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Paul Hill

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ABELIAN GROUP PAIRS HAVING A TRIVIAL COGALOIS GROUP

Paul Hill, Auburn

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Abstract. Torsion-free covers are considered for objects in the category q_2. Objects in the category q_2 are just maps in R-Mod. For $R = \mathbb{Z}$, we find necessary and sufficient conditions for the coGalois group $G(A \longrightarrow B)$, associated to a torsion-free cover, to be trivial for an object $A \longrightarrow B$ in q_2. Our results generalize those of E. Enochs and J. Rado for abelian groups.

Keywords: coGalois group, torsion-free covers, pairs of modules

MSC 2010: 16D10, 16G20

1. Introduction

Recall that, for abelian groups, if C is torsion free, then a mapping $C \xrightarrow{\varphi} A$ is called a torsion-free *precovering* of A if, whenever B is torsion free, every map $B \xrightarrow{\pi} A$ factors through $C \xrightarrow{\varphi} A$. That is, there exists a map $B \xrightarrow{\varrho} C$ that satisfies $\varphi\varrho = \pi$. A torsion-free precovering $C \xrightarrow{\varphi} A$ is said to be a *covering* if $\ker(\varphi)$ contains no nontrivial pure subgroup of C. We refer to [1] for general results about covers and envelopes.

In [2], the abelian groups were determined that have a trivial coGalois group relative to the covering class of torsion-free abelian groups. It was shown that the coGalois group $G(A)$ of an abelian group A is trivial if and only if A is p-divisible for each relevant prime p of A, where p is called a relevant prime if A has an element of order p. The criterion in [2] for $G(A) = 1$ is stated slightly differently, but is obviously equivalent to the above more succinct statement. In this connection, note that the condition stated above implies that the torsion subgroup $A(t)$ of A is divisible and therefore must split out. Thus, the above condition implies that $A = A(t) \oplus B$, where $B \cong A/A(t)$ is torsion free.

If $C \xrightarrow{\varphi} A$ is a torsion-free covering of the abelian group A and if $K = \ker(\varphi)$, then the coGalois group $G(A)$ of A consists of the automorphisms of C that map K into itself and induce the identity on C/K. Hence, an endomorphism $\pi\colon C \to C$ is an element of $G(A)$ if and only if $\pi = 1 + \sigma$, where σ is a homomorphism from C into the subgroup K. We hasten to add that if σ is a homomorphism from C into K, we know that $1 + \sigma$ is an automorphism of C since $C \xrightarrow{\varphi} A$ is a covering of A and since $\varphi(1 + \sigma) = \varphi$. We note that this description continues to hold for the category q_2 considered below.

In [3] and [4], torsion-free covers were studied for the category q_2 whose objects are maps $A \xrightarrow{f} A'$ in R-Mod. The maps in q_2 are pairs of maps (φ, φ') in R-Mod that make the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ B & \xrightarrow{g} & B' \end{array}$$

commutative, where $A \xrightarrow{f} A'$ and $B \xrightarrow{g} B'$ are objects in q_2. In this paper, we find necessary and sufficient conditions for an object $A \xrightarrow{f} A'$ in q_2 to have a trivial coGalois group. These conditions depend heavily on the mapping $f\colon A \longrightarrow A'$, not just on the groups A and A'. For example, the condition $G(A') = 1$ is both necessary and sufficient for $G(A \xrightarrow{f} A') = 1$ whenever f is monic, but if f is not monic this is by no means sufficient, and the required conditions depend on both the kernel and the image of f.

As in [2], in this paper modules are always abelian groups, that is, it is assumed that modules are always over the ring $\mathbb{Z}$.

2. Torsion free coverings for group pairs

Following [4], we say that an object $A \xrightarrow{f} A'$ is torsion in q_2 if A and A' are both torsion abelian groups. Actually, this is one of two torsion theories for q_2 studied in [4], but we consider only this one here.

