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Applications of Mathematics, Vol. 54 (2009), No. 6, 491–525

Persistent URL: `http://dml.cz/dmlcz/140381`

MODELLING AND CONTROL IN PSEUDOPLATE PROBLEM WITH DISCONTINUOUS THICKNESS*

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(Received July 17, 2006, in revised version January 6, 2007)

Abstract. This paper concerns an obstacle control problem for an elastic (homogeneous) and isotropic) pseudoplate. The state problem is modelled by a coercive variational inequality, where control variable enters the coefficients of the linear operator. Here, the role of control variable is played by the thickness of the pseudoplate which need not belong to the set of continuous functions. Since in general problems of control in coefficients have no optimal solution, a class of the extended optimal control is introduced. Taking into account the results of G-convergence theory, we prove the existence of an optimal solution of extended control problem. Moreover, approximate optimization problem is introduced, making use of the finite element method. The solvability of the approximate problem is proved on the basis of a general theorem. When the mesh size tends to zero, a subsequence of any sequence of approximate solutions converges uniformly to a solution of the continuous problem.

Keywords: control of variational inequalities, optimal design, minimization, pseudoplate with obstacles, cost functional, thickness, G-convergence, coercive variational inequality, approximate optimization problem, finite element

MSC 2010: 49J40, 49K30, 93C30, 65N30

Introduction

The paper is devoted to an optimal control problem for an elastic pseudoplate (a plate with small bending rigidity). The bending of the pseudoplate is described by means of a shear model, the plate is deformed only by shear force (see [2]). In classical elasticity theory a pseudoplate is a plate offering resistance to bending when only a shear is acting. We assume that a homogeneous and isotropic pseudoplate occupying a domain $\Omega \times (-e, e)$ of the space $\mathbb{R}^3$ is loaded by a transversal distributed

* This work was supported by Grant 1/3312/06 of the Grant Agency of Slovak Republic.

force $q(x_1, x_2)$ perpendicular to the plane Ox_1x_2. The pseudoplate is supported unilaterally by a finite number of rigid obstacles (punches). The role of control variable is played by the thickness of the pseudoplate. The control variable has to belong to a compact subset of $L_\infty(\Omega)$. The cost functional represents a norm of the deflection function. The state problem is modelled by a variational inequality, where the control variable influences the coefficients of the linear monotone operator. Existence of an optimal control is proven on the abstract level for a class of extended optimal control. We prove the continuous dependence of the solutions of state variational inequalities with respect to the G-convergence of the operators involved.

Next the optimal control problem has to be solved approximately. We restrict ourselves to a pseudoplate-beam problem. We employ the simplest kind of finite elements, namely piecewise linear functions over beam elements. In this way the space of state functions and the set of admissible control variables are discretized. We prove that approximate control problem has at least one solution on the basis of the general theorem for discretization problem. Finally, we study the convergence of approximate solutions when the mesh size tends to zero.

1. Setting of the problem

Let us assume that the midplane of the pseudoplate occupies a given bounded and simply connected domain $\Omega \subset \mathbb{R}^2$ with regular boundary $\partial\Omega$. We denote the standard Sobolev function spaces by $H^k(\Omega) \equiv W_2^k(\Omega)$, $k = 1, 2$. Let the norm in $H^1(\Omega)$ be denoted by $\|\cdot\|_{H^1(\Omega)}$, where $\|v\|_{H^1(\Omega)} := \left(\int_\Omega (|v|^2 + |\mathbf{grad}\, v|^2)\, \mathrm{d}\Omega\right)^{1/2}$.

Here $H^k(\Omega)$, $k = 1, 2$ is a Hilbert space with the scalar product:

$$(v, z)_{H^k(\Omega)} := \sum_{|\alpha| \leqslant k} \int_\Omega D^\alpha v\, D^\alpha z\, \mathrm{d}\Omega, \quad |\alpha| = \alpha_1 + \alpha_2, \quad D^\alpha = \partial^{|\alpha|}/\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2.$$

Further we denote $H_0^k(\Omega) = \{v \in H^k(\Omega)\colon\ v = D^\alpha v = 0 \text{ on } \partial\Omega \text{ for } |\alpha| \leqslant k-1\}$, the Hilbert space with the following scalar product: $(v, z)_{H_0^k(\Omega)} := \sum\limits_{|\alpha| = k} \int_\Omega D^\alpha v\, D^\alpha z\, \mathrm{d}\Omega$.

The expression $|v|_{H^k(\Omega)} := \sqrt{\sum\limits_{|\alpha|=k} \int_\Omega [D^\alpha v]^2\, \mathrm{d}\Omega}$, defines the seminorm in $H^k(\Omega)$.

In the following $L_2(\Omega)$ and $L_\infty(\Omega)$ denote the space of square Lebesgue integrable functions on Ω and the space of measurable essentially bounded functions on Ω, respectively. The functional $\|\cdot\|_{L_\infty(\Omega)}$ defined by $\|\cdot\|_{L_\infty(\Omega)} := \operatorname*{ess\,sup}\limits_{[x_1,x_2]\in\Omega} |v(x_1, x_2)|$ is a norm on $L_\infty(\Omega)$.

We shall denote by $C^m(\overline{\Omega})$ the space of m-times continuously differentiable real valued functions for which all the derivatives up to order m are continuous in $\overline{\Omega}$ and the space $C_0^m(\overline{\Omega})$ given by

$$C_0^m(\Omega) := \{v \in C_0^m(\overline{\Omega})\colon \operatorname{supp}(v) \text{ is a compact subset of } \Omega\}.$$

The Lipschitz boundary $\partial\Omega$ is decomposed as follows:

$$\partial\Omega = \partial\overline{\Omega}_{\text{displacement}} \cup \partial\overline{\Omega}_{\text{contact}},$$

where $\partial\Omega_{\text{displacement}}$ and $\partial\Omega_{\text{contact}}$ are open, non-overlapping parts of $\partial\Omega$, $\partial\Omega_{\text{displacement}} \neq \emptyset$, $\partial\Omega_{\text{contact}} \neq \emptyset$. On each part of $\partial\Omega$ different boundary conditions will be prescribed. On $\partial\Omega_{\text{displacement}}$ the kinematic condition is given

$$\gamma_{\square} v = 0 \quad \text{on } \partial\Omega_{\text{displacement}},$$

where $\gamma_{\square}$ is a linear, continuous and compact trace operator: $H^1(\Omega) \to L_2(\partial\Omega)$ such that $\gamma_{\square} v = v|_{\partial\Omega}$ for $v \in C(\overline{\Omega})$.

Moreover, the pseudoplate is subject to contact with friction on a part $\partial\Omega_{\text{contact}}$ of the boundary $\partial\Omega$, with friction bound $\mathscr{F}\colon \partial\Omega_{\text{contact}} \to \mathbb{R}^+$. The transversal displacements v belong to the space

$$V(\Omega) := \{v \in H^1(\Omega)\colon \gamma_{\square} v = 0 \text{ on } \partial\Omega_{\text{displacement}}\}.$$

For derivation of the load-deflection relation we consider a vertical equilibrium of a "shear element" cut out by the surfaces $[x_1, x_1 + \mathrm{d}x_1]$ and $[x_2, x_2 + \mathrm{d}x_2]$ as shown in Fig. 1.

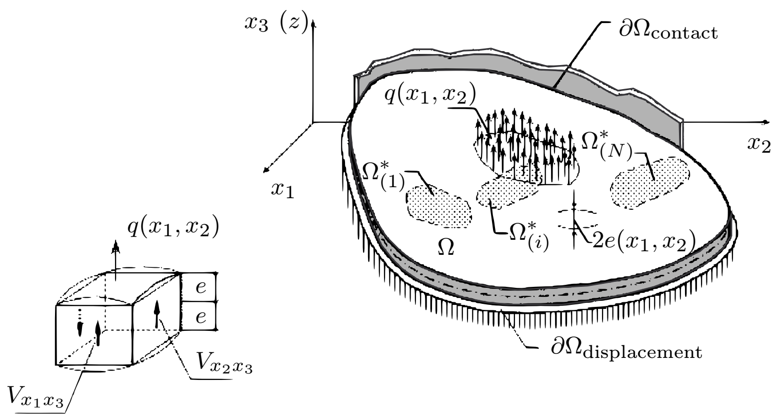

Figure 1. Geometry of a pseudoplate with curved boundary.

Let $v(x_1,x_2)$ and $[\tau_{x_1x_3}(x_1,x_2),\tau_{x_2x_3}(x_1,x_2)]$ denote arbitrary transversal displacement and stress field in the pseudoplate. We consider Hookean elastic material. For stress-strain relations we have the following relations

$$\begin{cases} \tau_{x_1x_3}(x_1,x_2) := KG\varepsilon_{x_1x_3}(x_1,x_2) = \dfrac{1}{2}KG\,\dfrac{\partial v(x_1,x_2)}{\partial x_1}, \\ \tau_{x_2x_3}(x_1,x_2) := KG\varepsilon_{x_2x_3}(x_1,x_2) = \dfrac{1}{2}KG\,\dfrac{\partial v(x_1,x_2)}{\partial x_2}, \end{cases}$$

where the term K is shear correction factor (positive constant) and G is a constant elastic shear modulus.

Then shear forces per unit length of the pseudoplate are given by

$$\begin{cases} V_{x_1x_3}(x_1,x_2) := \displaystyle\int_{-e(x_1,x_2)}^{e(x_1,x_2)} \tau_{x_1x_3}(x_1,x_2)\,\mathrm{d}x_3 = KGe\,\frac{\partial v(x_1,x_2)}{\partial x_1}, \\ V_{x_2x_3}(x_1,x_2) := \displaystyle\int_{-e(x_1,x_2)}^{e(x_1,x_2)} \tau_{x_2x_3}(x_1,x_2)\,\mathrm{d}x_3 = KGe\,\frac{\partial v(x_1,x_2)}{\partial x_2}. \end{cases}$$

The equilibrium equation for a pseudoplate (see the figure) without any internal obstacles has the form

$$\frac{\partial V_{x_1x_3}(x_1,x_2)}{\partial x_1} + \frac{\partial V_{x_2x_3}(x_1,x_2)}{\partial x_2} + q(x_1,x_2) = 0 \quad \text{in } \Omega$$
$$\Rightarrow KG\Big[\frac{\partial}{\partial x_1}\Big(e(x_1,x_2)\frac{\partial v(x_1,x_2)}{\partial x_1}\Big) + \frac{\partial}{\partial x_2}\Big(e(x_1,x_2)\frac{\partial v(x_1,x_2)}{\partial x_2}\Big)\Big] = -q(x_1,x_2).$$

Then one has

$$KGe(\partial v/\partial n) = V_{nx_3} \quad \text{on } \partial\Omega,$$

where n is the unit outward normal to $\partial\Omega$ and $\partial/\partial n$ is the normal derivative, $V_{nx_3} = \sum_{i=1}^{2} V_{x_ix_3} n_{x_i}$.

Now on the part $\partial\Omega_{\text{contact}}$ we prescribe a slip limit $\mathscr{F}$ and the following friction conditions: If the reaction force $|V_{nx_3}| = KGe|\partial v/\partial n|$ is below a certain value, the friction is not overcome and there is no displacement v; if it reaches this value, there is a displacement in the direction opposite to the force

$$\begin{cases} |V_{nx_3}| < \mathscr{F} \Rightarrow v = 0 \text{ on } \partial\Omega_{\text{contact}}, \\ |V_{nx_3}| = \mathscr{F} \Rightarrow \text{there exists } \lambda \geqslant 0 \text{ such that } v = -\lambda V_{nx_3}(v) \text{ for some } \lambda \geqslant 0. \end{cases}$$

This condition is the friction condition. At each point of $\partial\Omega_{\text{contact}}$ either the surface force $KG|\partial v/\partial n|$ is less than the friction bound $\mathscr{F}$ and then the pseudoplate remains in its original position because of friction, or $KGe|\partial v/\partial n| = \mathscr{F}$, the force is limiting, the pseudoplate may slip and the new equilibrium position is in the opposite direction to the friction force. The part of $\partial\Omega_{\text{contact}}$, where $KGe|\partial v/\partial n| < \mathscr{F}$ is called the stick region and the other part where $KGe|\partial v/\partial n| = \mathscr{F}$ is the slip region.

We consider several unilateral inner obstacles on Ω as follows.

For mutually disjoint subdomains $\overline{\Omega}_{0\langle i\rangle} \subset \Omega$, $\overline{\Omega}_{0\langle i\rangle} \cap \overline{\Omega}_{0\langle j\rangle} = \emptyset$ for $i \neq j$, $i = 1, 2, \ldots, N$, we introduce the set of admissible transversal displacements (the closed convex subset of $H^1(\Omega)$ associated with the Dirichlet data on $\partial\Omega_{\text{displacement}}$ and the unilateral inner obstacles)

$$\begin{aligned}\mathscr{K}(\Omega) := \{v \in V(\Omega)\colon\ v(x_1, x_2) \geqslant 0 \text{ for a.e. } [x_1, x_2] \in \Omega_{0\langle i\rangle},\\ i = 1, 2, \ldots, N\}.\end{aligned} \tag{1.1}$$

Now we describe the optimal control problem considered here. First, let $\mathscr{U}(\Omega) = L_\infty(\Omega)$ (control space) and consider the set of admissible control functions, given by

$$\mathscr{U}_{\text{ad}}(\Omega) := \{e \in \mathscr{U}(\Omega)\colon\ 0 < M_1 \leqslant e(x_1, x_2) \leqslant M_2 \text{ for a.e. } [x_1, x_2] \in \Omega\} \tag{1.2}$$

with given positive constants M_1 and M_2.

Here $\mathscr{U}_{\text{ad}}(\Omega)$ is nonempty, convex, bounded and weakly star (and hence strongly) closed in $\mathscr{U}(\Omega)$. Then $\mathscr{U}_{\text{ad}}(\Omega)$ is a compact subset of $L_\infty(\Omega)$ with respect to the L_∞-weakly star convergence.

For an arbitrary fixed $e \in \mathscr{U}_{\text{ad}}(\Omega)$ let the state function of the control system be given by solution of the nonlinear elliptic boundary value problem

$$\begin{cases} \mathscr{R}(e)u \geqslant q & \text{in the space of measures } \mathscr{M}(\Omega),\\ u = 0 & \text{on } \partial\Omega_{\text{displacement}},\\ & (\text{in } H^{1/2}(\partial\Omega_{\text{displacement}})),\\ u \geqslant 0 & \text{a.e. on } \Omega_{0\langle i\rangle},\ i = 1, 2, \ldots, N,\\ (\mathscr{R}(e)u - q)u = 0 & \text{a.e. on } \Omega_{0\langle i\rangle},\ i = 1, 2, \ldots, N,\\ |\partial u/\partial n| \leqslant \mathscr{F} & \text{on } \partial\Omega_{\text{contact}},\\ |\partial u/\partial n| < \mathscr{F} \Rightarrow u = 0, & \\ |\partial u/\partial n| = \mathscr{F} \Rightarrow u = -\lambda\partial u/\partial n & \text{for some } \lambda \geqslant 0, \end{cases} \tag{1.3}$$

where

$$\mathscr{R}(e)v = -\operatorname{div}(KGe\,\mathbf{grad}\,v).$$

When no more regularity that $H^1(\Omega)$ is available for the solution u, the normal derivative $\partial u/\partial n$ does exist in $H^{-1/2}(\partial\Omega_{\text{contact}})$ as $q \in L_2(\Omega)$ (since $\mathscr{R}(e)u \in L_2(\Omega)$), but cannot be expressed pointwise.

We turn to the variational formulation of (1.3). We define the friction functional $\Phi\colon H^1(\Omega)\to\mathbb{R}^+$ by

$$\Phi(v)=\int_{\partial\Omega_{\text{contact}}}\mathscr{F}|\gamma_{\square}v|\,\mathrm{d}S, \tag{1.4}$$

where $\mathscr{F}\in L_p(\partial\Omega)$, $2\leqslant p<\infty$, $\mathscr{F}\geqslant 0$.

