Shirong Li; Yangming Li

On S-quasinormal and c-normal subgroups of a finite group

Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 4, 1083–1095

Persistent URL: http://dml.cz/dmlcz/140441

Czechoslovak Mathematical Journal, 58 (133) (2008), 1083–1095

ON S-QUASINORMAL AND c-NORMAL SUBGROUPS OF A FINITE GROUP

Shirong Li, Guangxi, and Yangming Li, Guangzhou

(Received November 17, 2006)

Abstract. Let $\mathscr{F}$ be a saturated formation containing the class of supersolvable groups and let G be a finite group. The following theorems are presented: (1) $G \in \mathscr{F}$ if and only if there is a normal subgroup H such that $G/H \in \mathscr{F}$ and every maximal subgroup of all Sylow subgroups of H is either c-normal or S-quasinormally embedded in G. (2) $G \in \mathscr{F}$ if and only if there is a normal subgroup H such that $G/H \in \mathscr{F}$ and every maximal subgroup of all Sylow subgroups of $F^*(H)$, the generalized Fitting subgroup of H, is either c-normal or S-quasinormally embedded in G. (3) $G \in \mathscr{F}$ if and only if there is a normal subgroup H such that $G/H \in \mathscr{F}$ and every cyclic subgroup of $F^*(H)$ of prime order or order 4 is either c-normal or S-quasinormally embedded in G.

Keywords: S-quasinormally embedded subgroup, c-normal subgroup, p-nilpotent group, the generalized Fitting subgroup, saturated formation

MSC 2010: 20D10

1. Introduction

All groups considered in this paper are finite. Let G be a group and let $\mathscr{M}(G)$ be the set of all maximal subgroups of the Sylow subgroups of G. An interesting problem in group theory is to study the influence of the elements of $\mathscr{M}(G)$ on the structure of G. A typical result in this direction is due to Srinivasan [13]. It states that G is supersolvable provided that every member of $\mathscr{M}(G)$ is normal in G. This result has been widely generalized.

A subgroup H of G is called *S-quasinormal* in G provided H permutes with all Sylow subgroups of G, i.e, $HS = SH$ for any Sylow subgroup S of G. This concept

Supported by the Natural Science Foundation of China and the Natural Science Foundation of Guangxi Autonomous Region (No. 0249001).

Corresponding author. Supported in part by the Natural Science Foundation of China (10571181), NSF of Guangdong Province (06023728) and ARF(GDEI).

was introduced by Kegel in [4] and has been studied extensively by Deskins [3] and Schmid [12]. More recently, Ballester-Bolinches and Pedraza-Aguilera [2] generalized S-quasinormal subgroups to *S-quasinormally embedded subgroups.* A subgroup H of G is said to be S-quasinormally embedded in G provided every Sylow subgroup of H is a Sylow subgroup of some S-quasinormal subgroup of G. In [2], Ballester-Bolinches and Pedraza-Aguilera showed that if every subgroup in $\mathscr{M}(G)$ is S-quasinormally embedded in G, then G is supersolvable. M. Asaad and A. A. Heliel [1] showed that a group G is p-nilpotent for the smallest prime p dividing $|G|$ if and only if all members of $\mathscr{M}(G_p)$ are S-quasinormally embedded in G. In the same paper, they showed that a group G belongs to $\mathscr{F}$, a saturated formation containing all supersolvable groups, if and only if there is a normal subgroup H such that $G/H \in \mathscr{F}$ and every member of $\mathscr{M}(H)$ is S-quasinormally embedded in G. In the paper [10], the research in this direction has been continued further by considering a subset $\mathscr{M}_d(G)$ of $\mathscr{M}(G)$. In [11], Li and Wang have proved that $G \in \mathscr{F}$, a saturated formation containing all supersolvable groups, if and only if there is a normal subgroup H such that $G/H \in \mathscr{F}$ and every member of $\mathscr{M}(F^*(H))$, where $F^*(H)$ is the generalized Fitting subgroup of H, is S-quasinormally embedded in G.

As another generalization of normality, Wang [15] introduced the following concept: A subgroup H of G is called *c-normal* in G if there is a normal subgroup K such that $G = HK$ and $H \cap K \leqslant H_G$, where H_G is the normal core of H in G. In [15], Wang showed that G is supersolvable if every member of $\mathscr{M}(G)$ is c-normal in G. Wang's result has been generalized by some authors(see [5], [8], [9], [16], [17], etc). For example, Guo and Shum showed in [5] the following result. Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}(P)$ is c-normal, then G is p-nilpotent. In [17], Wei, Wang and Li showed that $G \in \mathscr{F}$ if there is a normal subgroup H such that $G/H \in \mathscr{F}$ and if every member of $\mathscr{M}(F^*(H))$ is c-normal in G (see [17]). The research on c-normal subgroups has formed a series, which is similar to the series of S-quasinormal subgroups, but the two series are independent of each other.