The torsion-free covering for an object $A \xrightarrow{f} A'$ in q_2 was described in [4] as follows. Let $C \xrightarrow{\varphi} A$ be a torsion-free covering of A in the category of abelian groups. Likewise, let $C' \xrightarrow{\varphi'} A'$ be a torsion-free covering of A'. Denote the kernel of φ and of φ', respectively, by K and K'. Let $P \subseteq C \oplus C'$ denote the pullback of $C \xrightarrow{f\circ\varphi} A'$ and $C' \xrightarrow{\varphi'} A'$. Let ϱ and ϱ', respectively, denote the projections of $C \oplus C'$

onto C and C' (usually restricted here to $P \subseteq C \oplus C'$). Finally, let

$$K^0 = \ker(f \circ \varphi) = \{x \in C\colon\ \varphi(x) \in A_0\},$$

where $A_0 = \ker(f)$.

Note that if M is a subgroup of K and if we let $\overline{\varrho}$ denote the projection of $P/M \subseteq C/M \oplus C'$ onto C/M, then $\varphi \circ \overline{\varrho}$ maps P/M onto A, where we continue to denote the map *induced* by φ from C/M to A simply by φ. It is easily verified that $(\varphi \circ \overline{\varrho}, \varphi')$ is a map in q_2 from $P/M \xrightarrow{\varrho'} C'$ to $A \xrightarrow{f} A'$.

Theorem 2.1 ([4])**.** *Using the above notation, we have that*

$$\begin{array}{ccc} P/M & \xrightarrow{\varrho'} & C' \\ {\scriptstyle \varphi\circ\overline{\varrho}}\downarrow & & \downarrow{\scriptstyle \varphi'} \\ A & \xrightarrow{f} & A' \end{array}$$

is a torsion-free covering in q_2 *of* $A \xrightarrow{f} A'$*, whenever* M *is any maximal subgroup contained in* K *that is pure in* K^0.

As was shown in [4], the kernel of the covering map described in the preceding theorem is

$$K/M \oplus K' \xrightarrow{\varrho'} K'.$$

It should be observed that the covering map $(\varphi \circ \overline{\varrho}, \varphi')$ is onto in the sense that for every element $(a, f(a))$ belonging to $A \xrightarrow{f} A'$, there is an element in the cover $P/M \xrightarrow{\varrho'} C'$ that maps onto it. This follows immediately from the fact that $\mathbb{Z} \xrightarrow{j} \mathbb{Z}$ is a torsion free object in q_2, where j is the identity map. And if π and π' are maps from $\mathbb{Z}$ into A and A' that map 1 onto a and $f(a)$, respectively, then (π, π') is a map in q_2 that maps $(1, 1)$ onto $(a, f(a))$. Therefore, $(a, f(a))$ must be in the image of the covering map.

The following lemma establishes a necessary condition for the coGalois group $G(A \xrightarrow{f} A')$ to be trivial. An object $B \xrightarrow{f} B'$ in q_2 is said to be *bounded* if B and B' are both bounded abelian groups, $nB = 0 = nB'$ for some positive integer n. A subobject $B \xrightarrow{f} B'$ of $A \xrightarrow{f} A'$ is called a bounded endomorphic image of $A \xrightarrow{f} A'$ if it is bounded and if there is a map in q_2 from $A \xrightarrow{f} A'$ onto $B \xrightarrow{f} B'$.

Lemma 2.2. *If* $G(A \xrightarrow{f} A') = 1$, *then* $A \xrightarrow{f} A'$ *can contain no nonzero bounded endomorphic image.*

P r o o f. The proof of this lemma becomes more transparent if we use single letters to denote the objects and maps in q_2. Objects in q_2 will be denoted by bold letters. Thus, we will let $\mathbb{A}$ and $\mathbb{B}$, respectively, stand for the objects $A \xrightarrow{f} A'$ and $B \xrightarrow{g} B'$ in q_2. And whenever the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ B & \xrightarrow{g} & B' \end{array}$$

is commutative, we denote the map (φ, φ') in q_2 from $\mathbb{A}$ to $\mathbb{B}$ simply by φ. Further, the preceding commutative diagram is denoted by $\mathbb{A} \xrightarrow{\varphi} \mathbb{B}$. While we do not distinguish maps in q_2 by using bold letters, it is easy to determine from the bold letters used for their domain and codomain or from the context whether or not a map is in q_2.