Since the traces of functions in $H^1(\Omega)$ are in $H^{1/2}(\partial\Omega_{\text{contact}})$, the functional $\Phi(v)$ is well defined. Let $v\in V(\Omega)$, $V^*(\Omega)$ be the dual space, $\langle\cdot,\cdot\rangle_{V(\Omega)}$ the dual pairing between $V(\Omega)$ and $V^*(\Omega)$. We associate with $\mathscr{U}_{\text{ad}}(\Omega)$ the bilinear form $a(e,\cdot,\cdot)$ by

$$a(e,v,z):=\int_\Omega KGe[(\partial v/\partial x_1)(\partial z/\partial x_1)+(\partial v/\partial x_2)(\partial z/\partial x_2)]\,\mathrm{d}\Omega \tag{1.5}$$

for all $[v,z]\in V(\Omega)$, $e\in\mathscr{U}_{\text{ad}}(\Omega)$, or

$$a(e,v,z)=\int_\Omega\langle KGe[E_{ij}]\cdot\mathbf{grad}\,v,\mathbf{grad}\,z\rangle_{\mathbb{R}^2}\,\mathrm{d}\Omega, \tag{1.6}$$

where $[E_{ij}]_{i,j=1,2}$ is the unit 2×2 matrix. Moreover, we introduce the functional $L\in V^*(\Omega)$ by means of

$$\langle L,v\rangle_{V(\Omega)}:=\langle q,v\rangle_{L_2(\Omega)}\,. \tag{1.7}$$

Denote by $\mathcal{A}(e)\in L(V(\Omega),V^*(\Omega))$, $e\in\mathscr{U}_{\text{ad}}(\Omega)$, the elliptic operator is associated with the bilinear form $a(e,\cdot,\cdot)$, i.e.,

$$\langle\mathcal{A}(e)v,z\rangle_{V(\Omega)}=a(e,v,z)\quad\text{for all }\ v,z\in V(\Omega). \tag{1.8}$$

On the basis of the virtual displacements principle, we may define the *state problem*:

$$\begin{cases}\text{Given any } e\in\mathscr{U}_{\text{ad}}(\Omega),\ \text{find } u(e)\in\mathscr{K}(\Omega) \text{ such that}\\ \langle\mathcal{A}(e)u(e),v-u(e)\rangle_{V(\Omega)}+\Phi(v)-\Phi(u(e))\geqslant\langle L,v-u(e)\rangle_{V(\Omega)}\\ \text{for all } v\in\mathscr{K}(\Omega).\end{cases} \tag{1.9}$$

We introduce the cost functional of the type

$$\mathscr{L}(u)=\int_\Omega[u-z_{\text{ad}}]^2\,\mathrm{d}\Omega, \tag{1.10}$$

where $z_{\text{ad}}\in L_2(\Omega)$ is a given element.

Here we define the following optimal control problem: Find

$$e_{\langle\square\rangle}=\operatorname*{Arg\,Min}_{e\in\mathscr{U}_{\text{ad}}(\Omega)}\mathscr{L}(u(e)), \tag{1.11}$$

where $u(e)$ denotes the solution of the state problem (1.9).

2. Existence of a solution to the optimal control problems

First of all, we consider a general class of nonsmooth optimal control problems. Let $\mathscr{U}(\Omega)$ denote a Banach space and let $\mathscr{U}_{\text{ad}}(\Omega)$ be a subset of $\mathscr{U}(\Omega)$. Let a Hilbert space $V(\Omega)$ be given with a norm $\|\cdot\|_{V(\Omega)}$ and $\langle\cdot,\cdot\rangle_{V(\Omega)}$ the duality pairing between $V(\Omega)$ and $V^*(\Omega)$. Next let $\mathscr{K}(\Omega)$ be a convex closed subset of $V(\Omega)$ and $0 \in \mathscr{K}(\Omega)$. Moreover, the space $V(\Omega)$ is considered in a real Hilbert space $H(\Omega)$, where the inclusion from $V(\Omega)$ into $H(\Omega)$ is compact and $H(\Omega)$ is identified with its own dual.

For a Hilbert space $\mathscr{H}(\Omega)$ we denote by $L(\mathscr{H}(\Omega), \mathscr{H}^*(\Omega))$ the space of all linear continuous operators from $\mathscr{H}(\Omega)$ into $\mathscr{H}^*(\Omega)$ endowed with the usual operator norm. On the other hand for two positive constants $[\alpha, M]$ we denote $\mathcal{B}_{\mathscr{H}(\Omega)}([\alpha, M])$ the set of all symmetric elements Π of $L(\mathscr{H}(\Omega), \mathscr{H}^*(\Omega))$ for which the inequalities

$$\begin{aligned} \alpha\|v\|^2_{\mathscr{H}(\Omega)} &\leqslant \langle \Pi v, v\rangle_{\mathscr{H}(\Omega)}, \\ \|\Pi v\|_{\mathscr{H}^*(\Omega)} &\leqslant M\|v\|_{\mathscr{H}(\Omega)} \end{aligned} \tag{2.1}$$

hold for all $v \in \mathscr{H}(\Omega)$.

Definition. Let Q_k and Q be in $\mathcal{B}_{V(\Omega)}([\alpha, M])$, $k \in \mathbb{N}$. We say that $\{Q_k\}_{k\in\mathbb{N}}$ is G-convergent to Q (in symbols $Q_k \xrightarrow{G} Q$) for $k \to \infty$ if, for any f and g in $V^*(\Omega)$,

$$\lim_{k\to\infty} \langle g, Q_k^{-1} f\rangle_{V(\Omega)} = \langle g, Q^{-1} f\rangle_{V(\Omega)}.$$

We introduce the system $\{\mathcal{A}(\mathscr{O})\}$ of linear symmetric operators $\mathcal{A}(\mathscr{O}) \in L(V(\Omega), V^*(\Omega))$, $\mathscr{O} \in \mathscr{U}_{\text{ad}}(\Omega)$ satisfying the following assumptions

$$\text{(H1)}\quad \begin{cases} 1^\circ & \{\mathcal{A}(\mathscr{O})\}_{\mathscr{O}\in\mathscr{U}_{\text{ad}}(\Omega)} \subset \mathcal{B}_{V(\Omega)}([\alpha_{\langle\square\rangle}, M_{\langle\square\rangle}]). \\ 2^\circ & \text{Any sequence of linear operators } \{\mathcal{A}(\mathscr{O}_n)\}_{\mathscr{O}_n\in\mathscr{U}_{\text{ad}}(\Omega)} \text{ contains a} \\ & G\text{-convergent subsequence } \{\mathcal{A}(\mathscr{O}_{n_k})\}_{\mathscr{O}_{n_k}\in\mathscr{U}_{\text{ad}}(\Omega)} \text{ such that} \\ & \mathcal{A}(\mathscr{O}_{n_k}) \xrightarrow{G} \mathcal{A}(\mathscr{O}_{\langle\square\rangle}),\ \text{where } \mathscr{O}_{\langle\square\rangle} \in \mathscr{U}_{\text{ad}}(\Omega). \\ 3^\circ & \text{For every sequence } \{\mathscr{O}_n\}_{n\in\mathbb{N}}\ (\subset \mathscr{U}_{\text{ad}}(\Omega)) \text{ and } \{v_n\}_{n\in\mathbb{N}}\ (\subset V(\Omega)) \\ & \text{a sequence } \{\mathcal{A}(\mathscr{O}_n)v_n\}_{n\in\mathbb{N}}\ (\subset V^*(\Omega)) \text{ is relative compact in } V^*(\Omega) \\ & \text{strongly.} \\ 4^\circ & \mathcal{A}^{-1}(\mathscr{O}_{n_k})\mathcal{K}_k \to \mathcal{A}^{-1}(\mathscr{O}_{\langle\square\rangle})\mathcal{K} \text{ (strongly) in } H(\Omega) \\ & \text{as } \mathcal{A}(\mathscr{O}_{n_k}) \xrightarrow{G} \mathcal{A}(\mathscr{O}_{\langle\square\rangle}) \text{ and } \mathcal{K}_k \to \mathcal{K} \text{ (strongly) in } V^*(\Omega), \\ & \text{where } [\mathscr{O}_{n_k}, \mathscr{O}_{\langle\square\rangle}] \in \mathscr{U}_{\text{ad}}(\Omega) \text{ and } [\mathcal{K}_k, \mathcal{K}] \in V^*(\Omega). \end{cases}$$

Let $\Phi\colon V(\Omega)\to\mathbb{R}$ be a strongly continuous functional such that

$$\text{(H2)}\qquad \begin{cases}\Phi(v+z)\leqslant\Phi(v)+\Phi(z)\text{ for all }[v,z]\in V(\Omega),\\ \Phi(\alpha v)\leqslant|\alpha|\Phi(v)\text{ for all }\alpha\in\mathbb{R},\text{ and for all }v\in V(\Omega).\end{cases}$$

Then from (H2) it follows that $\Phi(\cdot)$ is convex and hence weakly lower semicontinuous. Let us consider the equation

$$\mathcal{A}(\mathscr{O})u+\partial\chi_{\mathscr{K}(\Omega)}(u)+\partial\Phi(u)\ni L, \tag{2.2}$$

where $\chi_{\mathscr{K}(\Omega)}$ is the indicator function of a closed convex set $\mathscr{K}(\Omega)$ $(\subset V(\Omega))$, $\Phi\colon V(\Omega)\to\mathbb{R}$, due to (H2) is proper, convex and lower semicontinuous functional on $V(\Omega)$ and $\partial\chi_{\mathscr{K}(\Omega)}$ (or $\partial\Phi$): $V(\Omega)\to V^*(\Omega)$ is the subdifferential of $\chi_{\mathscr{K}(\Omega)}$ (or Φ, respectively), and $L\in V^*(\Omega)$.

As seen earlier, (2.2) can be rewritten as the variational inequality

$$\begin{cases}u(\mathscr{O})\in\mathscr{K}(\Omega),\\ \langle\mathcal{A}(\mathscr{O})u(\mathscr{O}),v-u(\mathscr{O})\rangle_{V(\Omega)}+\Phi(v)-\Phi(u(\mathscr{O}))\geqslant\langle L,v-u(\mathscr{O})\rangle_{V(\Omega)}\\ \text{for any }v\in\mathscr{K}(\Omega).\end{cases} \tag{2.3}$$

The equation (2.2) (or the inequation (2.3)) itself will be referred to as the state system or control system. It is well known (see [5]) that for every $\mathscr{O}\in\mathscr{U}_{\rm ad}(\Omega)$ there exists a unique solution of (2.3).

On the other hand, since $\mathcal{A}(\mathscr{O}_n)\in\mathcal{B}_{V(\Omega)}(\alpha_{\langle\square\rangle},M_{\langle\square\rangle})$ for any sequence $\{\mathscr{O}_n\}_{n\in\mathbb{N}}\subset\mathscr{U}_{\rm ad}(\Omega)$, we have

$$\lim_{\|v\|_{V(\Omega)}\to\infty}[\langle\mathcal{A}(\mathscr{O}_n)v,v\rangle_{V(\Omega)}+\Phi(v)]/\|v\|_{V(\Omega)}=+\infty. \tag{2.4}$$

Indeed, due to the assumption (H2) one has

$$\begin{cases}\Phi(0)=0,\ \Phi(v)\geqslant 0\\ |\Phi(v)-\Phi(z)|\leqslant\Phi(v-z)\text{ for all }[v,z]\in V(\Omega).\end{cases} \tag{2.5}$$

Thus, by virtue of $((\mathrm{H1}),1^\circ)$ and (2.5) we can write

$$[\langle\mathcal{A}(\mathscr{O}_n)v,v\rangle_{V(\Omega)}+\Phi(v)]\geqslant\alpha_{\langle\square\rangle}\|v\|^2_{V(\Omega)}$$

which gives (2.4).

Let us consider a cost functional

$$\mathscr{L}(\mathscr{O},u(\mathscr{O}))\colon\ \mathscr{U}_{\rm ad}(\Omega)\times V(\Omega)\to\mathbb{R}.$$

We can define $J(\mathscr{O}) := \mathscr{L}(\mathscr{O}, u(\mathscr{O}))$, where a state function $u(\mathscr{O})$ denotes the solution of the state inequality (2.3). Assume that the cost functional $\mathscr{L}(\mathscr{O}, v)$ satisfies the following condition:

$$
\text{(E1)} \qquad \begin{cases} \text{For any minimizing sequence } \{\mathscr{O}_n\}_{n\in\mathbb{N}} \ (\subset \mathscr{U}_{\mathrm{ad}}(\Omega)) \\ \text{i.e., if } \mathscr{O}_n \in \mathscr{U}_{\mathrm{ad}}(\Omega) \text{ for any } n\colon \\ \quad \lim\limits_{n\to\infty} J(\mathscr{O}_n) = \inf\{J(\mathscr{O}),\ \mathscr{O} \in \mathscr{U}_{\mathrm{ad}}(\Omega)\}, \\ \quad \liminf\limits_{n\to\infty} \mathscr{L}(\mathscr{O}_n, v_n) \geqslant \mathscr{L}(\mathscr{O}_{\langle\square\rangle}, u(\mathscr{O}_{\langle\square\rangle})), \\ \text{as } v_n \to u(\mathscr{O}_{\langle\square\rangle}) \text{ weakly in } V(\Omega). \end{cases}
$$

The optimal control problem $(\mathscr{P})$ can be set in the following general form:

$$
(\mathscr{P}) \qquad \begin{cases} \textit{minimize} \text{ the cost functional } J(\mathscr{O}) \text{ on the set } \mathscr{U}_{\mathrm{ad}}(\Omega);\ \text{that is:} \\ \text{find an element } \mathscr{O}_{\langle\square\rangle} \in \mathscr{U}_{\mathrm{ad}}(\Omega) \text{ such that } J(\mathscr{O}_{\langle\square\rangle}) = \inf\{J(\mathscr{O}), \\ \mathscr{O} \in \mathscr{U}_{\mathrm{ad}}(\Omega)\}. \end{cases}
$$

Theorem 1. *Let the assumptions* (H1), (H2) *and* (E1) *be fulfilled. Then there exists at least one solution to* $(\mathscr{P})$.

Proof. Consider a minimizing sequences $\{\mathscr{O}_n\}_{n\in\mathbb{N}}$ of $J(\mathscr{O})$, i.e.,

$$
\begin{cases} \mathscr{O}_n \in \mathscr{U}_{\mathrm{ad}}(\Omega) \text{ for each } n, \\ \lim\limits_{n\to\infty} J(\mathscr{O}_n) = \inf\{J(\mathscr{O}),\, \mathscr{O} \in \mathscr{U}_{\mathrm{ad}}(\Omega)\}. \end{cases}
$$

We can write (in view of (2.5))

$$
\text{(2.6)} \qquad \langle \mathcal{A}(\mathscr{O}_n)u(\mathscr{O}_n) - L, u(\mathscr{O}_n) - \theta\rangle_{V(\Omega)} \leqslant \Phi(\theta) - \Phi(u(\mathscr{O}_n)) \leqslant \Phi(\theta)
$$

for fixed $v = \theta$ in (2.3).

Then due to $((\mathrm{H1}), 1^\circ)$ and by the Schwarz inequality we obtain from (2.6) the estimate

$$
\begin{aligned} \alpha_{\langle\square\rangle} \|u(\mathscr{O}_n)\|^2_{V(\Omega)} \leqslant \|u(\mathscr{O}_n)\|_{V(\Omega)}&[M_{\langle\square\rangle}\|\theta\|_{V(\Omega)} + \|L\|_{V^*(\Omega)}] \\ &+ (\|L\|_{V^*(\Omega)}\|\theta\|_{V(\Omega)} + \text{constant}). \end{aligned}
$$

Therefore, we deduce that $\|u(\mathscr{O}_n)\|_{V(\Omega)} \leqslant$ constant for all n.

This means that $\{u(\mathscr{O}_n)\}_{n\in\mathbb{N}}$ is a bounded sequence. Hence, there exists a subsequence $\{u(\mathscr{O}_{n_k})\}_{k\in\mathbb{N}}$ and an element $u_{\langle\square\rangle} \in V(\Omega)$ such that

$$
\text{(2.7)} \qquad u(\mathscr{O}_{n_k}) \to u_{\langle\square\rangle} \quad \text{weakly in } V(\Omega).
$$

Moreover, one has $u_{\langle\square\rangle} \in \mathscr{K}(\Omega)$ (as $\mathscr{K}(\Omega)$ is a weakly closed set). Thus due to the assumption $((\mathrm{H}1), 3^\circ)$ there exists a subsequence $\{\mathcal{A}(\mathscr{O}_{n_k} u(\mathscr{O}_{n_k})\}_{k\in\mathbb{N}}$ such that

$$\mathcal{A}(\mathscr{O}_{n_k})u(\mathscr{O}_{n_k}) \to Q \quad \text{strongly in } V^*(\Omega) \text{ as } k \to \infty. \tag{2.8}$$

On the other hand (by virtue of $((\mathrm{H}1), 1^\circ, 2^\circ)$, the operator $\mathcal{A}(\mathscr{O}_{\langle\square\rangle})$ is surjective (where $\mathscr{O}_{\langle\square\rangle} \in \mathscr{U}_{\mathrm{ad}}(\Omega)$) and the class of $[\mathcal{A}(\mathscr{O})]$ is G-compact on the set $\mathscr{U}_{\mathrm{ad}}(\Omega)$ and $\mathcal{A}(\mathscr{O}_{n_k}) \xrightarrow{G} \mathcal{A}(\mathscr{O}_{\langle\square\rangle})$; that is, there exists an element $u_o \in V(\Omega)$ such that

$$\mathcal{A}(\mathscr{O}_{\langle\square\rangle})u_o = Q. \tag{2.9}$$

It results from (2.8) and (2.9) that

$$\mathcal{A}(\mathscr{O}_{n_k})u(\mathscr{O}_{n_k}) \to \mathcal{A}(\mathscr{O}_{\langle\square\rangle})u_o \quad \text{strongly in } V^*(\Omega). \tag{2.10}$$

Next due to the condition $((\mathrm{H}1),\ 4^\circ)$ we obtain $\lim\limits_{k\to\infty} \|u(\mathscr{O}_{n_k}) - u_o\|_{H(\Omega)} = 0$. But by (2.7) (since the embedding of $V(\Omega)$ into $H(\Omega)$ is compact), we have $\lim\limits_{k\to\infty} \|u(\mathscr{O}_{n_k}) - u_{\langle\square\rangle}\|_{H(\Omega)} = 0$. This means that $u_o = u_{\langle\square\rangle}$.

