some maximal filter M. Suppose that $F \subseteq P_1$ but $F \nsubseteq P_2$. Since $U(F)$ is stable, $M \in U(F)$ i.e. $F \nsubseteq M$, which is a contradiction with $F \subseteq P_1 \subseteq M$. □

**Corollary 4.10.** Let $F$ be a pure filter of $A$ and $M \in \mathrm{Max}(A)$. Then for all $P \in \widehat{W}_M$ either $F \subseteq P$ or $F \nsubseteq P$.

Proof. An immediate consequence of Proposition 4.4. □

We recall that $\sigma(F) = \{a \in A \,|\, x \wedge y = 0 \text{ for some } x \in F \text{ and } y \in a^{\perp}\}$ where $F$ is a filter of $L(A)$ [9].

**Proposition 4.5.** (Leustean [9]) Let $F$ be a filter of $L(A)$. Then $\sigma(F)$ is a filter of $A$ and $\sigma(F) \subseteq F$.

Based on Proposition 4.5, since any filter of $A$ is a filter of $L(A)$, $\sigma(F)$ is a filter of $A$ for every filter $F$ of $A$.

**Corollary 4.11.** Let $F$ be a filter of $A$ such that $\sigma(F) = F$. Then $U(F)$ is stable and $F$ is a pure filter.

Proof. By Corollary 4.9, It is enough to show that $U(F) = \bigcup_{x \in F} V(x^{\perp})$. Let $P \in U(F)$. Then $F \nsubseteq P$, i.e. there exists $x \in F - P$. Therefore $x^{\perp} \subseteq P$. Thus $P \in V(x^{\perp})$ for some $x \in F$ and hence $P \in \bigcup_{x \in F} V(x^{\perp})$. Conversely, we must show that $\bigcup_{x \in F} V(x^{\perp}) \subseteq U(F)$. Let $P \notin U(F)$ but $P \in V(x^{\perp})$ for some $x \in F$, i.e. $x^{\perp} \subseteq P$ for some $x \in F$. Since $\sigma(F) = F$, $x \in \sigma(F)$. Thus there exists $y \in x^{\perp}$ and $z \in F$ such that $y \wedge z = 0$. Since $x^{\perp} \subseteq P$ and $P \notin U(F)$, i.e. $F \subseteq P$, we conclude that $y \in P$ and $z \in P$. Therefore $y \odot z \in P$. But $y \odot z \leqslant y \wedge z = 0$. Hence $z \odot y = 0$, i.e. $0 \in P$ which is a contradiction. □

It is easy to see that $\sigma(\{1\}) = \{1\}$ and $\sigma(A) = A$. Then $A$ and $\{1\}$ are pure filters of $A$. Also we know that if $A$ is a nontrivial BL-algebra, then $\mathrm{Rad}(A)$ i.e. the intersection of all maximal filters of $A$ is a proper filter. Now let $A$ be a semisimple BL-algebra, that is $\mathrm{Rad}(A) = \{1\}$ then $\mathrm{Rad}(A)$ is a pure filter.

**Proposition 4.6.** (Leustean [9]) For each proper filter $F$ of a local BL-algebra $A$, $\sigma(F) = \{1\}$.

**Corollary 4.12.** Let $A$ be a local BL-algebra. Then $\{1\}$ is the unique proper pure filter of $A$.

Proof. It is an immediate consequence of Proposition 4.6 and Corollary 4.11 since $\{1\}$ is the only filter $F$ satisfying $\sigma(F) = \{1\}$. We conclude that in each local BL-algebra $A$, the only pure filters are $A$ and $\{1\}$. □

**Corollary 4.13.** Let $A$ be a BL-chain. Then $A$ and $\{1\}$ are the only pure filters.

Proof. Since each BL-chain is a local BL-algebra ([11]), the Corollary follows from Corollary 4.11 □

**Corollary 4.14.** In each local BL-algebra $A$, the stable topology is trivial.

Proof. Let $T = U(F)$ for some filter $F$ of $A$ be a stable open set of $\mathrm{Spec}(A)$. Then by Corollary 4.12, we have $T = U(\{1\})$ or $T = U(A)$. Therefore $T = \emptyset$ or $T = \mathrm{Spec}(A)$. □

**Corollary 4.15.** Let $A$ and $B$ be BL-algebras, $h : A \longrightarrow B$ a surjective BL-morphism and $F$ be a proper filter of $A$ such that $\sigma(F) = F$. Then we have $\sigma(h(F)) = h(\sigma(F)) = h(F)$. In other words, $U(h(F))$ is stable and hence $h(F)$ is a pure filter of $B$.

Proof. Suppose that $F$ is a proper filter of $A$. By Proposition 2.1, since $h$ is surjective, $h(F)$ is a proper filter of $B$. Now by Proposition 4.5, $\sigma(h(F)) \subseteq h(F)$. But $\sigma(F) = F$. Therefore, $\sigma(h(F)) \subseteq h(\sigma(F))$. Conversely, let $x \in h(\sigma(F))$. Then $x = h(k)$ for some $k \in \sigma(F)$, i.e. $y \wedge z = 0$ for some $y \in k^{\perp}$ and some $z \in F$. Take $l = h(y)$ and $s = h(z)$. It is easy to see that $s \in h(F)$ and $l \in h(k)^{\perp}$. Since $y \in k^{\perp}$, $y \vee k = 1$ and $l \vee s = h(y) \vee h(k) = h(y \vee k) = h(1) = 1$. On the other hand, $l \wedge s = h(y) \wedge h(z) = h(y \wedge z) = h(0) = 0$. This means that $x = h(k) \in \sigma(h(F))$. $\square$

We like to give another proof for Corollary 4.15 from Corollary 4.9, i.e. it is enough to show that $U(h(F)) = \bigcup_{l\in h(F)} V(l^{\perp})$.

Let $P$ be a prime filter of $B$, $P \in U(h(F))$. Then $h(F) \nsubseteq P$, i.e. $h(x) \notin P$ for some $l = h(x) \in h(F)$. Thus $l^{\perp} = h(x)^{\perp} \subseteq P$, i.e. $P \in V(l^{\perp})$ and hence $P \in \bigcup_{l\in h(F)} V(l^{\perp})$. Conversely, let $P \in \mathrm{Spec}(B)$ and $P \notin U(h(F))$ but $P \in \bigcup_{l\in h(F)} V(l^{\perp})$, that is $h(F) \subseteq P$ and $l^{\perp} \subseteq P$ for some $l \in h(F)$. Since $l \in h(F)$, $l = h(x)$ for some $x \in F$. By Proposition 2.1, $h^{-1}(P) \in \mathrm{Spec}(A)$. Now we claim that $x^{\perp} \subseteq h^{-1}(P)$. Suppose that $v \in x^{\perp}$. Then $v \vee x = 1$ and hence $h(v \vee x) = h(v) \vee h(x) = 1$. Therefore, $h(v) \in h(x)^{\perp}$ and by above $h(v) \in P$, i.e. $v \in h^{-1}(P)$. Thus $h^{-1}(P) \in V(x^{\perp}) \subseteq \bigcup_{m\in F} V(m^{\perp})$. Since $\sigma(F) = F$, by Corollary 4.11, $U(F)$ is stable and by Corollary 4.9 we conclude that $\bigcup_{m\in F} V(m^{\perp}) = U(F)$. Thus, $h^{-1}(P) \in U(F)$ and $F \nsubseteq h^{-1}(P)$. This means that $h(F) \nsubseteq P$ and $P \in U(h(F))$, which contradicts our assumption.

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