Naturally, $n\varphi$ stands for $(n\varphi, n\varphi')$, which is again a map in q_2 from $\mathbb{A}$ to $\mathbb{B}$. Now suppose that $\mathbb{A} \xrightarrow{\pi} \mathbb{B}$ is a map in q_2 from $\mathbb{A}$ onto $\mathbb{B}$, where $\mathbb{B}$ is a nonzero bounded subobject of $\mathbb{A}$. Let $n\mathbb{B} = 0$ and let $\mathbb{C} \xrightarrow{\varphi} \mathbb{A}$ be a torsion-free covering of $\mathbb{A}$. Since $\mathbb{C} \xrightarrow{\varphi} \mathbb{A}$ is a torsion-free covering of $\mathbb{A}$ and since $\mathbb{B}$ is a subobject of $\mathbb{A}$, there must be an endomorphism σ of $\mathbb{C}$ that satisfies $\varphi \circ \sigma = \pi \circ \varphi$. Since φ and π are both epimorphisms, clearly σ cannot be the zero map. Moreover, since $\mathbb{C}$ is torsion free, $n\sigma$ cannot be zero either. However, $n\pi = 0$ because $n\mathbb{B} = 0$. Therefore,

$$\varphi \circ n\sigma = n\pi \circ \varphi = 0,$$

so $n\sigma$ must map $\mathbb{C}$ into the kernel of φ. Thus $G(A \xrightarrow{f} B)$ is not trivial. $\square$

Corollary 2.3. *If A and A' are bounded but not both zero, then the coGalois group of* $A \xrightarrow{f} A'$ *is not trivial.*

Another necessary condition for the coGalois group to be trivial is the following useful lemma.

Lemma 2.4. *A necessary condition for* $G(A \xrightarrow{f} A') = 1$ *is that* $G(A') = 1$ *and therefore that A' is p-divisible for every relevant prime p of A'.*

P r o o f. Assume that $G(A') \neq 1$. Then there is a nonzero map δ' from C' into K', where $C' \xrightarrow{\varphi'} A'$ is a torsion-free covering of A' and where $K' = \ker(\varphi')$. Recall from Theorem 1 that $P/M \xrightarrow{\varrho'} C'$, together with the covering map $(\varphi \circ \overline{\varrho}, \varphi')$ from

$P/M \xrightarrow{\varrho'} C'$ onto $A \xrightarrow{f} A'$, is the torsion-free cover in q_2 of $A \xrightarrow{f} A'$. And the kernel of the covering map is $K/M \oplus K' \xrightarrow{\varrho'} K'$. Let $\delta = \delta' \circ \varrho'$. Then (δ, δ') is a nonzero map from the cover $P/M \xrightarrow{\varrho'} C'$ into the kernel, $K/M \oplus K' \xrightarrow{\varrho'} K'$, of the covering map. Therefore, we conclude that $G(A \xrightarrow{f} A') \neq 1$. $\square$

The next result shows that A' being p-divisible for each relevant prime p of A' is not only necessary but also sufficient for the coGalois group of $A \xrightarrow{f} A'$ to be trivial whenever f is monic.

Theorem 2.5. *If* $\ker(f)$ *is torsion free, in particular if* f *is monic, then* $G(A \xrightarrow{f} A') = 1$ *if and only if* A' *is* p*-divisible for each relevant prime* p *of* A'.

Proof. In view of the preceding lemma, we only need to prove the sufficiency. Thus, suppose that A' is p-divisible for each relevant prime p of A'. We want to show that $G(A \xrightarrow{f} A') = 1$ whenever $\ker(f)$ is torsion free. As before, consider the torsion-free cover $P/M \xrightarrow{\varrho'} C'$ of $A \xrightarrow{f} A'$ with a covering map $(\varphi \circ \overline{\varrho}, \varphi')$ in q_2 with the kernel $K/M \oplus K' \xrightarrow{\varrho'} K'$.