Then we may write

$$\mathcal{A}(\mathscr{O}_{n_k})u(\mathscr{O}_{n_k}) \to \mathcal{A}(\mathscr{O}_{\langle\square\rangle})u_{\langle\square\rangle} \quad \text{strongly in } V^*(\Omega). \tag{2.11}$$

Hence, we can now pass to the limit in the inequality (due to (2.7), (2.11) and taking into account a weak lower semicontinuity of $\Phi(\cdot)$)

$$\begin{aligned}&\langle \mathcal{A}(\mathscr{O}_{n_k})u(\mathscr{O}_{n_k}), u(\mathscr{O}_{n_k})\rangle_{V(\Omega)} + \Phi(u(\mathscr{O}_{n_k}))\\ &\qquad \leqslant \langle L, u(\mathscr{O}_{n_k}) - v\rangle_{V(\Omega)} + \Phi(v) + \langle \mathcal{A}(\mathscr{O}_{n_k})u(\mathscr{O}_{n_k}), v\rangle_{V(\Omega)}.\end{aligned}$$

Consequently, we obtain

$$\big\langle \mathcal{A}(\mathscr{O}_{\langle\square\rangle})u_{\langle\square\rangle}, v - u_{\langle\square\rangle}\big\rangle_{V(\Omega)} + \Phi(v) - \Phi(u_{\langle\square\rangle}) \geqslant \big\langle L, v - u_{\langle\square\rangle}\big\rangle_{V(\Omega)}$$

for any $v \in \mathscr{K}(\Omega)$.

As the element $v \in \mathscr{K}(\Omega)$ is chosen arbitrary we get $u_o = u(\mathscr{O}_{\langle\square\rangle})$ and

$$u(\mathscr{O}_n) \to u(\mathscr{O}_{\langle\square\rangle}) \quad \text{weakly in } V(\Omega) \tag{2.12}$$

(the whole sequence $\{u(\mathscr{O}_n)\}_{n\in\mathbb{N}}$ converges to $u(\mathscr{O}_{\langle\square\rangle})$ weakly in $V(\Omega)$). Thus by virtue of (E1) and (2.7), (2.12) we obtain (since $\mathscr{O}_{\langle\square\rangle} \in \mathscr{U}_{\mathrm{ad}}(\Omega)$)

$$\mathscr{L}(\mathscr{O}_{\langle\square\rangle}, u(\mathscr{O}_{\langle\square\rangle})) \leqslant \liminf_{n\to\infty} \mathscr{L}(\mathscr{O}_n, u(\mathscr{O}_n)) := \{\inf \mathscr{L}(\mathscr{O}, u(\mathscr{O}),\ \mathscr{O} \in \mathscr{U}_{\mathrm{ad}}(\Omega)\}.$$

Hence, one has

$$J(\mathscr{O}_{\langle\square\rangle})(= \mathscr{L}(\mathscr{O}_{\langle\square\rangle}, u(\mathscr{O}_{\langle\square\rangle})) = \{\inf J(\mathscr{O})(= \mathscr{L}(\mathscr{O}, u(\mathscr{O})),\ \mathscr{O} \in \mathscr{U}_{\mathrm{ad}}(\Omega)\}$$

which completes the proof. $\square$

Further, we can apply Theorem 1 to the proof of existence of solutions to the optimal control problem (1.1).

Lemma 1. *The set* $\mathscr{K}(\Omega)$ *defined by* (1.1) *is a nonempty, closed and convex subset of* $V(\Omega)$.

P r o o f. Since the element $v = 0$ belongs to $\mathscr{K}(\Omega)$, we see that $\mathscr{K}(\Omega)$ is nonempty. The convexity is obvious. If $\{v_n\}_{n\in\mathbb{N}} \subset \mathscr{K}(\Omega)$ and $v_n \to v$ strongly in $V(\Omega)$, then $v_n \to v$ strongly in $L_2(\Omega)$, where $v \in V(\Omega)$. Therefore, we can extract a subsequence $\{v_{n_k}\}_{k\in\mathbb{N}}$ such that $v_{n_k} \to v$ a.e. on Ω. Since $v_{n_k} \in \mathscr{K}(\Omega)$, $v_{n_k} \geqslant 0$ a.e. on $\Omega_{\langle 0\rangle i}$, $i = 1, 2, \ldots, N$. Therefore, $v \geqslant 0$ a.e. on $\Omega_{\langle 0\rangle i}$, $i = 1, 2, \ldots, N$. Hence, $v \in \mathscr{K}(\Omega)$ which shows that $\mathscr{K}(\Omega)$ is closed.

Let $\boldsymbol{\mathcal{O}} = [\mathcal{O}_{ij}] \in [L_\infty(\Omega)]^4$ be a given (2×2) symmetric matrix function, i.e.

$$\mathcal{O}_{ij}(x_1, x_2) = \mathcal{O}_{ji}(x_1, x_2), \tag{2.13}$$

for a.e. $[x_1, x_2] \in \Omega$, $i, j = 1, 2$.

Assume that

$$M_{\langle 1\rangle}|\xi|^2 \leqslant \sum_{i,j=1}^{2} \mathcal{O}_{ij}(x_1, x_2)\xi_i\xi_j \leqslant M_{\langle 2\rangle}|\xi|^2, \tag{2.14}$$

for a.e. $[x_1, x_2] \in \Omega$ and for every $\xi = [\xi_1, \xi_2]$, where $M_{\langle 1\rangle}$ and $M_{\langle 2\rangle}$ are given constants, $0 < M_{\langle 1\rangle}$, and $|\xi|^2 = \sum\limits_{i=1}^{2} \xi_i^2$.

For $[x_1, x_2] \in \Omega$ denote by $\lambda_1(\boldsymbol{\mathcal{O}}, [x_1, x_2])$, $\lambda_2(\boldsymbol{\mathcal{O}}, [x_1, x_2])$ the eigenvalues of the matrix $\boldsymbol{\mathcal{O}}([x_1, x_2]) = [\mathcal{O}_{ij}(x_1, x_2)]$. By virtue of (2.14) one has

$$M_{\langle 1\rangle} \leqslant \lambda_1(\boldsymbol{\mathcal{O}}, [x_1, x_2]) \leqslant \lambda_2(\boldsymbol{\mathcal{O}}, [x_1, x_2]) \leqslant M_{\langle 2\rangle}. \tag{2.15}$$

The sequence of matrices $\{\boldsymbol{A}_n\}_{n\in\mathbb{N}}$ ($\boldsymbol{A}_n \subset [L_\infty(\Omega)]^4$), $n = 1, 2, \ldots$, is called G-convergent to the matrix $\boldsymbol{A}$ in the domain Ω ($\boldsymbol{A}_n \xrightarrow{G} \boldsymbol{A}$), if for any $f \in H^{-1}(\Omega)$ the solution u_n of the Dirichlet problem

$$\begin{cases} \operatorname{div}(\boldsymbol{A}_n \operatorname{\mathbf{grad}} u_n) = f,\ u_n \in H_0^1(\Omega) \text{ satisfy the relations} \\ u_n \to u \text{ weakly in } H_0^1(\Omega), \\ p_n = \boldsymbol{A}_n \operatorname{\mathbf{grad}} u_n \to p = \boldsymbol{A} \operatorname{\mathbf{grad}} u \text{ weakly in } [L_2(\Omega)]^2, \\ \quad \text{where } u \text{ is the solution of the Dirichlet problem} \\ \operatorname{div}(\boldsymbol{A} \operatorname{\mathbf{grad}} u) = f,\ u \in H_0^1(\Omega). \end{cases} \tag{2.16}$$

We define: $\mathcal{U}^{\Box}(\Omega) = \{\boldsymbol{\mathcal{O}} \in [L_\infty(\Omega)]^4 \colon \boldsymbol{\mathcal{O}}$ satisfies (2.14)$\}$ and the subset (a bounded closed subset of $\mathcal{U}^{\Box}(\Omega)$)

$$\begin{aligned} \mathcal{U}^{\Box}_{\mathrm{ad}}(\Omega) := \{&\boldsymbol{\mathcal{O}} \in \mathcal{U}^{\Box}(\Omega),\ \boldsymbol{\mathcal{O}} = [\mathcal{O}_{ij}],\ \mathcal{O}_{ij}(x_1,x_2) = \mathcal{O}_{ji}(x_1,x_2) \\ &\text{for a.a. } [x_1,x_2] \in \Omega,\ i,j = 1,2, \\ &-\infty < [\mathcal{O}_{ij}]_{\min} \leqslant \mathcal{O}_{ij} \leqslant [\mathcal{O}_{ij}]_{\max},\ \text{ and} \\ &M_{\langle 1\rangle} \leqslant [M_{\langle 1\rangle} M_{\langle 2\rangle}/(M_{\langle 1\rangle} + M_{\langle 2\rangle} - \lambda_2(\mathcal{O}_{ij},[x_1,x_2]))] \\ &\leqslant \lambda_2(\mathcal{O}_{ij},[x_1,x_2]) \leqslant M_{\langle 2\rangle} \ \text{ for a.e. } [x_1,x_2] \in \Omega\}, \end{aligned}$$

where $([\mathcal{O}_{ij}]_{\min}, [\mathcal{O}_{ij}]_{\max})$ are given constant symmetric 2×2 matrices.

We note that the physical interpretation of the set $\mathcal{U}^{\Box}_{\mathrm{ad}}(\Omega)$ is given in [14].

On the open set Ω we now define a bilinear form $Q(\boldsymbol{\mathcal{O}},\cdot,\cdot)\colon V(\Omega) \times V(\Omega) \to \mathbb{R}$ (for all $\boldsymbol{\mathcal{O}} \in \mathcal{U}^{\Box}_{\mathrm{ad}}(\Omega)$) by

$$Q(\boldsymbol{\mathcal{O}},v,z) := \sum_{i,j=1}^{2} \int_\Omega \mathcal{O}_{ij}(x_1,x_2) \frac{\partial v}{\partial x_i} \frac{\partial z}{\partial x_j}\, \mathrm{d}\Omega \quad \text{for all } [v,z] \in V(\Omega). \tag{2.17}$$

We define the operator $\mathcal{A}(\boldsymbol{\mathcal{O}}) \colon\ V(\Omega) \to V^*(\Omega)$ for $\boldsymbol{\mathcal{O}} \in \mathcal{U}^{\Box}_{\mathrm{ad}}(\Omega)$, $[v,z] \in V(\Omega)$ by the relation

$$\langle \mathcal{A}(\boldsymbol{\mathcal{O}})v, z\rangle_{V(\Omega)} = Q(\boldsymbol{\mathcal{O}},v,z). \tag{2.18}$$

Furthermore, we see from (2.17), (2.18), and (2.14) that: $\mathcal{A}(\boldsymbol{\mathcal{O}}) \in L(V(\Omega), V^*(\Omega))$.

Moreover, in view of [19] for every sequence $\{\boldsymbol{\mathcal{O}}_n\}_{n\in\mathbb{N}}$ ($\boldsymbol{\mathcal{O}}_n \in \mathcal{U}^{\Box}_{\mathrm{ad}}(\Omega)$, $n = 1,2,\ldots$), there exists an element $\boldsymbol{\mathcal{O}} \in \mathcal{U}^{\Box}_{\mathrm{ad}}(\Omega)$ and a subsequence $\{\boldsymbol{\mathcal{O}}_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\boldsymbol{\mathcal{O}}_{n_k} \xrightarrow{G} \boldsymbol{\mathcal{O}} \quad \text{in } \Omega. \tag{2.19}$$

On the other hand we associate with $\mathscr{U}_{\mathrm{ad}}(\Omega)$ (given in (1.2)) the bilinear form $a(e,\cdot,\cdot)$ by the relation

$$\begin{aligned} a(e,v,z) := \int_\Omega \sum_{i=1,2} e E_{ij} \frac{\partial v(x_1,x_2)}{\partial x_i} \frac{\partial z(x_1,x_2)}{\partial x_j}\, \mathrm{d}\Omega, \\ \text{for } [v,z] \in V(\Omega),\ e \in \mathscr{U}_{\mathrm{ad}}(\Omega), \end{aligned} \tag{2.20}$$

where $\mathbf{E} = [E_{ij}]_{i=1,2}$ is the unit 2×2 matrix.

Along with (2.20) we denote by $\mathcal{A}(e) \in L(V(\Omega), V^*(\Omega))$, $e \in \mathscr{U}_{\mathrm{ad}}(\Omega)$, the elliptic operator which is associated with the bilinear form $a(e,\cdot,\cdot)$, i.e.

$$\langle \mathcal{A}(e)v, z\rangle_{V(\Omega)} = a(e,v,z), \quad \text{for } [v,z] \in V(\Omega). \tag{2.21}$$

Observe that due to (2.19) for every sequence $\{e_n\}_{n\in\mathbb{N}}$ ($e_n \in \mathscr{U}_{\mathrm{ad}}(\Omega)$, $n \in 1, 2, \ldots$) there exist a subsequence $\{e_{n_k}\}_{k\in\mathbb{N}}$ and a matrix $\mathbf{B} \in \mathcal{U}^{\square}_{\mathrm{ad}}(\Omega)$ such that

$$(2.22) \qquad e_{n_k}\mathbf{E} \xrightarrow{G} \mathbf{B} \quad \text{in } \Omega.$$

Furthermore, if we define a set $G\mathcal{U}^{\square}_{\mathrm{ad}}(\Omega)$ by

$$(2.23) \quad G\mathcal{U}^{\square}_{\mathrm{ad}}(\Omega) = \{\mathbf{B} \in \mathcal{U}^{\square}_{\mathrm{ad}}(\Omega)\colon \text{ there exists } \{e_n\}_{n\in\mathbb{N}} \subset \mathscr{U}_{\mathrm{ad}}(\Omega) \text{such that } e_n\mathbf{E} \xrightarrow{G} \mathbf{B} \text{ in } \Omega\},$$

then $G\mathcal{U}^{\square}_{\mathrm{ad}}(\Omega) = \mathcal{U}^{\square}_{\mathrm{ad}}(\Omega)$ for $\Omega \subset \mathbb{R}^2$ (due to [14]).

By virtue of (2.17) and (2.18) we have

$$(2.23)_{\square} \quad \begin{aligned} \langle \mathcal{A}(\boldsymbol{\mathcal{O}})v, v\rangle_{V(\Omega)} &= Q(\boldsymbol{\mathcal{O}}, v, v) = \sum_{i,j=1}^{2} \mathcal{O}_{ij}(x_1, x_2)[\partial v/\partial x_i)(\partial v/\partial x_j)]\,\mathrm{d}\Omega \\ &\geqslant M_{\langle 1\rangle} \sum_{i=1}^{2} \int (\partial v/\partial x_i)^2\,\mathrm{d}\Omega \geqslant \mathrm{constant}_F M_{\langle 1\rangle}\|v\|^2_{V(\Omega)} \\ &= M_{1\langle\square\rangle}\|v\|^2_{V(\Omega)}, \end{aligned}$$

for all $v \in V(\Omega)$, since we can employ the Friedrichs inequality.

Moreover, one has

$$|\langle \mathcal{A}(\boldsymbol{\mathcal{O}})v, v\rangle_{V(\Omega)}| \leqslant M_{\langle 2\rangle}\left(\sum_{i=1}^{2}\int_{\Omega} (\partial v/\partial x_i)^2\,\mathrm{d}\Omega\right) \leqslant M_{\langle 2\rangle}\|v\|^2_{V(\Omega)}.$$

We have therefore shown that in this case $\mathcal{A}(\boldsymbol{\mathcal{O}}) \in \mathcal{B}_{V(\Omega)}(M_{1\langle\square\rangle}, M_2)$, and consequently, $((\mathrm{H}1), 1^\circ)$ is fulfilled. Since G-convergence is a combination of two types of convergence, namely, the convergence of solution of the Dirichlet problem and that of the corresponding flows (see (2.16)), by virtue of (2.19) we may write: $\mathcal{A}^{-1}(\boldsymbol{\mathcal{O}}_n)f \to \mathcal{A}^{-1}(\boldsymbol{\mathcal{O}})f$ weakly in $V(\Omega)$ (G-convergence of the operators $\mathcal{A}(\boldsymbol{\mathcal{O}}_n)$ to the operator $\mathcal{A}(\boldsymbol{\mathcal{O}})$). But this means that the condition $((\mathrm{H}1), 2^\circ)$ is fulfilled.

Due to Minty's lemma ([12]) the variational inequality

$$(2.24) \quad \langle \mathcal{A}(\boldsymbol{\mathcal{O}}_n)u(\boldsymbol{\mathcal{O}}_n), v - u(\boldsymbol{\mathcal{O}}_n)\rangle_{V(\Omega)} + \Phi(v) - \Phi(u(\boldsymbol{\mathcal{O}}_n)) \geqslant \langle L, v - u(\boldsymbol{\mathcal{O}}_n)\rangle_{V(\Omega)}$$

may be rewritten in the following form

$$(2.25) \quad \langle \mathcal{A}(\boldsymbol{\mathcal{O}}_n)v, v - u(\boldsymbol{\mathcal{O}}_n)\rangle_{V(\Omega)} - \langle L, v - u(\boldsymbol{\mathcal{O}}_n)\rangle_{V(\Omega)} \geqslant \Phi(u(\boldsymbol{\mathcal{O}}_n)) - \Phi(v), \quad \text{for any } v \in \mathcal{K}(\Omega).$$

Let $v_{[\varepsilon]n}$ be the solution of the following boundary value problem

$$\begin{aligned} v_{[\varepsilon]n} + \varepsilon \mathcal{A}(\mathcal{O}_n) v_{[\varepsilon]n} &= u(\mathcal{O}_n), \\ \gamma_{\Box} v_{[\varepsilon]n} &= 0 \quad \text{on } \partial\Omega_{\text{displacement}} \end{aligned} \tag{2.26}$$

for any $\varepsilon > 0$.

Problem (2.26) has a unique solution in $V(\Omega)$, and if $\partial\Omega$ is smooth enough $v_{[\varepsilon]n}$ belongs to $H^2(\Omega)$ (see [8]). Since: $u(\mathcal{O}_n) \geqslant 0$ a.e. on Ω (if $\Omega = \Omega_0$), by the maximum principle for second-order elliptic differential operators (cf. [17]), we have $v_{[\varepsilon]n} \geqslant 0$ on Ω. Hence, $v_{[\varepsilon]n} \in \mathscr{K}(\Omega)$.

Thus one has

$$v_{[\varepsilon]n} = (I + \varepsilon \mathcal{A}(\mathcal{O}_n))^{-1} u(\mathcal{O}_n) \in \mathscr{K}(\Omega).$$

Recalling that $(I + \varepsilon \mathcal{A}(\mathcal{O}_n))^{-1}$ is nonexpansive in $L_2(\Omega)$, we have

$$\|(I + \varepsilon \mathcal{A}(\mathcal{O}_n))^{-1} u(\mathcal{O}_n)\|_{L_2(\Omega)} < \|u(\mathcal{O}_n)\|_{L_2(\Omega)}.$$

On the other hand, the map $v \to |v|$ is continuous from $V(\Omega)$ into $V(\Omega)$. Thus we get the estimate (by virtue of the continuity, surjectivity and injectivity of the trace operator):

$$\|\gamma_{\Box}(I + \varepsilon \mathcal{A}(\mathcal{O}_n))^{-1} u(\mathcal{O}_n)\|_{H^{1/2}(\partial\Omega_{\text{contact}})} \leqslant \|\gamma_{\Box} u(\mathcal{O}_n)\|_{H^{1/2}(\partial\Omega_{\text{contact}})}.$$

But then one has

$$\Phi((I + \varepsilon \mathcal{A}(\mathcal{O}_n))^{-1} u(\mathcal{O}_n)) \leqslant \Phi(u(\mathcal{O}_n)). \tag{2.27}$$

On the other hand inserting $v = v_{[\varepsilon]n}$ into (2.25) (due to (2.27)), we obtain

$$\big\langle \mathcal{A}(\mathcal{O}_n) v_{[\varepsilon]n} - L, -\mathcal{A}(\mathcal{O}_n) v_{[\varepsilon]n} \big\rangle_{V(\Omega)} \geqslant 0.$$

We thus have (according to [6, Theorem I.1]),

$$\begin{cases} \mathcal{A}(\mathcal{O}_n) v_{[\varepsilon]n} \in L_2(\Omega), \\ \|\mathcal{A}(\mathcal{O}_n) v_{[\varepsilon]n}\|_{L_2(\Omega)} \leqslant \|L\|_{L_2(\Omega)}. \end{cases}$$

Hence (if $L \in L_2(\Omega)$) the sequence $\{\mathcal{A}(\mathcal{O}) v_{[\varepsilon]n}\}_{n\in\mathbb{N}}$ is weakly relatively compact in $L_2(\Omega)$. Thus (by virtue of Rellich's theorem) the sequence is strongly relatively compact in $V^*(\Omega)$ and the assumption $((\mathrm{H1}), 3^\circ)$ follows.

Let us assume that $\{\mathcal{Z}_k\}_{k\in\mathbb{N}}$ and $\mathcal{Z} \in V^*(\Omega)$ are such that $\lim\limits_{k\in\infty} \|\mathcal{Z}_k - \mathcal{Z}\|_{V^*(\Omega)} = 0$, then using Corollary 1 of [23], we obtain

$$\lim_{k\to\infty} \|\mathcal{A}^{-1}(\boldsymbol{\mathcal{O}}_k)\mathcal{Z}_k - \mathcal{A}^{-1}(\boldsymbol{\mathcal{O}})\mathcal{Z}\|_{L_2(\Omega)} = 0 \quad \text{as } \mathcal{A}(\boldsymbol{\mathcal{O}}_k) \xrightarrow{G} \mathcal{A}(\boldsymbol{\mathcal{O}}).$$

This means that the condition $((\mathrm{H1}), 4^\circ)$ is fulfilled.

As a consequence of the above results we conclude that the assumptions (H1) of Theorem 1 are fulfilled. $\square$

Denote by $u(e,\cdot) \in \mathscr{K}(\Omega)$ the unique solution of the variational inequality

$$(2.28)\qquad \begin{cases} \langle \mathcal{A}(\boldsymbol{\mathcal{O}})u, v-u\rangle_{V(\Omega)} + \Phi(v) - \Phi(u) \geqslant \langle L, v-u\rangle_{V(\Omega)} \\ \quad \text{for any } v \in \mathscr{K}(\Omega), \\ \text{corresponding to the matrix } \boldsymbol{\mathcal{O}} = e\mathbf{E} \in \mathcal{U}^{\square}_{\mathrm{ad}}(\Omega) \\ \quad \text{where } e \in \mathscr{U}_{\mathrm{ad}}(\Omega)\ (\not\equiv \mathscr{U}^{\square}_{\mathrm{ad}}(\Omega)). \end{cases}$$

Let $u_n \in \mathscr{K}(\Omega)$, $n = 1, 2, \ldots$, be the unique solution of (2.28) (for $\boldsymbol{\mathcal{O}}_n \in \mathcal{U}^{\square}_{\mathrm{ad}}(\Omega)$, $\boldsymbol{\mathcal{O}}_n \xrightarrow{G} \boldsymbol{\mathcal{O}}_{\square}$) then for $n \to \infty$ we have (due to [3])

$$(2.29)\qquad u_n \to u_{\langle\square\rangle} \quad \text{weakly in } V(\Omega),$$

where $u_{\langle\square\rangle} \in \mathscr{K}(\Omega)$ is the unique solution of the variational inequality (2.28) with $\boldsymbol{\mathcal{O}} \equiv \boldsymbol{\mathcal{O}}_{\langle\square\rangle}$.

Consider the optimal control problem:

$(\mathscr{P})$ *minimize* the cost functional

$$(2.30)\qquad J(e) = \int_\Omega [u(e,[x_1,x_2]) - z_{\mathrm{ad}}([x_1,x_2])]^2 \, \mathrm{d}\Omega$$

over the set $\mathscr{U}_{\mathrm{ad}}(\Omega)$, where $z_{\mathrm{ad}} \in L_2(\Omega)$ is a given element. In the following, we denote by $\mathscr{Z}(\Omega)$ $(\subset V(\Omega))$ the bounded set

$$\mathscr{Z}(\Omega) := \{u \in V(\Omega)\colon \text{there exists } \boldsymbol{\mathcal{O}} \in \mathcal{U}^{\square}_{\mathrm{ad}}(\Omega),\ u([x_1,x_2]) = u(\boldsymbol{\mathcal{O}},[x_1,x_2]), \\ [x_1,x_2] \in \Omega\},$$

where $u(\boldsymbol{\mathcal{O}},\cdot) \in \mathscr{K}(\Omega)$ denotes the solution of (2.28). Moreover, due to (2.19) and (2.29) this set is weakly closed in the space $V(\Omega)$ (a weakly compact subset of $V(\Omega)$). Then it follows from [15], [22] that for every $u \in \mathscr{Z}(\Omega)$ there exists sequence $\{e_n\}_{n\in\mathbb{N}}$ ($e_n \in \mathscr{U}_{\mathrm{ad}}(\Omega)$, $n = 1, 2, \ldots$), $\{u(e_n)\}_{n\in\mathbb{N}}$ ($u(e_n) \in \mathscr{K}(\Omega)$) such that for $n \to \infty$

$$(2.31)\qquad u(e_n) \to u \quad \text{weakly in } V(\Omega)$$

and

$$\langle \mathcal{A}(e_n)u(e_n), v - u(e_n)\rangle_{V(\Omega)} + \Phi(v) - \Phi(u(e_n)) \geqslant \langle L, v - u(e_n)\rangle_{V(\Omega)}\,, \tag{2.32}$$

for any $v \in \mathscr{K}(\Omega)$.

Thus, in view of (2.31) and (2.32) we may write

$$\inf\{J(e)\colon\ e \in \mathscr{U}_{\mathrm{ad}}(\Omega)\} = \min\Big\{\int_\Omega [u - z_{\mathrm{ad}}]^2 \,\mathrm{d}\Omega\colon\ u \in \mathscr{Z}(\Omega)\Big\}, \tag{2.33}$$

which holds independently of the choice of elements $z_{\mathrm{ad}} \in L_2(\Omega)$ and $L \in V^*(\Omega)$. We define an extension of the problem $(\mathscr{P})$ (we need to extend the set of the bilinear forms in such a way that the family of operators defined by this bilinear forms is G-closed). In the following we define an extension of the problem $(\mathscr{P})$

$(\mathscr{P}_{\langle\square\rangle})$ *minimize* cost functional

$$J_{\langle\square\rangle}(\boldsymbol{\mathcal{O}}) = \int_\Omega [u(\boldsymbol{\mathcal{O}}, [x_1, x_2]) - z_{\mathrm{ad}}([x_1, x_2])]^2 \,\mathrm{d}\Omega \tag{2.34}$$

over the set $\mathscr{U}^{\square}_{\mathrm{ad}}(\Omega)$.

On the other hand, one has:

$$J_{\langle\square\rangle}(e\mathbf{E}) = J(e), \tag{2.35}$$

for every element $e \in \mathscr{U}_{\mathrm{ad}}(\Omega)$.

In order to formulate necessary optimality conditions for the problem $(\mathscr{P}_{\langle\square\rangle})$, differentiablity properties of nonsmooth cost functional (2.34) should be investigated. Let $\boldsymbol{\mathcal{O}} \in \mathscr{U}^{\square}_{\mathrm{ad}}(\Omega)$ be a given matrix function, and let $u(\boldsymbol{\mathcal{O}}, \cdot)$ denote the corresponding solution of (2.28). Denote by $S_{\langle o\rangle}(\Omega)$ a cone of the form

$$\begin{aligned} S_{\langle o\rangle}(\Omega) = \{z \in H^1_0(\Omega)\colon\ &z(x_1, x_2) \geqslant 0 \ \text{q.e. on}\ \Omega(u(\boldsymbol{\mathcal{O}}, \cdot)),\\ &\langle \mathcal{A}(\boldsymbol{\mathcal{O}})u(\boldsymbol{\mathcal{O}}), z\rangle_{V(\Omega)} = \langle L, z\rangle_{V(\Omega)}\}, \end{aligned} \tag{2.36}$$

where $\Omega(u(\boldsymbol{\mathcal{O}})) = \{[x_1, x_2] \in \Omega_{0\langle i\rangle}\colon\ u(\boldsymbol{\mathcal{O}}, [x_1, x_2]) = 0 \text{ for } i = 1, 2, \ldots, N\}$. Here q.e. means everywhere, possibly except for a set zero capacity (see [15]).

Further for the matrix function $\mathbf{Q} \in [L_2(\Omega)]^4$ denote by $H(\mathbf{Q}, \cdot) \in H^1_0(\Omega)$ the unique solution to variational inequality

$$\begin{cases} H(\mathbf{Q}, \cdot) \in S_{\langle o\rangle}(\Omega),\\ \langle \mathcal{A}(\boldsymbol{\mathcal{O}})H(\mathbf{Q}, \cdot), z - H(\mathbf{Q}, \cdot)\rangle_{V(\Omega)} \geqslant -\langle \mathcal{A}(\mathbf{Q})u(\boldsymbol{\mathcal{O}}), z - H(\mathbf{Q}, \cdot)\rangle_{V(\Omega)}\,, \end{cases} \tag{2.37}$$

for any $z \in S_{\langle o\rangle}(\Omega)$.

On the other hand we introduce the linear subspace of $H_0^1(\Omega)$ in the following way:

$$V_{\mathbf{Q}}(\Omega)=\{z\in H_0^1(\Omega)\colon\ z(x_1,x_2)=0 \text{ q.e. on } \Omega_{\mathbf{Q}},\quad \langle \mathcal{A}(\boldsymbol{\mathcal{O}})u(\boldsymbol{\mathcal{O}}),z\rangle_{V(\Omega)}=\langle L,z\rangle_{L_2(\Omega)}\},\tag{2.38}$$

where

$$\Omega_{\mathbf{Q}}=\{[x_1,x_2]\in\Omega_{0\langle i\rangle}\colon\ H(\mathbf{Q}[x_1,x_2])=0 \text{ for } i=1,2,\dots,N\}.\tag{2.39}$$

Furthermore [22], [15] for any matrix function $\mathbf{Q}\in[L_2(\Omega)]^4$, the directional derivative of the cost functional $J_{\langle\square\rangle}(\boldsymbol{\mathcal{O}})$ takes the form

$$\mathrm{d}J_{\langle\square\rangle}(\boldsymbol{\mathcal{O}},\mathbf{Q})=\sum_{i,j=1}^{2}\int_\Omega Q_{ij}(x_1,x_2)(\partial u(\boldsymbol{\mathcal{O}},[x_1,x_2])/\partial x_i)\times(\partial\Pi(\mathbf{Q},[x_1,x_2])/\partial x_j)\,\mathrm{d}\Omega,\tag{2.40}$$

where

$$\begin{cases}\Pi=\Pi(\mathbf{Q},\cdot)\in V_{\mathbf{Q}}(\Omega),\\ \langle\mathcal{A}(\boldsymbol{\mathcal{O}})\Pi,z\rangle_{V(\Omega)}=\int_\Omega[u(\boldsymbol{\mathcal{O}},[x_1,x_2])-z_{\mathrm{ad}}(x_1,x_2)]z(x_1,x_2)\,\mathrm{d}\Omega,\\ \text{for any } z\in V_{\mathbf{Q}}(\Omega).\end{cases}\tag{2.41}$$

Since the set $\mathscr{Z}(\Omega)$ is weakly compact in the space $V(\Omega)$, there exists at least one optimal solution $\boldsymbol{\mathcal{O}}\in\mathscr{U}_{\mathrm{ad}}^{\square}(\Omega)$ to $(\mathscr{P}_{\langle\square\rangle})$. Then the matrix function $\boldsymbol{\mathcal{O}}$ satisfies the necessary optimality conditions

$$\sum_{i,j=1}^{2}\int_\Omega[Q_{ij}(x_1,x_2)-\mathcal{O}_{i,j}(x_1,x_2)](\partial u(\boldsymbol{\mathcal{O}}[x_1,x_2])/\partial x_i)\times(\partial\Pi([\mathbf{Q}-\boldsymbol{\mathcal{O}}],[x_1,x_2])/\partial x_j)\,\mathrm{d}\Omega\geqslant 0\tag{2.42}$$

for any $\mathbf{Q}\in\mathscr{U}_{\mathrm{ad}}^{\square}(\Omega)$.

On the other hand, as $e_{\square}\in\mathscr{U}_{\mathrm{ad}}(\Omega)$ is an optimal solution to $(\mathscr{P})$ then one has

$$\sum_{i,j=1}^{2}\int_\Omega Q_{ij}(x_1,x_2)(\partial u(e_{\square},[x_1,x_2])/\partial x_i)(\partial\Pi([\mathbf{Q}-e_{\square}\mathbf{E}],[x_1,x_2])/\partial x_j)\,\mathrm{d}\Omega \geqslant\sum_{i,j=1}^{2}\int_\Omega e_{\square}(x_1,x_2)(\partial u(e_{\square},[x_1,x_2])/\partial x_i)(\partial\Pi([\mathbf{Q}-e_{\square}\mathbf{E}],[x_1,x_2])/\partial x_j)\,\mathrm{d}\Omega,\tag{2.43}$$

for any $\mathbf{Q}\in\mathscr{U}_{\mathrm{ad}}^{\square}(\Omega)$.

Here, the necessary optimality conditions (2.42) and (2.43) follow from the expression (2.40). Note that $e_{\square} \in \mathscr{U}_{\rm ad}(\Omega)$ is an optimal solution to $(\mathscr{P})$ if the matrix $e_{\square}\mathbf{E}$ is an optimal solution to $(\mathscr{P}_{\langle\square\rangle})$.

3. Approximation of the optimal control problem by discretization

We now pass to an approximation of $(\mathscr{P})$. Let $h > 0$ be a discretization parameter tending to zero. With any $h > 0$ finite dimensional spaces $V_h(\Omega) \subset V(\Omega)$ and $\mathscr{U}_h(\Omega) \subset \mathcal{U}_{\square}(\Omega)$ will be associated. The symbol $\mathcal{U}_{\square}(\Omega)$ stands for another Banach space such that $\mathscr{U}(\Omega) \subseteq \mathcal{U}_{\square}(\Omega)$. The reason for introducing $\mathcal{U}_{\square}(\Omega)$ is simple: sometimes it is more convenient to work with approximations $\mathscr{U}_{\mathrm{ad},\langle h\rangle}(\Omega)$ of $\mathscr{U}_h(\Omega)$ that do not belong to the original space $\mathscr{U}(\Omega)$.