Assume that (δ, δ') is a map in q_2 from the cover $P/M \xrightarrow{\varrho'} A'$ into the kernel, $K/M \oplus K' \xrightarrow{\varrho'} K'$, of the covering map. This means that the diagram

$$\begin{array}{ccc} P/M & \xrightarrow{\varrho'} & C' \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'} \\ K/M \oplus K' & \xrightarrow{\varrho'} & K' \end{array}$$

is commutative. We wish to show that (δ, δ') is the zero map in q_2. The divisibility on A' implies that $\delta' = 0$, but the argument for $\delta = 0$ is entirely different since δ maps into $K/M \oplus K'$, not K. Furthermore, no restriction has been placed on A. Since the diagram is commutative and since $\delta' = 0$, we see at once that δ must map into K/M.

Recall that earlier we defined $K^0 = \ker(f \circ \varphi)$ and that K denotes the kernel of φ. Thus, $K^0/K \cong \ker(f)$, so by hypothesis, K^0/K is torsion free. Thus, K is pure in K^0, and M must be equal to K since M is a maximal pure subgroup of K^0 contained in K. Since $K/M = 0$ and since δ maps into K/M, clearly δ must be zero. Therefore, we have the desired result that (δ, δ') is the zero map in q_2. $\square$

An immediate corollary of the preceding theorem is that the condition $G(A) = 1$ is not necessary for $G(A \xrightarrow{f} A') = 1$. Indeed, we have the following.

Corollary 2.6. *If A is arbitrary and E is its injective envelope, then $A \subseteq E$ always has a trivial coGalois group.*

We have seen that unlike the condition $G(A') = 1$, the corresponding condition $G(A) = 1$ is not always necessary for the coGalois group to be trivial. In the other direction, the next simple example demonstrates that the conditions $G(A) = 1$ and $G(A') = 1$ are not sufficient for $G(A \xrightarrow{f} A')$ to have a trivial coGalois group.

3. Example

Example. Let $A = \mathbb{Z}(p^\infty)$, and let $C \xrightarrow{\varphi} A$ be the torsion-free covering of the abelian group A. As usual, set $K = \ker(\varphi)$. It is convenient here to take $A' = A$. Then $C' = C$, and $K' = K$. Certainly, $G(A) = 1 = G(A')$ since $A = A'$ is divisible. We take f to be the zero map from A to A'. Then $K^0 = C$, so $M = 0$ since there is no nonzero pure subgroup of C contained in K. We conclude that $P/M = P = C \oplus K'$. The kernel of the covering map is $K \oplus K' \xrightarrow{\varrho'} K'$. In order to show that $G(A \xrightarrow{f} A')$ is nontrivial, we need to find a nonzero map (δ, δ') in q_2 for which the diagram

$$\begin{array}{ccc} C \oplus K' & \xrightarrow{\varrho'} & C' \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'} \\ K \oplus K' & \xrightarrow{\varrho'} & K' \end{array}$$

is commutative. There is no choice about δ'; it must be zero. However, even though there is no nonzero map from C into K, there obviously is a nonzero map δ from $C \oplus K'$ into K since $K' = K$. With this choice of δ (that maps K' onto K), the diagram commutes, so (δ, δ') is the desired nonzero map. Therefore, $G(A \xrightarrow{f} A') \neq 1$.

Using exactly the same argument as in the preceding example, one can prove the following.

Proposition 3.1. *Suppose that $G(A) = 1 = G(A')$. Let $C \xrightarrow{\varphi} A$ and $C' \xrightarrow{\varphi'} A'$ be coverings of A and A' with kernels K and K', respectively. Let $f\colon A \to A'$ be the zero map. Then $G(A \xrightarrow{f} A') = 1$ if and only if* $\operatorname{Hom}(K', K) = 0$.

Since $G(A) = 1 = G(A')$ is not sufficient for $G(A \xrightarrow{f} A') = 1$, we need to find other necessary conditions. This is accomplished in the next section.