The family of $\{\mathscr{K}_h(\Omega)\}_h$ is supposed to satisfy the following conditions (we introduce a concept of convergence in the sense of Glowinski) $\mathscr{K}_{h_n}(\Omega) \xrightarrow{\mathrm{GL}} \mathscr{K}(\Omega)$:

$$(\mathrm{M1})_h \qquad \begin{cases} 1^\circ & \text{Let } \{v_{h_n}\}_{n\in\mathbb{N}} \text{ be a bounded subset of } V(\Omega),\ v_{h_n} \in \mathscr{K}_{h_n}(\Omega). \\ & \text{Then weak cluster points of } \{v_{h_n}\}_{n\in\mathbb{N}} \text{ belong to } \mathscr{K}(\Omega). \\ 2^\circ & \text{There exists } \Lambda(\Omega) \subset V(\Omega),\ \mathrm{cl}\,\Lambda(\Omega) = \mathscr{K}(\Omega), \text{ and} \\ & \mathscr{R}_h\colon\ \Lambda(\Omega) \to \mathscr{K}_h(\Omega) \text{ such that } \lim\limits_{h_n\to 0} \mathscr{R}_{h_n} v = v \\ & \text{strongly in } V(\Omega) \text{ for any } v \in \Lambda(\Omega). \end{cases}$$

Let us note that we do not necessarily have $\mathscr{K}_h(\Omega) \subset \mathscr{K}(\Omega)$ and $\mathscr{U}_{\mathrm{ad}\langle h\rangle}(\Omega) \subset \mathscr{U}_{\rm ad}(\Omega)$. If, however, this is true for any $h \in (0,1)$, we say that we have an internal approximation of $\mathscr{K}(\Omega)$, $\mathscr{U}_{\rm ad}(\Omega)$, respectively.

Remark 1. If $\mathscr{K}_h(\Omega) \subset \mathscr{K}(\Omega)$ for any $h \in (0,1)$, then $((\mathrm{M1})_h, 1^\circ)$ is trivially satisfied, because $\mathscr{K}(\Omega)$ is weakly closed. We have $\bigcup\limits_{h_n} \mathscr{K}_{h_n}(\Omega) \subset \mathscr{K}(\Omega)$. A useful variant of the condition $((\mathrm{M1})_h, 2^\circ)$ for $\mathscr{R}_h$ is the following:

There exists a subset $\Lambda(\Omega) \subset V(\Omega)$ such that $\mathrm{cl}\,\Lambda(\Omega) = \mathscr{K}(\Omega)$ and $\mathscr{R}_h\colon \Lambda(\Omega) \to V_h(\Omega)$ having the property that for each $v \in \Lambda(\Omega)$ there exists $h_0 = h_0(v)$ with $\mathscr{R}_h v \in \mathscr{K}_h(\Omega)$ for all $h \leqslant h_0(v)$ and $\lim\limits_{h_n\to 0} \mathscr{R}_{h_n} v = v$ strongly in $V(\Omega)$.

The approximation of the state inequation (2.3) is now defined by means of the Ritz-Galerkin procedure on $\mathscr{K}_h(\Omega)$, using elements of $\mathscr{U}_{\mathrm{ad}\langle h\rangle}(\Omega)$ as controls. Thus for any $e_h \in \mathscr{U}_{\mathrm{ad}\langle h\rangle}(\Omega)$ we define the following approximate state problem

$$(3.1) \qquad \begin{cases} u_h(e_h) \in \mathscr{K}_h(\Omega), \\ \langle \mathcal{A}_h(e_h)u_h(e_h), v_h - u_h(e_h)\rangle_{V_h(\Omega)} + \Phi_h(v_h) - \Phi_h(u_h(e_h)) \\ \qquad\qquad\qquad \geqslant \langle L_h, v_h - u_h(e_h)\rangle_{V_h(\Omega)} \\ \text{for any } v_h \in \mathscr{K}_h(\Omega) \text{ and } e_h \in \mathscr{U}_{\mathrm{ad}\langle h\rangle}(\Omega). \end{cases}$$

The approximate functionals $\Phi_h(\cdot)\colon V_h \to \mathbb{R}^+$ satisfy the condition (H2).

The approximation of $(\mathscr{P})$ is now stated as follows:

$$(\mathscr{P}_h) \qquad \begin{cases} \text{Find } e_{\langle\square\rangle h} \in \mathscr{U}_{\mathrm{ad}\langle h\rangle}(\Omega) \text{ such that} \\ \mathscr{L}(e_{\langle\square\rangle h}, u_h(e_{\langle\square\rangle h})) \leqslant \mathscr{L}(e_h, u_h(e_h)), \text{ for any } e_h \in \mathscr{U}_{\mathrm{ad}\langle h\rangle}(\Omega), \end{cases}$$

where $u_h(e_h)$ solves (3.1).

Denote by

$$\begin{cases} \mathscr{Z}_\Delta(\Omega) := \{e \in \mathscr{U}_{\mathrm{ad}}(\Omega)\colon\ u(e) \in \mathscr{K}(\Omega)\}, \\ \mathscr{Z}_{\Delta h}(\Omega) := \{e \in \mathscr{U}_{\mathrm{ad}\langle h\rangle}(\Omega)\colon\ u_h(e_h) \in \mathscr{K}_h(\Omega)\} \end{cases} \tag{3.2}$$

where $u(e)$ is the solution of (1.9) corresponding to e and $u_h(e_h)$ is given similarly by (3.1). We assume that $\mathscr{Z}_{\Delta h}(\Omega) \neq 0$ for any $h > 0$.

By discretization of $(\mathscr{P})$ we mean the problem

$$(\mathscr{P}_h) \qquad \min \mathscr{L}(e_h, u_h(e_h)) \text{ for } e_h \in \mathscr{Z}_{\Delta h}(\Omega) \text{ and } u_h(e_h) \text{ as above,}$$

which is equivalent to the approximation of $(\mathscr{P})$.

In what follows, we shall study the relation between optimal pairs of $(\mathscr{P}_h)$ and $(\mathscr{P}_\square)$ (cf. (2.34)) as $h_n \to 0_+$. For the analysis of the relation between (1.9) and (3.1) we shall need the following hypotheses.

In what follows, let $h_n \to 0_+$ as $n \to \infty$.

$$(\mathrm{M2})_h \quad \begin{cases} 1^\circ & \mathcal{A}_h(e_h) \in \mathcal{B}_{V_h(\Omega)}(\alpha_{\mathcal{A}_\square}, M_{\mathcal{A}_\square}), \text{ for any } h \in (0,1) \\ & \text{and any } e_h \in \mathscr{U}_{\mathrm{ad}\langle h\rangle}(\Omega). \\ 2^\circ & \big\langle \mathcal{A}_{h_n}(e_{h_n}) v_{h_n}, z_{h_n} \big\rangle_{V_{h_n}(\Omega)} \to \langle \mathcal{A}(e)v, z\rangle_{V(\Omega)}, \text{ if } e_{h_n} \to e \\ & \text{strongly in } \mathscr{U}_\square(\Omega), \\ & e_{h_n} \in \mathscr{U}_{\mathrm{ad}\langle h_n\rangle}(\Omega),\ e \in \mathscr{U}_{\mathrm{ad}}(\Omega),\ v_{h_n} \to v \text{ weakly in } V(\Omega), \\ & z_{h_n} \to z \text{ strongly in } V(\Omega). \\ 3^\circ & \liminf\limits_{h_n \to 0} \big\langle \mathcal{A}_{h_n}(e_{h_n}) v_{h_n}, v_{h_n} \big\rangle_{V_{h_n}(\Omega)} \geqslant \langle \mathcal{A}(e)v, v\rangle_{V(\Omega)}, \\ & \text{if } e_{h_n} \to e \text{ strongly in } \mathcal{U}_\square(\Omega),\ v_{h_n} \to v \text{ weakly in } V(\Omega). \\ 4^\circ & \mathcal{A}(e_{h_n}) \to \mathcal{A}(e) \text{ in } L(V(\Omega), V^*(\Omega)), \text{ if } e_{h_n} \to e \\ & \text{strongly in } \mathcal{U}_\square(\Omega),\ e_{h_n} \in \mathscr{U}_{\mathrm{ad}\langle h\rangle}(\Omega),\ e \in \mathscr{U}_{\mathrm{ad}}(\Omega). \\ 5^\circ & v_{h_n} \to v \text{ weakly in } V(\Omega) \Rightarrow \liminf\limits_{h_n \to 0} \Phi_{h_n}(v_{h_n}) \geqslant \Phi(v), \\ & \text{where } \Phi\colon V(\Omega) \to \mathbb{R}^+. \end{cases}$$

6° $v_{h_n} \to v$ strongly in $V(\Omega),\ v_{h_n} \in V_{h_n}(\Omega) \Rightarrow\ \Phi_{h_n}(v_{h_n}) \to \Phi(v)$.

7° For any $e \in \mathscr{U}_{\mathrm{ad}}(\Omega)$ there exists a sequence $\{e_{h_n}\}_{n\in\mathbb{N}}$,
$e_{h_n} \in \mathscr{U}_{\mathrm{ad}\langle h_n\rangle}(\Omega)$, such that $e_{h_n} \to e$ strongly in $\mathcal{U}_{\square}(\Omega)$.

8° For any sequence $\{e_{h_n}\}_{n\in\mathbb{N}},\ e_{h_n} \in \mathscr{U}_{\mathrm{ad}\langle h_n\rangle}(\Omega)$, there exists its subsequence and an element $e \in \mathscr{U}_{\mathrm{ad}}(\Omega)$ such that $e_{h_{n_k}} \to e$ strongly in $\mathcal{U}_{\square}(\Omega)$.

9° There is a positive constant such that
$\langle L_h, v_h\rangle_{V_h(\Omega)} \leqslant \text{constant}\, \|v_h\|_{V_h(\Omega)}$,
for any $v_h \in V_h(\Omega)$ and for any $h \in (0,1)$.

10° $v_{h_n} \to v$ weakly in $V(\Omega) \Rightarrow \langle L_{h_n}, v_{h_n}\rangle_{V_{h_n}(\Omega)} \to \langle L, v\rangle_{V(\Omega)}$
as $h_n \to 0,\ e_{h_n} \in \mathscr{U}_{\mathrm{ad}\langle h\rangle}(\Omega)$.

11° For $e_{h_n} \to e$ strongly in $\mathcal{U}_{\square}(\Omega),\ v_{h_n} \to v$ strongly in $V(\Omega)$,
where $e_{h_n} \in \mathscr{U}_{\mathrm{ad}\langle h\rangle}(\Omega),\ e \in \mathscr{U}_{\mathrm{ad}}(\Omega),\ v_{h_n} \in V_{h_n}(\Omega),\ v \in V(\Omega)$
one has $\lim\limits_{n\to\infty} \mathscr{L}(e_{h_n}, v_{h_n}) \to \mathscr{L}(e, v)$.

In view of the assumptions ($(\mathrm{M2})_h$, 1°, 5°, 6°, 9°), we can use Theorem 6.1 (see [8]) to conclude that the discrete state problem (3.1) has a unique solution $u_h(e_h)$.

Theorem 2. *Let the assumptions* ($(\mathrm{M1})_h$, $(\mathrm{M2})_h$, 1° to 6° and 9°, 10°) *be satisfied and* $\{e_{h_n}\}_{n\in\mathbb{N}}$, $e_{h_n} \in \mathscr{U}_{\mathrm{ad}\langle h_n\rangle}(\Omega)$ *be a sequence such that* $e_{h_n} \to e$ *strongly in* $\mathcal{U}_{\square}(\Omega)$, $e \in \mathscr{U}_{\mathrm{ad}}(\Omega)$.

Then we have

$$u_{h_n}(e_{h_n}) \to u(e) \quad \text{strongly in } V(\Omega), \text{ as } n \to \infty, \tag{3.3}$$

where $u(e)$ *and* $u_{h_n}(e_{h_n})$ *are the solutions of* (2.3) *(for* $\mathscr{O} = e$*) and* (3.1), *respectively.*

P r o o f. Let $a \in \mathscr{K}(\Omega)$ be a fixed element. From ($(\mathrm{M1})_h, 2°$) the existence of $\{a_{h_n}\}_{n\in\mathbb{N}}$, $a_{h_n} \in \mathscr{K}_{h_n}(\Omega)$ such that

$$a_{h_n} \to a \quad \text{strongly in } V(\Omega), \text{ as } n \to \infty, \tag{3.4}$$

follows.

From the definition of (3.1), ((M2)$_h$, 1°, 6°, 9°) we deduce that

$$
\begin{aligned}
&\alpha_{\mathcal{A}_\square}\|u_{h_n}(e_{h_n})\|^2_{V(\Omega)}\\
&\qquad\leqslant \alpha_{\mathcal{A}_\square}\|u_{h_n}(e_{h_n})\|^2_{V(\Omega)} + \Phi_h(u_{h_n}(e_{h_n}))\\
&\qquad\leqslant \langle \mathcal{A}_h(e_h)u_{h_n}(e_{h_n}), u_{h_n}(e_{h_n})\rangle_{V_h(\Omega)} + \Phi_h(u_{h_n}(e_{h_n}))\\
&\qquad\leqslant \langle L_{h_n}, u_{h_n}(e_{h_n}) - a_{h_n}\rangle_{V_h(\Omega)} + \langle \mathcal{A}_h(e_{h_n})u_{h_n}(e_{h_n}), a_{h_n}\rangle_{V_h(\Omega)} + \Phi_h(a_{h_n})\\
&\qquad\leqslant M_{\mathcal{A}_\square}\|u_{h_n}(e_{h_n})\|_{V(\Omega)}\|a_{h_n}\|_{V(\Omega)} + \|L_{h_n}\|_{V^*(\Omega)}(\|u_{h_n}(e_{h_n})\|_{V(\Omega)}\\
&\qquad\quad + \|a_{h_n}\|_{V(\Omega)}) + \text{constant}\\
&\qquad\leqslant \varepsilon\Big(\frac{1}{2}M_{\mathcal{A}_\square} + \frac{1}{2}\Big)\|u_{h_n}(e_{h_n})\|^2_{V(\Omega)} + \Big(\frac{M_{\mathcal{A}_\square}}{2\varepsilon}\Big)\|a_{h_n}\|^2_{V(\Omega)} + \text{constant}.
\end{aligned}
$$

Thus this estimate implies (for $\varepsilon > 0$ sufficiently small) the boundedness of $\{u_{h_n}(e_{h_n})\}_{n\in\mathbb{N}}$. Therefore, one can pass to a subsequence such that

$$u_{h_{n_k}}(e_{h_{n_k}}) \to u_\diamondsuit \quad \text{weakly in } V(\Omega), \text{ as } k\to\infty. \tag{3.5}$$

But then in view of ((M1)$_h$, 1°) it follows that $u_\diamondsuit \in \mathscr{K}(\Omega)$.

Next taking $v_{h_k} = a_{h_k}$ in the state inequality (3.1), we have that

$$
\begin{aligned}
&\langle \mathcal{A}_{h_{n_k}}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), a_{h_{n_k}} - u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V_h(\Omega)}\\
&\qquad + \Phi_h(a_{h_{n_k}}) - \Phi_h(u_{h_{n_k}}(e_{h_{n_k}}))\\
&\qquad\geqslant \langle L_{h_{n_k}}, a_{h_{n_k}} - u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V_h(\Omega)}.
\end{aligned}
\tag{3.6}
$$

We can show that for $k\to\infty$

$$\liminf_{k\to\infty}\langle \mathcal{A}(e_{h_{n_k}})u_{h_{n_k}}(e_{hn_k}), u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V(\Omega)} \geqslant \langle \mathcal{A}(e)u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega)}. \tag{3.7}$$

Indeed, since for $e\in\mathscr{U}_{\rm ad}(\Omega)$, $\langle \mathcal{A}(e)v, v\rangle_{V(\Omega)}$ is a lower weakly semicontinuous functional on $V(\Omega)$, due to (3.5) we conclude that

$$\liminf_{k\to\infty}\langle \mathcal{A}(e)u_{h_{n_k}}(e_{h_{n_k}}), u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V_h(\Omega)} \geqslant \langle \mathcal{A}(e)u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega)}. \tag{3.8}$$

Further making use of ((M2)$_h$, 4°) and (3.5) we derive that

$$|\langle \mathcal{A}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V(\Omega)} - \langle \mathcal{A}(e)u_{h_{n_k}}(e_{h_{n_k}}), u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V(\Omega)}| \to 0$$

for $k\to\infty$.

Therefore,

$$
\begin{aligned}
&\liminf_{k\to\infty}\,\langle \mathcal{A}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V(\Omega)}\\
&\qquad\geqslant \liminf_{k\to\infty}\langle \mathcal{A}(e)u_{h_{n_k}}(e_{h_{n_k}}), u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V(\Omega} \geqslant \langle \mathcal{A}(e)u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega)}.
\end{aligned}
$$

On the other hand, using the assumption $((\mathrm{M2})_h,\ 3^\circ)$ and (3.5), we may write

$$\liminf_{k\to\infty}\langle \mathcal{A}_{h_{n_k}}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V_h(\Omega)} \geqslant \langle \mathcal{A}(e)u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega)}. \tag{3.9}$$

Next, taking into account (3.6), we see that

$$\begin{aligned}
&\limsup_{k\to\infty}\langle \mathcal{A}_{h_{n_k}}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V_h(\Omega)}\\
&\qquad\leqslant \lim_{k\to\infty}\langle L_{h_{n_k}}, u_{h_{n_k}}(e_{h_{n_k}}) - a_{h_{n_k}}\rangle_{V_h(\Omega)} + \lim_{k\to\infty}\Phi_{h_{n_k}}(a_{h_{n_k}})\\
&\qquad\quad - \liminf_{k\to\infty}\Phi_{h_{n_k}}(u_{h_{n_k}}(e_{h_{n_k}}))\\
&\qquad\quad + \lim_{k\to\infty}\langle \mathcal{A}_{h_{n_k}}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), a_{h_{n_k}},\rangle_{V_h(\Omega)}.
\end{aligned}$$

Hence taking the limit as $k\to\infty$, we conclude (we use the hypotheses $((\mathrm{M2})_h$, 2°, 5° and 6°, 10°))

$$\begin{aligned}
&\limsup_{k\to\infty}\langle \mathcal{A}_{h_{n_k}}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V_h(\Omega)}\\
&\qquad\leqslant \langle L, u_\diamondsuit - a\rangle_{V_h(\Omega)} + \Phi(a) - \Phi(u_\diamondsuit) + \langle \mathcal{A}(e)u_\diamondsuit, a\rangle_{V(\Omega)}.
\end{aligned} \tag{3.10}$$

Since $a\in\mathscr{K}(\Omega)$ was arbitrary, we may insert $a = u_\diamondsuit$ into the estimate (3.10). Then from (3.10) it follows that

$$\limsup_{k\to\infty}\langle \mathcal{A}_{h_{n_k}}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V_h(\Omega)} \leqslant \langle \mathcal{A}(e)u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega)}. \tag{3.11}$$

Thus combining (3.9) and (3.11), we arrive at

$$\lim_{k\to\infty}\langle \mathcal{A}_{h_{n_k}}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V_h(\Omega)} = \langle \mathcal{A}(e)u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega)} \tag{3.12}$$

and consequently,

$$\begin{aligned}
&\lim_{k\to\infty}\langle \mathcal{A}_{h_{n_k}}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), a_{h_{n_k}} - u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V_h(\Omega)}\\
&\qquad = \langle \mathcal{A}(e)u_\diamondsuit, a - u_\diamondsuit\rangle_{V(\Omega)}.
\end{aligned} \tag{3.13}$$

Furthermore, from the assumptions $((\mathrm{M2})_h$, 5°, 6°, and 10°) and the relation (3.4) we have (passing to the limit with $k\to\infty$)

$$\langle L_{h_{n_k}}, a_{h_{n_k}} - u_{h_{n_k}}(e_{h_k})\rangle_{V_h(\Omega)} \to \langle L, a - u_\diamondsuit\rangle_{V(\Omega)} \tag{3.14}$$

and

$$\begin{cases} \liminf_{k\to\infty}\Phi_{h_{n_k}}(u_{h_{n_k}}(e_{h_{n_k}})) \geqslant \Phi(u_\diamondsuit),\\ \lim_{k\to\infty}\Phi_{h_{n_k}}(a_{h_{n_k}}) = \Phi(a). \end{cases} \tag{3.15}$$

Thus, we arrive at the inequality (by virtue of (3.6), (3.13), (3.14), and (3.15))

$$\langle \mathcal{A}(e)u_\diamondsuit, a - u_\diamondsuit\rangle_{V(\Omega)} + \Phi(a) - \Phi(u_\diamondsuit) \geqslant \langle L, a - u_\diamondsuit\rangle_{V(\Omega)}.$$

Since $a \in \mathscr{K}(\Omega)$ was arbitrary and the inequality (2.3) has a unique solution, $u_\diamondsuit = u(e)$ and the whole sequence $\{u_{h_n}(e_{h_n})\}_{n\in\mathbb{N}}$ tends to $u(e)$ weakly in $V(\Omega)$.

Finally, it remains to show the strong convergence. By ((M1)$_h$, 2°) there exists a sequence $\{\theta_{h_n}\}_{n\in\mathbb{N}}$, $\theta_{h_n} \in \mathscr{K}_{h_n}(\Omega)$, such that

$$\theta_{h_n} \to u(e) \text{ strongly in } V(\Omega) \text{ as } n \to \infty. \tag{3.16}$$

Then one has

$$\begin{aligned}
\limsup_{n\to\infty} \alpha_{\mathcal{A}_\square} \|u_{h_n}(e_{h_n}) - \theta_{h_n}\|^2_{V(\Omega)} \\
&\leqslant \limsup_{n\to\infty} \langle \mathcal{A}_{h_n}(e_{h_n})(u_{h_n}(e_{h_n}) - \theta_{h_n}), u_{h_n}(e_{h_n}) - \theta_{h_n}\rangle_{V_h(\Omega)} \\
&\leqslant \limsup_{n\to\infty} \langle L_{h_n}, u_{h_n}(e_{h_n}) - \theta_{h_n})\rangle_{V(\Omega)} \\
&\quad + \limsup_{n\to\infty} \langle \mathcal{A}_{h_n}(e_{h_n})\theta_{h_n}, u_{h_n}(e_{h_n}) - \theta_{h_n}\rangle_{V_h(\Omega)} \\
&\quad + \limsup_{n\to\infty} \Phi_{h_n}(\theta_{h_n}) - \liminf_{n\to\infty} \Phi_{h_n}(u_{h_n}(e_{h_n})) \leqslant 0,
\end{aligned}$$

as follows from the definition of (3.1), (3.5), and (3.16) ((M2)$_h$, 2°, 5°, 6°, 10°).

Therefore, one has

$$\|u_{h_n}(e_{h_n}) - \theta_{h_n}\|_{V(\Omega)} \to 0. \tag{3.17}$$

Making use of the triangle inequality, (3.16), and (3.17), we arrive at the assertion

$$\|u_{h_n}(e_{h_n}) - u(e)\|_{V(\Omega)} \leqslant \|u_{h_n}(e_{h_n}) - \theta_{h_n}\|_{V(\Omega)} + \|\theta_{h_n} - u(e)\|_{V(\Omega)} \to 0 \text{ as } n \to \infty.$$

□

On the basis of Theorem 2, we prove the following convergence result.

Theorem 3. *Let* (M1)$_h$ *and* (M2)$_h$ *be satisfied. Then for every sequence* $\{e_{\langle\square\rangle h_n}, u_{h_n}(e_{\langle\square\rangle h_n})\}_{n\in\mathbb{N}}$ *of optimal pairs of* $(\mathscr{P}_{h_n})$, $n \to \infty$, *there exists its subsequence such that*

$$\begin{cases} e_{\langle\square\rangle h_{n_k}} \to e_{\langle\square\rangle} & \textit{strongly in } \mathcal{U}_\square(\Omega), \\ u_{h_{n_k}}(e_{\langle\square\rangle h_{n_k}}) \to u(e_{\langle\square\rangle}) & \textit{strongly in } V(\Omega). \end{cases} \tag{3.18}$$

In addition, $[e_{\langle\square\rangle}, u(e_{\langle\square\rangle})]$ *is an optimal pair of* $(\mathscr{P})$. *Furthermore, any accumulation point of* $\{e_{\langle\square\rangle h_n}, u_{h_n}(e_{\langle\square\rangle h_n})\}_{n\in\mathbb{N}}$ *in the sense of* (3.18) *possesses this property.*

Proof. Let $\mathcal{O} \in \mathscr{U}_{\mathrm{ad}}(\Omega)$ be an arbitrary element. From $((\mathrm{M2})_h,\ 7^\circ)$ the existence of a sequence $\{\mathcal{O}_{h_n}\}_{n\in\mathbb{N}}$, $\mathcal{O}_{h_n} \in \mathscr{U}_{\mathrm{ad}\langle h_n\rangle}(\Omega)$ such that $\mathcal{O}_{h_n} \to \mathcal{O}$ strongly in $\mathcal{U}_{\square}(\Omega)$ as $n \to \infty$, follows.

On the other hand, using $((\mathrm{M2})_h,\ 8^\circ)$ one can find a subsequence $\{e_{\langle\square\rangle h_{n_k}}\}_{k\in\mathbb{N}}$ of $\{e_{\langle\square\rangle h_n}\}_{n\in\mathbb{N}}$ such that $e_{\langle\square\rangle h_{n_k}} \to e_{\langle\square\rangle}$ strongly in $\mathcal{U}_{\square}(\Omega)$, as $k \to \infty$, where $e_{\langle\square\rangle} \in \mathscr{U}_{\mathrm{ad}}(\Omega)$.

Thus, at the same time

$$\begin{cases} u_{h_{n_k}}(e_{\langle\square\rangle h_{n_k}}) \to u(e_{\langle\square\rangle}), \\ u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}) \to u(\mathcal{O}) \ \text{strongly in } V(\Omega), \ \text{as } k \in \infty, \end{cases} \tag{3.19}$$

where $u(e_{\langle\square\rangle})$ and $u(\mathcal{O})$ are solutions of (2.3), as follows from Theorem 2.

Next the definition of $(\mathscr{P}_h)$ yields

$$\mathscr{L}(e_{\langle\square\rangle h_{n_k}}, u_{h_{n_k}}(e_{\langle\square\rangle h_{n_k}})) \leqslant \mathscr{L}(\mathcal{O}_{h_{n_k}}, u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})) \quad \text{for any } k \in \mathbb{N}. \tag{3.20}$$

Then letting $h_{n_k} \to 0_+$ in (3.20), using previous convergences of $\{e_{\langle\square\rangle h_{n_k}}, u_{h_{n_k}}(e_{\langle\square\rangle h_{n_k}})\}_{k\in\mathbb{N}}$, $\{\mathcal{O}_{h_{n_k}}, u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})\}_{k\in\mathbb{N}}$ and the assumption $((\mathrm{M2})_h,\ 11^\circ)$, we may conclude that

$$\mathscr{L}(e_{\langle\square\rangle}, u(e_{\langle\square\rangle})) \leqslant \mathscr{L}(\mathcal{O}, u(\mathcal{O})) \quad \text{for any } \mathcal{O} \in \mathscr{U}_{\mathrm{ad}}(\Omega).$$

□

4. Approximate optimal control of pseudo-beam

The bending of the pseudo-beam is described by means of a shear model: the beam is deformed only by shear forces. We assume that a homogeneous and isotropic pseudo-beam occupying a domain: $\Omega\times(-e,e)$ of the space $\mathbb{R}^2$ is loaded by a transversal distributed force $p(x)$ perpendicular to the axis Ox.

The transversal displacements v belong to the space $V(\Omega) = \{v \in H^1(\Omega)\colon\ v(0) = 0,\ v(L) = 0\}$, where $\Omega = (0, L)$, $L > 0$ is the length of the beam. We have the following stress-strain relation: $\tau_{xz} = KG\varepsilon_{xz} = \frac{1}{2}KG(\mathrm{d}v/\mathrm{d}x)$.

The shear force of pseudo-beam is given by the relation

$$V_{xz}(v) = \int_{-e}^{e} \tau_{xz}\,\mathrm{d}z = KGe\frac{\mathrm{d}v}{\mathrm{d}x}.$$

Then the equilibrium equation of the pseudo-beam (without any internal rigid obstacles) has the form: $\mathrm{d}V_{xz}(v)/\mathrm{d}x+p=0$ or: $\mathrm{d}(KGe(\mathrm{d}v/\mathrm{d}x))/\mathrm{d}x=-p$ in Ω. Moreover, we consider several unilateral inner obstacles as follows: $\Omega_{0\langle i\rangle}$, $i=1,2,\dots,N$, are mutually disjoint subdomains such that $\overline{\Omega}_{0\langle i\rangle}\subset\Omega$, $\overline{\Omega}_{0\langle i\rangle}\cap\overline{\Omega}_{0\langle j\rangle}=\emptyset$ for $i\neq j$. For the variational formulation of the pseudo-beam problem we introduce the set: $\mathscr{K}(\Omega)=\{v\in V(\Omega)\colon\ v\geqslant 0\ \text{ for any }\ x\in\Omega_{0\langle i\rangle}\ \text{ for all }\ i=1,2,\dots,N\}$. The equation of the virtual work can be written in the following form:

$$(1.5)_{\square}\qquad \langle\mathcal{A}(e)v,z\rangle_{V(\Omega)}=\int_{\Omega}\langle KGe\,\mathbf{grad}\,v,\mathbf{grad}\,z\rangle_{\mathbb{R}^1}\,\mathrm{d}\Omega.$$

Here $\mathcal{A}(e)\colon\ V(\Omega)\to V^*(\Omega)$ is the operator corresponding to the bending of an elastic pseudo-beam. On the basis of the virtual displacement principle we introduce the following state problem: Given any $e\in\mathscr{U}_{\mathrm{ad}}(\Omega)$, find $u(e)\in\mathscr{K}(\Omega)$ such that

$$(1.9)_{\square}\qquad \langle\mathcal{A}(e)u(e),v-u(e)\rangle_{V(\Omega)}\geqslant\langle L,v-u(e)\rangle_{V(\Omega)}\,,$$

for all $v\in\mathscr{K}(\Omega)$.

The family of the operators $\{\mathcal{A}(e)\}$, $e\in\mathscr{U}_{\mathrm{ad}}(\Omega)$, satisfies the following estimates

$$(1.10)_{\square}\qquad \langle\mathcal{A}(e)v,v\rangle_{V(\Omega)}\geqslant KGe_{\min}\int_{\Omega}|\mathbf{grad}\,\mathbf{v}|^2\,\mathrm{d}\Omega\geqslant\mathrm{const}_F KGe_{\min}\|v\|^2_{V(\Omega)}$$

for all $v\in V(\Omega)$ (we use the Friedrichs inequality),

$$(1.11)_{\square}\qquad |\langle\mathcal{A}(e)v,z\rangle_{V(\Omega)}|\leqslant KGe_{\max}\|v\|_{V(\Omega)}\|z\|_{V(\Omega)}.$$

Thus due to $(1.10)_{\square}$ and $(1.11)_{\square}$ the state variational inequality $(1.9)_{\square}$ has a unique solution.

Let $N_{\square}$ be an integer and $\mathscr{T}_h$ a partition of the interval $[0,L]$ into $N_{\square}$ subintervals $H_{\langle j\rangle}=[A_{\langle j-1\rangle},A_{\langle j\rangle}]$ of length h, $0=A_{\langle 0\rangle}<A_{\langle 1\rangle}\dots<A_{\langle N_{\square}\rangle}=L$. The step $h=(L/N_{\square})$, $A_{\langle j\rangle}=jh$, $j=1,2,\dots,N_{\square}(h)$.

Consider a regular family of partitions $\{\mathscr{T}_{h_n}\}_{n\in\mathbb{N}}$, $h_n\to 0_+$, of Ω, which are consistent with all subdomains $\Omega_{0\langle i\rangle}$. We introduce the finite element space of piecewise linear functions

$$H^0_h(\Omega)=\{v_h\in C(\overline{\Omega})\colon\ v_h|_H\in P_1(H_{\langle j\rangle})\ \text{ for all beam elements }\ H_{\langle j\rangle}\in\mathscr{T}_h\}$$

and the following sets

$$\begin{aligned}&V_h(\Omega)=H^0_h(\Omega)\cap V(\Omega)\quad\text{(an interior approximation to }V(\Omega)),\\&\mathcal{K}_h(\Omega)=\{v_h\in V_h(\Omega)\colon\ v_h(A)\geqslant 0\quad\text{for all nodes }\ A\in\Sigma_h\},\end{aligned}$$

where Σ_h denotes the set of all nodes A of pseudo-beam elements $H \in \mathscr{T}_h$, $H \subset (\overline{\Omega})_{0\langle i\rangle}$, $i = 1, 2, \ldots, N$.

Here the set $\mathcal{U}^{\square\mid}_{\mathrm{ad}\langle h\rangle}(\Omega)$ of discrete thickness distributions is defined as follows

$$
(4.1)\qquad \begin{aligned} \mathcal{U}^{\square\mid}_{\mathrm{ad}\langle h\rangle}(\Omega) = \{e_h \in L_\infty(\Omega)\colon\ & e_h|_{H_{\langle i\rangle}} \in P_0(H_{\langle i\rangle}) \\ & \text{for all beam elements } H_{\langle i\rangle} \in \mathscr{T}_h, \\ e_{\min} \leqslant e_h|_{H_{\langle i\rangle}} \leqslant e_{\max},\ & \text{a.e. in } \Omega,\ |e_h|_{H_{\langle i\rangle}} - e_h|_{H_{\langle j\rangle}}| \leqslant C_{\langle\square\rangle} h \\ & \text{for } i = j + 1\}. \end{aligned}
$$

This means that $\mathcal{U}^{\square\mid}_{\mathrm{ad}\langle h\rangle}(\Omega)$ consists of all piecewise constant functions on $H_{\langle i\rangle} \in \mathscr{T}_h$ satisfying the uniform boundedness. The uniform Lipschitz constraint is satisfied by the discrete values $e_h|_{H_{\langle i\rangle}}$, $H_{\langle i\rangle} \in \mathscr{T}_h$. We note that $\mathcal{U}^{\square\mid}_{\mathrm{ad}\langle h\rangle}(\Omega)$ is not a subset of $\mathcal{U}^{\triangle}_{\mathrm{ad}}(\Omega)$ (since it contains discontinuous functions), where

$$
(4.2)\qquad \mathcal{U}^{\triangle}_{\mathrm{ad}}(\Omega) = \{e \in C(\overline{\Omega})\colon\ 0 < e_{\min} \leqslant e \leqslant e_{\max} \text{ in } \Omega\}.
$$

On the other hand $\mathcal{U}^{\triangle}_{\mathrm{ad}}$ is not a compact subset of $\mathcal{U}^{\triangle}(\Omega) = C(\overline{\Omega})$. But one can still extend $\mathcal{U}^{\triangle}_{\mathrm{ad}}(\Omega)$ as follows

$$
(4.3)\qquad \mathcal{U}^{\square\cdot\cdot}_{\mathrm{ad}}(\Omega) = \{e \in L_\infty(\Omega)\colon\ 0 < e_{\min} \leqslant e \leqslant e_{\max} \text{ almost everywhere in } \Omega\}.
$$

Then $\mathcal{U}^{\square\cdot\cdot}_{\mathrm{ad}}(\Omega)$ is a compact subset of $L_\infty(\Omega)$ with respect to weakly star convergence.