4. Additional necessary conditions

Even though it is not necessary for A to be p-divisible for its relevant primes in order for $G(A \xrightarrow{f} A') = 1$ to hold, the next lemma shows that it is necessary for this condition to hold for $A_0 = \ker(f)$.

Theorem 4.1. *In order for $G(A \xrightarrow{f} A') = 1$, to hold it is necessary that A_0 be p-divisible for each relevant prime p of A_0.*

Proof. We know that $K^0 \xrightarrow{\varphi} A_0$ is a precover of A_0 since $K^0 = \varphi^{-1}(A_0)$ is the complete inverse image of A_0 under the covering map $C \xrightarrow{\varphi} A$. Clearly, the kernel of the precovering map $K^0 \xrightarrow{\varphi} A_0$ is $K = \ker(C \xrightarrow{\varphi} A)$.

Now, M is maximal among the pure subgroups of K^0 contained in K. Therefore, $K^0/M \xrightarrow{\varphi} A_0$ is a covering map for A_0 with the kernel K/M. If A_0 is not p-divisible for some relevant prime p of A_0, then there is a nonzero mapping π from the cover K^0/M into the kernel K/M. Since K^0/M is pure in P/M and since K/M is the kernel of a covering map, π can be extended to P/M by a special case of Wakamatsu's lemma [5]. So, we have a nonzero map

$$P/M \xrightarrow{\pi} K/M \subseteq K/M \oplus K'.$$

Thus, $(\pi, 0)$ is a nonzero map in q_2 from the cover $P/M \xrightarrow{\varrho'} C'$ of $A \xrightarrow{f} A'$ into the kernel $K/M \oplus K' \xrightarrow{\varrho'} K'$, and consequently $G(A \xrightarrow{f} A') \neq 1$. □

We have shown that both A' and $A_0 = \ker(f)$ being p-divisible for their relevant primes are necessary conditions for $A \xrightarrow{f} A'$ to have a trivial coGalois group, but the example in Section 3 demonstrates that these conditions are not sufficient. Therefore, we seek additional necessary conditions. Toward this end, let

$$A'_0 = f(A) \subseteq A'$$

denote the image of the mapping f. We will show that A'_0 must be p-divisible not for its own relevant primes but for the relevant primes p of A_0.

First we establish

Lemma 4.2. *If $G(A \xrightarrow{f} A') = 1$ and p is a relevant prime of A_0, then K/M has a rank 1 pure subgroup whose type is finite at the prime p but ∞ at every other prime.*

Proof. Let p be a relevant prime of A_0. Since $G(A \xrightarrow{f} A') = 1$, we know that A_0 is p-divisible by Theorem 4.1. Therefore, since p is a relevant prime, A_0

has $\mathbb{Z}(p^\infty)$ as a direct summand. Recall that $K^0/M \xrightarrow{\varphi} A_0$ is a cover of A_0 with the kernel K/M. Therefore, K^0/M has a nonzero divisible summand, namely the cover of $\mathbb{Z}(p^\infty)$. And consequently K/M must have a nonzero summand which is q-divisible for each prime $q \neq p$. However, this summand cannot be divisible because the kernel of a covering can never contain a nonzero divisible subgroup since the kernel has no nonzero subgroup that is pure in the cover. It follows that any rank one pure subgroup of this summand satisfies the conclusion of the lemma. □

Theorem 4.3. *If P/M has a torsion-free homomorphic image H that contains a rank 1 pure subgroup B with finite type at a relevant prime p of A_0, then $G(A \xrightarrow{f} A')$ is not trivial.*

Proof. Under the hypothesis, the preceding lemma implies that there is a nontrivial homomorphism σ from B into K/M because the type of B is less than or equal to the type of some pure subgroup of K/M. Since K/M is the kernel of a covering map (making it cotorsion) and since B is pure in H, we know that σ can be extended to a mapping of H into K/M. Therefore, since H is a homomorphic image of P/M, there is a nontrivial homomorphism from P/M into K/M, and the coGalois group of $A \xrightarrow{f} A'$ is not trivial. □

Theorem 4.4. *For the coGalois group of $A \xrightarrow{f} A'$ to be trivial, it is necessary that $A'_0 = f(A)$ be p-divisible whenever p is a relevant prime of A_0.*

Proof. Let $C'_0 = \varrho'(P/M) \subseteq C'$. It is easy to see that $C'_0 \xrightarrow{\varphi'} A'_0$ is a precover of A'_0. Suppose that A'_0 is not p-divisible for a relevant prime p of A_0. Then $\mathbb{Z}/p\mathbb{Z}$ is a homomorphic image of A'_0 and therefore of C'_0, and consequently $\mathbb{Z}/p\mathbb{Z}$ is a homomorphic image of P/M. Therefore, there must be a nontrivial homomorphism from P/M into the p-adic group since the p-adic group is the torsion-free cover of $\mathbb{Z}/p\mathbb{Z}$. This means that P/M cannot be p-divisible, so it must have a rank one pure subgroup whose type is finite at the prime p. According to the preceding theorem, there is a nontrivial homomorphism from P/M into K/M, which implies that $A \xrightarrow{f} A'$ does not have a trivial coGalois group. □

Once again, the example in Section 3 shows that the necessary conditions cited thus far are not sufficient conditions for $A \xrightarrow{f} A'$ to have a trivial co-Galois group. These conditions are:

(1) A' is p-divisible for each of its relevant primes.

(2) $A_0 = \ker(f)$ is p-divisible for each of its relevant primes.

(3) $A'_0 = \mathrm{Im}(f)$ is p-divisible for each relevant prime of A_0.

We need another necessary condition that precludes the example.

Lemma 4.5. *In order for* $G(A \xrightarrow{f} A') = 1$ *to hold, it is necessary that the following condition be satisfied:*

(4) $A'_0(p) = A'(p)$ *for each relevant prime* p *of* A_0.

P r o o f. Assume that $A'(p) \neq A'_0(p)$ for some relevant prime p of A_0, where $A(p)$ denotes the p-primary subgroup of A. Since $A'(p)$ and $A'_0(p)$ are both divisible, we can write

$$A' = \mathbb{Z}(p^\infty) \oplus A'_1,$$

where $A'_0 \subseteq A'_1$. Let $C'_1 \longrightarrow A'_1$ and $C^* \longrightarrow \mathbb{Z}(p^\infty)$ be the torsion-free coverings of A'_1 and $\mathbb{Z}(p^\infty)$, respectively. Further, let K^* be the kernel of the covering map $C^* \longrightarrow \mathbb{Z}(p^\infty)$. Then $C' = C^* \oplus C'_1$, and

$$C'_0 = K^* \oplus (C'_0 \cap C'_1).$$

Now, K^* cannot be divisible but is q-divisible if $q \neq p$ since it is the kernel of the covering map of the p-group $\mathbb{Z}(p^\infty)$. Therefore, C'_0 and consequently P/M cannot be p-divisible. Hence, as before, there is a nonzero map from P/M into K/M, and the coGalois group of $A \xrightarrow{f} A'$ is not trivial. $\square$

5. The sufficiency of conditions (1)–(4)

We have shown in the previous sections that conditions (1)–(4) are necessary for $A \xrightarrow{f} A'$ to have a trivial co-Galois group. The purpose of this section is to prove that they are also sufficient.

Theorem 5.1. *The coGalois group of the object* $A \xrightarrow{f} A'$ *in* q_2 *is trivial if and only if the following conditions are satisfied.*

(1) A' *is* p*-divisible for each of its relevant primes.*

(2) $A_0 = \ker(f)$ *is* p*-divisible for each of its relevant primes.*

(3) $A'_0 = \operatorname{Im}(f)$ *is* p*-divisible for each relevant prime of* A_0.

(4) $A'_0(p) = A'(p)$ *for each relevant prime* p *of* A_0.