Further we introduce the set

$$
(4.2)_{\square}\qquad \mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega) = \Big\{e \in C^{0,1}(\overline{\Omega})\colon\ 0 < e_{\min} \leqslant e \leqslant e_{\max} \text{ in } \overline{\Omega}, \Big|\frac{\mathrm{d}e}{\mathrm{d}x}\Big| \leqslant C_{\square} \text{ a.e.}\Big\},
$$

i.e. $\mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega)$ consists of functions that are uniformly bounded and uniformly Lipschitz continuous on $\overline{\Omega}$. Obviously, $\mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega) \neq \emptyset$. Furthermore, $\mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega)$ is a compact subset of $\mathcal{U}^{\triangle}(\Omega)$ as follows from the Ascoli-Arzela theorem. As we already know, the result of the optimization process depends on, among other factors, how large $\mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega)$ is.

As far as the approximation of $\mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega)$ is concerned we then have the case:

$$
(4.4)\qquad \begin{aligned} \mathscr{U}_{\mathrm{ad}\langle h\rangle}(\Omega) &= \mathcal{U}^{\square\mid}_{\mathrm{ad}\langle h\rangle}(\Omega) \quad \text{for any } h > 0 \text{ defined by (4.1)}, \\ \mathscr{U}(\Omega) &= C(\overline{\Omega}), \quad \mathcal{U}_{\square}(\Omega) = L_\infty(\Omega). \end{aligned}
$$

In the following we want to find the thickness distribution $e \in \mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega)$ with $\mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega)$ defined by $(4.2)_{\square}$ satisfying (1.11).

Now we may define the following approximate state problem for the pseudo-beam with discontinuous thickness:

$$(4.5)\qquad \begin{cases} \text{Given any } e_h \in \mathcal{U}^{\Box\mathrm{I}}_{\mathrm{ad}\langle h\rangle}(\Omega) \text{ find } u_h(e_h) \in \mathcal{K}_h(\Omega) \text{ such that} \\ \langle \mathcal{A}(e_h)u_h(e_h), v_h - u_h(e_h)\rangle_{V(\Omega)} \geqslant \langle L, v_h - u_h(e_h)\rangle_{V(\Omega)} \\ \text{for all } v_h \in \mathcal{K}_h(\Omega). \end{cases}$$

Here we introduce the mapping $\mathcal{A}(e_h)\colon\ V_h(\Omega) \to V_h^*(\Omega)$ by the relation

$$(4.6)\qquad \langle \mathcal{A}(e_h)v_h, z_h\rangle_{V(\Omega)} = a(e_h, v_h, z_h), \quad [v_h, z_h] \in V_h(\Omega),$$

where

$$a(e_h, v_h, z_h) = KG \sum_{H\subset\Omega} e_h(\Pi_H) \int_H \langle \mathbf{grad}\, v_h, \mathbf{grad}\, z_h\rangle_{\mathbb{R}_1}\, \mathrm{d}\Omega,$$

Π_H being the centroid of the beam element $H \subset \Omega$.

Hence, we deduce that (we employ the Fridrichs inequality)

$$(4.7)\qquad \begin{cases} \langle \mathcal{A}(e_h)v_h, v_h\rangle_{V(\Omega)} \geqslant \mathrm{const}_F KG e_{\min} \|v_h\|^2_{V(\Omega)}, \\ \langle \mathcal{A}(e_h)v_h, z_h\rangle_{V(\Omega)} \leqslant KG e_{\max} \|v_h\|_{V(\Omega)} \|z_h\|_{V(\Omega)}, \\ \text{for all } [v_h, z_h] \in V_h(\Omega). \end{cases}$$

Finally, let us define the cost functional

$$(4.8)\qquad \mathscr{L}_{\langle h\rangle} = \mathscr{L}.$$

Now we introduce the *approximate optimal control problem*: Given a fixed partition $\mathcal{T}_h$ of the interval $[0, L]$, find

$$(4.9)\qquad e_{\langle\Box\rangle\langle h\rangle} = \operatorname*{Arg\,Min}_{e_h \in \mathcal{U}^{\Box\mathrm{I}}_{\mathrm{ad}\langle h\rangle}(\Omega)} \mathscr{L}_{\langle h\rangle}(e_h, u_h(e_h)),$$

where $u_h(e_h)$ is the solution of the *approximate state problem* (4.5).

Lemma 2. *For fixed h, the set $\mathcal{K}_h(\Omega)$ is a closed convex subset of $V_h(\Omega)$ and $\mathcal{K}_{h\langle n\rangle}(\Omega)$ converges to $\mathcal{K}_h(\Omega)$ in the sense of Glowinski:*

$$\mathcal{K}_{h\langle n\rangle}(\Omega) \xrightarrow{\mathrm{GL}} \mathcal{K}_h(\Omega) \quad \text{as } h\langle n\rangle \to h.$$

P r o o f. The closedness and convexity are immediate. For any $v_h \in \mathcal{K}_h(\Omega)$ we construct a sequence $v_{h\langle n\rangle} = v_h$. Then one has $v_{h\langle n\rangle} \geqslant 0$ for $x \in \Omega_{0\langle i\rangle}$, $i = 1, 2, \ldots, N$, so that $v_{h\langle n\rangle} \in \mathcal{K}_h(\Omega)$. Moreover, we get $\|v_{h\langle n\rangle} - v_h\|_{V(\Omega)} \to 0$ as $n \to \infty$. Next, let $v_{h\langle n\rangle} \in \mathcal{K}_{h\langle n\rangle}(\Omega)$, $v_{h\langle n\rangle} \to v_h$ weakly in $V(\Omega)$. Here we have $v_{h\langle n\rangle} \to v_h$ strongly in $L_2(\Omega)$ and $v_{h\langle n\rangle} \geqslant 0$ for $x \in \Omega_{0\langle i\rangle}$. The Lebesgue theorem yields $v_h \geqslant 0$, so that $v_h \in \mathcal{K}_h(\Omega)$. □

Next, let $e_{h\langle n\rangle} \to e_h$ in $\mathcal{U}_{\square}(\Omega)$ as $n \to \infty$, $e_{h\langle n\rangle} \in \mathcal{U}^{\square\mathrm{l}}_{\mathrm{ad}\langle h\rangle}(\Omega)$. Then we may write

$$
\begin{aligned}
(4.10)\quad & |\langle \mathcal{A}(e_{h\langle n\rangle})v_h - \mathcal{A}(e_h)v_h, z_h\rangle_{V(\Omega)}| \\
&= \left| KG \sum_{H\subset\Omega} [e_{h\langle n\rangle}(\Pi_H) - e_h(\Pi_H)] \int_H \langle \mathbf{grad}\, v_h, \mathbf{grad}\, z_h\rangle_{\mathbb{R}^1}\,\mathrm{d}\Omega \right| \\
&\leqslant KG \sum_{H\subset\Omega} |e_{h\langle n\rangle}(\Pi_H) - e_h(\Pi_H)| \|v_h\|_{V(\Omega)} \|z_h\|_{V(\Omega)} \to 0.
\end{aligned}
$$

As a consequence one has

$$
\mathcal{A}(e_{h\langle n\rangle})v_h \to \mathcal{A}(e_h)v_h \tag{4.11}
$$

strongly in $V_h^*(\Omega)$ for all $v_h \in V_h(\Omega)$ and $e_{h\langle n\rangle} \to e_h$ in $\mathcal{U}_{\square}(\Omega)$.

Moreover, observe that

$$
\begin{aligned}
(4.12)\quad & |\langle \mathcal{A}(e_h)v_h, w_h\rangle_{V(\Omega)} - \langle \mathcal{A}(e_h)z_h, w_h\rangle_{V(\Omega)}| \\
&\leqslant \left| KG \sum_{H\subset\Omega} [e_h(\Pi_H)] \int_H \langle \mathbf{grad}[v_h - z_h], \mathbf{grad}\, w_h\rangle_{\mathbb{R}^1}\,\mathrm{d}\Omega \right| \\
&\leqslant KGe_{\max} \|v_h - z_h\|_{V(\Omega)} \|w_h\|_{V(\Omega)}.
\end{aligned}
$$

Hence, we deduce that the mapping $\mathcal{A}(e_h)\colon V_h(\Omega) \to V_h^*(\Omega)$ is Lipschitz-continuous in $V_h(\Omega)$, uniformly in $\mathcal{U}^{\square\mathrm{l}}_{\mathrm{ad}\langle h\rangle}(\Omega)$.

By virtue of (4.8) we may write

$$
\mathscr{L}_{\langle h\rangle}(v_{h\langle n\rangle}) = \int_\Omega [v_{h\langle n\rangle} - z_{\mathrm{ad}}]^2\,\mathrm{d}\Omega.
$$

Then for $v_{h\langle n\rangle} \to v_h$ in $V(\Omega)$ we deduce that

$$
\lim_{n\to\infty} \mathscr{L}_{\langle h\rangle}(v_{h\langle n\rangle}) = \mathscr{L}_{\langle h\rangle}(v_h). \tag{4.13}
$$

Based on Lemma 2 and the relations (4.6) to (4.13) the approximate state problem (4.5) has a unique solution $u_h(e_h)$ for any $e_h \in \mathcal{U}^{\square\mathrm{l}}_{\mathrm{ad}\langle h\rangle}(\Omega)$. Moreover, the approximate control problem (4.9) has at least one solution for any h.

Convergence results

Here we will study the convergence of finite element approximations when the mesh size tends to zero.

Lemma 3. *One has:* $\mathcal{K}_{h_n}(\Omega) \xrightarrow{\mathrm{GL}} \mathcal{K}(\Omega)$ *as* $h_n \to 0^+$.

P r o o f. The condition $((\mathrm{M1})_h, 1^\circ)$ is trivially satisfied, since $\mathcal{K}_{h_n}(\Omega) \subset \mathcal{K}(\Omega)$. Taking note of the density result: $\overline{\mathcal{K}(\Omega) \cap C^\infty(\overline{\Omega})} = \mathcal{K}(\Omega)$ (see [8]), it is natural to take $\Lambda(\Omega) = \mathcal{K}(\Omega) \cap C^\infty(\overline{\Omega})$. Define: $\mathscr{R}_{h_n}\colon V(\Omega) \cap C^\infty(\overline{\Omega}) \to V_{h_n}(\Omega)$ by

$$\begin{cases} \mathscr{R}_{h_n} v \in V_{h_n}(\Omega) \text{ for any } v \in V(\Omega) \cap C^\infty(\overline{\Omega}), \\ (\mathscr{R}_{h_n} v)(A) = v(A) \text{ for any } A \in \Sigma_{h_n}. \end{cases}$$

On the other hand, from the assumptions made on $\mathscr{T}_h$ we deduce that (cf. [7])

$$\|\mathscr{R}_{h_n} v - v\|_{V(\Omega)} \leqslant \operatorname{const} h_n \|v\|_{H^2(\Omega)}, \tag{4.14}$$

for any $v \in C^\infty(\overline{\Omega})$ with constant independent of h and v.

Hence, we have

$$\lim_{n\to\infty} \|\mathscr{R}_{h_n} v - v\|_{V(\Omega)} = 0, \quad \text{for any } v \in \Lambda(\Omega). \tag{4.15}$$

Further, observe that $\mathscr{R}_{h_n} v \in \mathscr{K}_{h_n}(\Omega)$ for any $v \in \Lambda(\Omega)$. This means that the condition $((\mathrm{M1})_h, 2^\circ)$ is satisfied. □

It is readily seen that the estimate

$$|\langle \mathcal{A}(e_h) v_h, v_h \rangle_{V(\Omega)}| \geqslant \operatorname{const}_{\langle\square\rangle} \|v_h\|^2_{V(\Omega)}, \quad v_h \in V_h(\Omega), \tag{4.16}$$

where the $\operatorname{const}_{\langle\square\rangle}$ is independent of h, e_h, and v_h, follows immediately from (4.6), $((4.7), 1^\circ)$, and the bounds for e_h.

Lemma 4. *For any sequence* $\{e_{h_n}\}_{n\in\mathbb{N}}$, $e_{h_n} \in \mathcal{U}^{\square\mathsf{I}}_{\mathrm{ad}\langle h\rangle}(\Omega)$, *there exists its subsequence and an element* $Q \in \mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega)$ *such that* $e_{h_{n_k}} \to Q$ *in* $L_\infty(\Omega)$.

P r o o f. Let $\{e_{h_n}\}_{n\in\mathbb{N}}$, $n \to \infty$, $e_{h_n} \in \mathcal{U}^{\square\mathsf{I}}_{\mathrm{ad}\langle h\rangle}(\Omega)$, be an arbitrary sequence. With any e_{h_n}, the following continuous piecewise linear functions Q_{h_n} defined on the partition $\mathcal{T}_{h_n\langle\square\rangle}$ will be associated.

Here one has

$$Q_{h_n}(A_{\langle i+1/2\rangle}) = e_{h_n}(H_{\langle i\rangle}), \quad i = 0, 1, \ldots, N-1,$$

and

$$Q_{h_n}(A_{\langle 0\rangle}) = e_{h_n}(H_{\langle 1\rangle}), \quad Q_{h_n}(A_{\langle N_\square\rangle}) = e_{h_n}(H_{\langle N\rangle}),$$

where $A_{\langle i-1/2\rangle}$ (or $\Pi_{H_{\langle i\rangle}}$) denotes the centroid of the pseudo-beam element $H_{\langle i\rangle}$ (the midpoint of the interval $[A_{\langle i-1\rangle}, A_{\langle i\rangle}]$), $i = 1, 2, \ldots, N_\square$.

Hence, from the construction of Q_{h_n} we see that $e_{\min} \leqslant Q_{h_n} \leqslant e_{\max}$ and $|\mathrm{d}Q_{h_n}/\mathrm{d}x| \leqslant C_{\langle\square\rangle}$, a.e. in $\overline{\Omega}$. Thus $\{Q_{h_{n_k}}\}_{k\in\mathbb{N}}$ is compact in $C(\overline{\Omega})$ so that there exists a subsequence $\{Q_{h_{n_k}}\}_{k\in\mathbb{N}}$ and an element $Q \in C(\overline{\Omega})$ such that

$$Q_{h_{n_k}} \to Q \quad \text{in } C(\overline{\Omega}) \quad \text{as } k \to \infty, \tag{4.17}$$

satisfying the estimate $e_{\min} \leqslant Q \leqslant e_{\max}$ in $\overline{\Omega}$ and $|\mathrm{d}Q/\mathrm{d}x| \leqslant C_{\langle\square\rangle}$ a.e. in $\overline{\Omega}$.