P r o o f. At this point, we only need to demonstrate that the conditions are sufficient for a coGalois group to be trivial. Hence, assume the conditions. Suppose that the diagram

$$\begin{array}{ccc} P/M & \xrightarrow{\varrho'} & C' \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'} \\ K/M \oplus K' & \xrightarrow{\varrho'} & K' \end{array}$$

is commutative. In view of what has gone before, the proof will be complete if we can show that δ and δ' both must be zero. Condition (1) implies that δ' is zero, so the proof is all about δ having to be zero. As we observed before, δ must map into K/M since δ' is zero. Since K^0/M is the cover of A_0 with the kernel K/M, condition (2) implies that δ must map K^0/M to 0. Therefore, δ induces a map from $P/K^0 \cong (P/M)/(K^0/M)$ into K/M, and it suffices to show that this map must be zero. But $P/K^0 \cong C_0'$, where $C_0' = \varrho'(P)$. Therefore, it suffices to show that $\mathrm{Hom}(C_0', K/M) = 0$.

Recall that K/M is the kernel of the covering $K^0/M \xrightarrow{\varphi} A_0$. We claim, therefore, that K/M cannot contain a nonzero subgroup which is p-divisible for every relevant prime of A_0. Suppose such a subgroup H exists. Then H would have to be p-pure in K^0/M for each relevant prime p of A_0. It follows that the purification H^* of H in K^0/M resides within K/M because $(K^0/M)/(K/M) \cong A_0$ has no nonzero element of order a power of q whenever q is a prime not relevant to A_0. But K/M cannot contain a nonzero pure subgroup of K^0/M since M is a maximal pure subgroup of K^0 contained in K. Hence, to verify that $\mathrm{Hom}(C_0', K/M) = 0$ and complete the proof of the theorem, it suffices to show that C_0' is p-divisible for each relevant prime of A_0.

We proceed now to show that C_0' is p-divisible for each relevant prime p of A_0. Henceforth, assume that p is a relevant prime of A_0.

In view of conditions (3) and (4), we can write

$$A' = A_0'(p) \oplus A^*,$$

where $A_0'(p)$ is the p-primary component of A' and A^* has no element of order p. So, since covers are unique, we have that

$$C' = C_{0,p}' \oplus C^*,$$

where $C_{0,p}'$ is a cover of $A_0'(p)$, and C^* is the cover of A^*. Further, we have that

$$C_0' = C_{0,p}' \oplus C_0^*,$$

where $C_0^* \subseteq C^*$ since $C_{0,p}' \subseteq C_0'$.

Since $C_{0,p}'$ is the cover of $A_0'(p)$ and since $A_0'(p)$ is divisible, we know that $C_{0,p}'$ is divisible. In particular, $C_{0,p}'$ is p-divisible, and therefore C_0' will be p-divisible if C_0^* is p-divisible.

We aim to show now that C_0^* is p-divisible. Toward this end, notice that $(A'/A_0')(p) = 0$ because A_0' is p-divisible and $A'(p) = A_0'(p)$ in view of conditions (3) and (4). Moreover, since

$$C_0' = \{x \in C' \colon \varphi'(x) \in A_0'\},$$

we have that $C'/C'_0 \cong A'/A'_0$. Thus, $(C'/C'_0)(p) = 0$, and C'_0 is p-pure in C'. Further, $C'_0 \xrightarrow{\varphi'} A'_0$ is a precover since $C' \xrightarrow{\varphi'} A'$ is a cover. Set $A^*_0 = A^* \cap A'_0$. Since $C^* \xrightarrow{\varphi'} A^*$ is a cover of A^* and since

$$C^*_0 = \{x \in C^* \colon \varphi'(x) \in A^*_0\},$$

it follows that C^*_0 is a precover of A^*_0. Therefore,

$$C^*_0 = C(A^*_0) \oplus L,$$

where $C(A^*_0)$ is a cover of A^*_0 and where $L \subseteq K^* = \ker(C^* \xrightarrow{\varphi'} A^*)$. Since C'_0 is p-pure in C', it follows that C^*_0 is p-pure in C^*. So, by transitivity, L is p-pure in C^* and consequently in K^*.

Recall that $A' = A'_0(p) \oplus A^*$, so condition (1) implies that $A^* = A^*(t) \oplus F$, where $A^*(t)$ is the torsion part of A^* and F is torsion free. Hence, $C^* = C(A^*(t)) \oplus F$, where $C(A^*(t))$ is the cover of $A^*(t)$. Since A^* is a summand of A' and since $A'(t)$ is divisible, so is $A^*(t)$. Therefore, $C(A^*(t))$ is divisible. But $K^* \subseteq C(A^*(t))$ and it is p-pure in $C(A^*(t))$ since $A^*(t)(p) = 0$. Therefore, K^* is p-divisible. As we have seen, L is p-pure in K^*, so L is p-divisible.

It remains only to show that $C(A^*_0)$ is p-divisible since $C^*_0 = C(A^*_0) \oplus L$. But A^*_0 is p-divisible since

$$A'_0 = A'_0(p) \oplus A^*_0$$

and A'_0 is p-divisible by condition (3). Thus, $C(A^*_0)$ must also be p-divisible. □

We have seen that in the special case where f is monic, then $A \xrightarrow{\varphi} A'$ has a trivial coGalois group if and only if A' is p-divisible for each relevant prime. We now consider the special case where f is epic. So, for a subgroup A_0 of A, we consider the object $A \longrightarrow A/A_0$, where the map is the natural map from A onto its quotient A/A_0.

The following result is easily established by specializing Theorem 5.1 to the case at hand.

Corollary 5.2. *The coGalois group of $A \longrightarrow A/A_0$ is trivial if and only if:*

(i) *A/A_0 is p-divisible for each of its own relevant primes and for the relevant primes of A_0.*

(ii) *A_0 is p-divisible for each of its relevant primes.*

6. Extensions having trivial coGalois group

In this section we consider the notion of an extension of a group A by a group C having a trivial coGalois group. More precisely, we consider a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and say that it has a trivial coGalois group if all the maps, together with their specified domains and codomains, have trivial coGalois groups as objects in the category q_2. The following theorem gives definitive results.

Theorem 6.1. *The short-exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

representing the extension of A by C has a trivial coGalois group if and only if each of the groups A, B, and C is p-divisible for each of its relevant primes.

Proof. Since each of the groups A, B, and C is the codomain of one of the maps in the sequence, it is clearly necessary according to Lemma 2.4 that these groups be p-divisible for their relevant primes in order for the sequence of maps to have trivial coGalois groups.

In order to prove sufficiency of the conditions, we begin by observing that the hypothesis that A and B are p-divisible for their relevant primes means that the monic maps $0 \longrightarrow A$ and $A \longrightarrow B$ have trivial coGalois groups by Theorem 2.5. Moreover, since the kernel of the mapping $C \longrightarrow 0$ is the group B and since B is p-divisible for its relevant primes, it follows quickly that $C \longrightarrow 0$ satisfies conditions (1)–(4) of Theorem 5.1. Therefore, $C \longrightarrow 0$ has a trivial coGalois group. It remains to show that $B \longrightarrow C$ has a trivial coGalois group.

Condition (1) of Theorem 5.1 is satisfied for $B \longrightarrow C$ because C is p-divisible for its relevant primes. And the kernel of the map is A, which is p-divisible for its relevant primes. So, condition (2) of Theorem 5.1 is also satisfied. But, in addition, we need to show that C is p-divisible for the relevant primes of A in order to verify condition (3). Toward this end, let p be a relevant prime of A. Since $A \longrightarrow B$ is monic, p must be a relevant prime of B, Therefore, by hypothesis, B is p-divisible. Since $B \longrightarrow C$ is epic, C must be p-divisible since B is. Finally, condition (4) is satisfied trivially since $B \longrightarrow C$ is an epimorphism. Therefore, $B \longrightarrow C$ satisfies conditions (1)–(4) of Theorem 5.1, and hence it has a trivial coGalois group. □

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Author's address: Hill P., Auburn University, Department of Mathematics, Auburn, AL 36849, USA, e-mail: hillpa@auburn.edu.