The function e_h can be viewed as the piecewise constant interpolant of $Q_h \in W^1_\infty(\Omega)$ implying that: $\|e_{h_{n_k}} - Q_{h_{n_k}}\|_{L_\infty(\Omega)} \leqslant (1/2)C_{\langle\square\rangle}h_{n_k}$. Then this, (4.17), and the triangle inequality yield the relation

$$\|e_{h_{n_k}} - Q\|_{L_\infty(\Omega)} \leqslant \|e_{h_{n_k}} - Q_{h_{n_k}}\|_{L_\infty(\Omega)} + \|Q_{h_{n_k}} - Q\|_{L_\infty(\Omega)} \to 0 \quad \text{as } k \to \infty.$$

$\square$

Lemma 5. *For any $e \in \mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega)$ there exists a sequence $\{e_{h_n}\}_{n\in\mathbb{N}}$, $e_{h_n} \in \mathcal{U}^{\square|}_{\mathrm{ad}\langle h\rangle}(\Omega)$ such that $e_{h_n} \to e$ strongly in $\mathscr{U}_\square(\Omega)$. (The density of $\mathcal{U}^{\square|}_{\mathrm{ad}\langle h\rangle}(\Omega)$ in $\mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega)$ in the $L_\infty(\Omega)$ norm.)*

Proof. Let $e \in \mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega)$ be given and define e_{h_n} as follows:

$$e_{h_n} = \sum_{H_{\langle i\rangle}\subset\Omega} \left(\frac{1}{\operatorname{diam} H_{\langle i\rangle}} \int_{H_{\langle i\rangle}} e\,\mathrm{d}\Omega\right)\chi_{H_{\langle i\rangle}}, \tag{4.18}$$

where $\chi_{H_{\langle i\rangle}}$ is the characteristic function of $H_{\langle i\rangle}$, $i = 1, 2, \ldots, N_\square$. Then $e_{h_n} \in \mathcal{U}^{\square|}_{\mathrm{ad}\langle h\rangle}(\Omega)$ and $e_{h_n} \to e$ in $L_\infty(\Omega)$ as $n \to \infty$, ($\|e_{h_n} - e\|_{L_\infty(\Omega)} \leqslant h_n C_{\langle\square\rangle}$). $\square$

Theorem 4. *Let $e_{h_n} \in \mathcal{U}^{\square|}_{\mathrm{ad}\langle h\rangle}(\Omega)$, $e \in \mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega)$ with $e_{h_n} \to e$ in $L_\infty(\Omega)$ as $n \to \infty$. Then one has*

$$u_{h_n}(e_{h_n}) \to u(e) \quad \text{strongly in } V(\Omega) \text{ as } n \to \infty. \tag{4.19}$$

Proof. We substitute $v_{h_n} = 0$ in the state inequality (4.5). Hence, we obtain the estimate

$$\langle \mathcal{A}(e_{h_n})u_{h_n}(e_{h_n}), u_{h_n}(e_{h_n})\rangle_{V(\Omega)} \leqslant \langle L, u_{h_n}(e_{h_n})\rangle_{V(\Omega)}. \tag{4.20}$$

From (4.20) and the estimate $(2.23)_\square$ we have

$$M_{1\langle\square\rangle}\|u_{h_n}(e_{h_n})\|^2_{V(\Omega)} \leqslant \text{const}\|u_{h_n}(e_{h_n})\|_{V(\Omega)},$$

so that $\|u_{h_n}(e_{h_n})\|_{V(\Omega)} \leqslant$ constant for all n. As a consequence, there exists $u_\diamondsuit \in V(\Omega)$ and a subsequence $\{u_{h_k}(e_{h_k}\}_{k\in\mathbb{N}}$ such that

$$u_{h_k}(e_{h_k}) \to u_\diamondsuit \quad \text{weakly in } V(\Omega). \tag{4.21}$$

From Lemma 3 it follows that $u_\diamondsuit \in \mathscr{K}(\Omega)$. Next we show that $u_\diamondsuit$ solves the limit state problem $(1.9)_\square$. Let $v \in \mathscr{K}(\Omega)$ be a fixed element. By virtue of Lemma 3 there exists a sequence $\{a_{h_k}\}_{k\in\mathbb{N}}$, $a_{h_k} \in \mathscr{K}_{h_k}(\Omega)$, such that

$$a_{h_k} \to v \quad \text{strongly in } V(\Omega) \text{ as } k \to \infty. \tag{4.22}$$

Then from (4.5) (we set $v_{h_k} = a_{h_k}$) it follows that

$$\langle \mathcal{A}(e_{h_k})u_{h_k}(e_{h_k}), a_{h_k} - u_{h_k}(e_{h_k})\rangle_{V(\Omega)} \geqslant \langle L, a_{h_k} - u_{h_k}(e_{h_k})\rangle_{V(\Omega)}. \tag{4.23}$$

Thus, passing to the limit as $k \to \infty$ ($h_k \to 0_+$) on the right-hand side and using (4.21) and (4.22) we obtain

$$\langle L, a_{h_k} - u_{h_k}(e_{h_k})\rangle_{V(\Omega)} \to \langle L, v - u_\diamondsuit\rangle_{V(\Omega)} \quad \text{as } k \to \infty.$$

We now pass to the limit on the left-hand side of (4.23).

The functional $v \to \langle \mathcal{A}(e)v, v\rangle_{V(\Omega)}$ is weakly lower semicontinuous, being convex and differentiable. Hence, we have

$$\liminf_{k\to\infty} \langle \mathcal{A}(e)u_{h_k}(e_{h_k}), u_{h_k}(e_{h_k})\rangle_{V(\Omega)} \geqslant \langle \mathcal{A}(e)u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega)}. \tag{4.24}$$

Next in view of Lemma 4 and $(1.5)_\square$, we arrive at the relation

$$\begin{aligned} &|\langle \mathcal{A}(e_{h_k})u_{h_k}(e_{h_k}), u_{h_k}(e_{h_k})\rangle_{V(\Omega)} - \langle \mathcal{A}(e)u_{h_k}(e_{h_k}), u_{h_k}(e_{h_k})\rangle_{V(\Omega)}| \\ &\quad = \left| KG \sum_{H\subset\Omega} \int_H [e_{h_k} - e]\langle \mathbf{grad}\, u_{h_k}(e_{h_k}), \mathbf{grad}\, u_{h_k}(e_{h_k})\rangle_{\mathbb{R}^1}\, \mathrm{d}\Omega \right| \\ &\quad \leqslant \mathrm{const}\|e_{h_k} - e\|_{L_\infty(\Omega)}\|u_{h_k}(e_{h_k})\|^2_{V(\Omega)} \to 0. \end{aligned} \tag{4.25}$$

Thus from (4.25) we conclude that

$$\begin{aligned} &\liminf_{k\to\infty} \langle \mathcal{A}(e_{h_k})u_{h_k}(e_{h_k}), u_{h_k}(e_{h_k})\rangle_{V(\Omega)} \\ &\quad = \liminf_{k\to\infty} [\langle \mathcal{A}(e)u_{h_k}(e_{h_k}), u_{h_k}(e_{h_k})\rangle_{V(\Omega)} \\ &\qquad + (\langle \mathcal{A}(e_{h_k})u_{h_k}(e_{h_k}), u_{h_k}(e_{h_k})\rangle_{V(\Omega)} - \langle \mathcal{A}(e)u_{h_k}(e_{h_k}), u_{h_k}(e_{h_k})\rangle_{V(\Omega)})] \\ &\quad \geqslant \langle \mathcal{A}(e)u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega)}. \end{aligned} \tag{4.26}$$

For any $v \in V(\Omega)$ we have

$$\lim_{k\to\infty} \langle \mathcal{A}(e_{h_k}) u_{h_k}(e_{h_k}), v\rangle_{V(\Omega)} = \langle \mathcal{A}(e) u_\diamondsuit, v\rangle_{V(\Omega)}. \tag{4.27}$$

In fact we may write

$$\begin{aligned} &|\langle \mathcal{A}(e_{h_k}) u_{h_k}(e_{h_k}), v\rangle_{V(\Omega)} - \langle \mathcal{A}(e) u_\diamondsuit, v\rangle_{V(\Omega)}| \\ &\quad \leqslant |\langle \mathcal{A}(e_{h_k}) u_{h_k}(e_{h_k}), v\rangle_{V(\Omega)} - \langle \mathcal{A}(e) u_{h_k}(e_{h_k}), v\rangle_{V(\Omega)}| \\ &\qquad + |\langle \mathcal{A}(e)[u_{h_k}(e_{h_k}) - u_\diamondsuit], v\rangle_{V(\Omega)} \to 0 \quad \text{as } k\to\infty, \end{aligned} \tag{4.28}$$

which follows from (4.25) and (4.21).

Then due to (4.28) we derive that

$$\lim_{k\to\infty} [\langle \mathcal{A}(e_{h_k}) u_{h_k}(e_{h_k}), v\rangle_{V(\Omega)} - \langle \mathcal{A}(e) u_\diamondsuit, v\rangle_{V(\Omega)}] = 0. \tag{4.29}$$

On the other hand, by virtue of (4.22) and Lemma 3, we conclude that

$$\begin{aligned} &|\langle \mathcal{A}(e_{h_k}) u_{h_k}(e_{h_k}), a_{h_k} - v\rangle_{V(\Omega)}| \\ &\quad \leqslant KGe_{\max} \|u_{h_k}(e_{h_k})\|_{V(\Omega)} \|a_{h_k} - v\|_{V(\Omega)} \to 0. \end{aligned} \tag{4.30}$$

By combining (4.30) with (4.27) we arrive at

$$\begin{aligned} &|\langle \mathcal{A}(e_{h_k}) u_{h_k}(e_{h_k}), a_{h_k}\rangle_{V(\Omega)} - \langle \mathcal{A}(e) u_\diamondsuit, v\rangle_{V(\Omega)}| \\ &\quad \leqslant |\langle \mathcal{A}(e_{h_k}) u_{h_k}(e_{h_k}), a_{h_k} - v\rangle_{V(\Omega)}| \\ &\qquad + |\langle \mathcal{A}(e_{h_k}) u_{h_k}(e_{h_k}), v\rangle_{V(\Omega)} - \langle \mathcal{A}(e) u_\diamondsuit, v\rangle_{V(\Omega)}| \to 0. \end{aligned} \tag{4.31}$$

It follows from (4.23) that

$$\begin{aligned} &\langle \mathcal{A}(e_{h_k}) u_{h_k}(e_{h_k}), u_{h_k}(e_{h_k})\rangle_{V(\Omega)} \\ &\quad \leqslant \langle \mathcal{A}(e_{h_k}) u_{h_k}(e_{h_k}), a_{h_k}\rangle_{V(\Omega)} + \langle L, u_{h_k}(e_{h_k}) - a_{h_k}\rangle_{V(\Omega)}. \end{aligned} \tag{4.32}$$

Let us pass to the $\liminf_{k\to\infty}$ on both sides in (4.32). Then by virtue of (4.26), the left-hand side is bounded below by $\langle \mathcal{A}(e) u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega)}$. The right-hand side possesses the following limit

$$\langle \mathcal{A}(e) u_\diamondsuit, v\rangle_{V(\Omega)} + \langle L, u_\diamondsuit - v\rangle_{V(\Omega)},$$

as follows from (4.31), (4.22), and (4.21).

From this we conclude that

$$\langle \mathcal{A}(e) u_\diamondsuit, u_\diamondsuit - v\rangle_{V(\Omega)} \leqslant \langle L, u_\diamondsuit - v\rangle_{V(\Omega)} \quad \text{for any } v \in \mathscr{K}(\Omega),$$

i.e. $u(e) := u_{\diamondsuit}$ solves $(1.9)_{\square}$. Since $u(e)$ is unique, the whole sequence $\{u_{h_n}(e_{h_n})\}_{n\in\mathbb{N}}$ tends weakly to $u(e)$ in $V(\Omega)$.

It remains to show strong convergence. On the basis of the variational inequality (4.5) and the estimate (4.26) we may write

$$
\begin{aligned}
(4.33)\quad & \langle \mathcal{A}(e)u(e), u(e)\rangle_{V(\Omega)} \\
& \leqslant \liminf_{n\to\infty} \langle \mathcal{A}(e_{h_n})u_{h_n}(e_{h_n}), u_{h_n}(e_{h_n})\rangle_{V(\Omega)} \\
& \leqslant \limsup_{n\to\infty} \langle \mathcal{A}(e_{h_n})u_{h_n}(e_{h_n}), u_{h_n}(e_{h_n})\rangle_{V(\Omega)} \\
& \leqslant \langle \mathcal{A}(e)u(e), v\rangle_{V(\Omega)} + \langle L, u(e)-v\rangle_{V(\Omega)} \quad \text{for all } v \in \mathscr{K}(\Omega).
\end{aligned}
$$

Put $v := u(e)$ in (4.33). Hence, one has

$$
(4.34)\qquad \lim_{n\to\infty} \langle \mathcal{A}(e_{h_n})u_{h_n}(e_{h_n}), u_{h_n}(e_{h_n})\rangle_{V(\Omega)} = \langle \mathcal{A}(e)u(e), u(e)\rangle_{V(\Omega)}.
$$

Next by (4.25) and (4.34) we have

$$
\begin{aligned}
(4.35)\quad & \lim_{n\to\infty} a(e, u_{h_n}(e_{h_n}), u_{h_n}(e_{h_n})) \\
& = \lim_{n\to\infty} \langle \mathcal{A}(e)u_{h_n}(e_{h_n}), u_{h_n}(e_{h_n})\rangle_{V(\Omega)} = a(e, u(e), u(e)).
\end{aligned}
$$

However, for $e \in \mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega)$ the estimate $(1.10)_{\square}$ holds.

This means that the bilinear form $a(e,\cdot,\cdot)$ can be taken for a scalar product in $V(\Omega)$ (in view of (4.34)). Then from (4.35) and the weak convergence of $\{u_{h_n}(e_{h_n})\}_{n\in\mathbb{N}}$ we conclude that: $\lim\limits_{n\to\infty} a(e, u_{h_n}(e_{h_n}) - u(e), u_{h_n}(e_{h_n}) - u(e)) = 0$ which in turn implies that $u_{h_n}(e_{h_n}) \to u(e)$ strongly in the space $V(\Omega)$. $\square$

Lemma 6. *Let* $e_{h_n} \in \mathcal{U}^{\square|}_{\mathrm{ad}\langle h\rangle}(\Omega)$ *with* $e_{h_n} \to e$ *in* $L_\infty(\Omega)$, $e \in \mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega)$, *as* $n \to \infty$ $(h_n \to 0_+)$. *Then one has*

$$
\lim_{n\to\infty} \mathscr{L}_{\langle h\rangle}(e_{h_n}, u_{h_n}(e_{h_n})) = \mathscr{L}(e, u(e)).
$$

Proof. By Theorem 4 it is readily seen that

$$
\begin{aligned}
(4.36)\quad & |\mathscr{L}_{\langle h\rangle}(e_{h_n}, u_{h_n}(e_{h_n})) - \mathscr{L}(e(u(e))| \\
& = \left| \int_\Omega [(u_{h_n}(e_{h_n}) - z_{\mathrm{ad}})^2 - (u(e) - z_{\mathrm{ad}})^2] \,\mathrm{d}\Omega \right| \\
& \leqslant \|u_{h_n}(e_{h_n}) - u(e)\|_{L_2(\Omega)} \|u_{h_n}(e_{h_n}) + u(e) - 2z_{\mathrm{ad}}\|_{L_2(\Omega)} \to 0.
\end{aligned}
$$

$\square$

Theorem 5. *Let* $\{e_{\langle\square\rangle\langle h_n\rangle}\}_{n\in\mathbb{N}}$, $n\to\infty$, $(h_n\to 0_+)$, *be a sequence of solutions to the approximate optimal control problem* (4.9). *Then there exists a subsequence*

$$\{e_{\langle\square\rangle\langle h_{n_k}\rangle}\}_{k\in\mathbb{N}}\subset\{e_{\langle\square\rangle\langle h_n\rangle}\}_{n\in\mathbb{N}},$$

such that

$$e_{\langle\square\rangle\langle h_{n_k}\rangle}\to e_{\langle\square\rangle}\ \textit{strongly in}\ \mathcal{U}_{\square}(\Omega), \tag{4.37}$$

$$u(e_{\langle\square\rangle\langle h_{n_k}\rangle})\to u(e_{\langle\square\rangle})\ \textit{strongly in}\ V(\Omega), \tag{4.38}$$

$$\mathscr{L}(e_{\langle\square\rangle\langle h_{n_k}\rangle},u(e_{\langle\square\rangle\langle h_{n_k}\rangle}))\to\mathscr{L}(e_{\langle\square\rangle},u(e_{\langle\square\rangle})), \tag{4.39}$$

where $e_{\langle\square\rangle}\in\mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega)$ *is a solution of the optimal control problem* (1.11). *The limit of each subsequence of* $\{e_{\langle\square\rangle\langle h_k\rangle}\}_{k\in\mathbb{N}}$ *converging in* $L_\infty(\Omega)$ *is a solution of the latter problem and the analogue of* (4.38) *holds.*

P r o o f. Due to Lemma 4, there exists a sequence $\{e_{\langle\square\rangle\langle h_{n_k}\rangle}\}_{n\in\mathbb{N}}, e_{\langle\square\rangle\langle h_{n_k}\rangle}\in\mathcal{U}^{\square|}_{\mathrm{ad}\langle\square\rangle\langle h_{n_k}\rangle}(\Omega)$, $e_{\langle\square\rangle}\in\mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega)$, $k\to\infty$ $(h_{n_k}\to 0_+)$, such that (4.37) holds. Consider an element $e\in\mathscr{U}_{\mathrm{ad}\langle\square\rangle}(\Omega)$. By virtue of Lemma 5, there exists a sequence of $e_{\langle h_n\rangle}\in\mathcal{U}^{\square|}_{\mathrm{ad}\langle h\rangle}(\Omega)$ such that $e_{\langle h_n\rangle}\to e$ in $L_\infty(\Omega)$ as $n\to\infty$ $(h_n\to 0_+)$. By definition, we have

$$\mathscr{L}(e_{\langle\square\rangle\langle h_{n_k}\rangle},u_{h_{n_k}}(e_{\langle\square\rangle\langle h_{n_k}\rangle}))\leqslant\mathscr{L}(e_{\langle h_{n_k}\rangle},u_{h_{n_k}}(e_{\langle h_{n_k}\rangle})).$$

Thus letting $k\to\infty$ $(h_{n_k}\to 0_+)$ and applying Lemma 6 to both sides of this inequality, we arrive at

$$\mathscr{L}(e_{\langle\square\rangle},u(e_{\langle\square\rangle}))\leqslant\mathscr{L}(e,u(e)),$$

so that $e_{\langle\square\rangle}$ is a solution of the original *optimal control problem*. Making use of Theorem 4 and Lemma 6, we obtain (4.38). This line of thought may be repeated for any uniformly convergent subsequence $\{e_{\langle\square\rangle\langle h_{n_k}\rangle}\}_{k\in\mathbb{N}}$. $\square$

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