The aim of this article is to unify and improve the results of [1], [11], [17] and some of [5]. Our results are more general. At the end, we also consider the influences of minimal subgroups of G on the structure of G.

A class $\mathscr{F}$ of finite groups is called *a formation* if $G \in \mathscr{F}$ and $N \trianglelefteq G$ imply $G/N \in \mathscr{F}$, and $G/N_i \in \mathscr{F}$ $(i = 1, 2)$ implies $G/N_1 \cap N_2 \in \mathscr{F}$. If, in addition, $G/\Phi(G) \in \mathscr{F}$ implies $G \in \mathscr{F}$, then $\mathscr{F}$ is called saturated. An interesting example of a saturated formation is the class of all supersolvable groups, which is denoted by $\mathscr{U}$. For a formation $\mathscr{F}$, each group G has a smallest normal subgroup N such that $G/N \in \mathscr{F}$. This uniquely determined normal subgroup of G is called *the $\mathscr{F}$-residual subgroup* of G and is denoted by $G^{\mathscr{F}}$.

The following notation is used in the paper. If H is a subgroup of the group G, then by H_G we denote *the normal core* of H in G, the largest normal subgroup of G which is contained in H. Also, G_p denotes always a Sylow p-subgroup of G for some prime $p \in \pi(G)$.

2. Preliminaries

We first collect some results related to the S-quasinormal subgroup.

Lemma 2.1.
(a) *An S-quasinormal subgroup of G is subnormal.*
(b) *If $H \leqslant K \leqslant G$ and H is S-quasinormal in G, then H is S-quasinormal in K.*
(c) *If H is an S-quasinormal subgroup of G, then H/H_G is nilpotent, where H_G is the core of H in G.*
(d) *Suppose that H is a nilpotent subgroup of G. Then H is S-quasinormal in G if and only if the Sylow subgroups of H are S-quasinormal in G.*
(e) *If both H and K are S-quasinormal subgroups of G, then both $H \cap K$ and $\langle H, K\rangle$ are S-quasinormal subgroups of G.*
(f) *A p-subgroup H of G is S-quasinormal in G if and only if $N_G(H) \geqslant O^p(G)$ for some prime $p \in \pi(G)$.*
(g) *Let G_p be a Sylow p-subgroup of G and let P be a maximal subgroup of G_p for some prime $p \in \pi(G)$. Then P is normal in G if and only if P is S-quasinormal in G.*

Proof. For the proof of (a) and (b), see Kegel [4]; for (c), see Deskins [3]; for (d), (e) and (f), see Schmid [12]; for (g), see Asaad and Heliel [1]. $\square$

The following lemma is related to S-quasinormally embedded subgroups.

Lemma 2.2. *Suppose that U is an S-quasinormally embedded subgroup of G and that K is a normal subgroup of G. Then*
(a) *U is S-quasinormally embedded in H whenever $U \leqslant H \leqslant G$.*
(b) *UK is S-quasinormally embedded in G and UK/K is S-quasinormally embedded in G/K.*
(c) *Suppose that $p \in \pi(G)$ and P is a maximal subgroup of a Sylow p-subgroup G_p of G. If P is S-quasinormally embedded in G, then P is normally embedded in G.*

Proof. For the proof of (a) and (b), see Ballester-Bolinches, Pedraza-Aguilera [2]. Now we prove (c).

By definition, there is an S-quasinormal subgroup M of G such that P is a Sylow p-subgroup of M. Then M/M_G is S-quasinormal in G/M_G and M/M_G is nilpotent by Lemma 2.1(c). Hence every Sylow subgroup of M/M_G is S-quasinormal in G/M_G by Lemma 2.1(d). Now, because PM_G/M_G is a Sylow p-subgroup of M/M_G, it follows that PM_G/M_G is S-quasinormal in G/M_G. By Lemma 2.1(g), PM_G/M_G is normal in G/M_G. It is easy to see that P is a Sylow p-subgroup of PM_G and PM_G is normal in G. □

Lemma 2.3. *Let G be a group and p a prime dividing the order of G such that $(|G|, p-1) = 1$. If G_p is cyclic, then G is p-nilpotent.*

Proof. Suppose $|G_p| = p^n$. Since G_p is cyclic, $|\mathrm{Aut}(G_p)| = p^{n-1}(p-1)$. We know that $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(G_p)$, hence $|N_G(P)/C_G(P)|$ divides $(|G|, p-1) = 1$. Therefore $N_G(P) = C_G(P)$. Then $N_G(P) = C_G(P)$. Applying the Burnside Theorem, we have that G is p-nilpotent. □

The following lemma is related to c-normal subgroups.

Lemma 2.4. *Let $X \leqslant H \leqslant G$ and $N \trianglelefteq G$. Then*

(a) *If X is c-normal in G, then X is also c-normal in H.*

(b) *Let π be a set of primes, let N be a normal π-subgroup of G and X be a π'-subgroup of G. If X is c-normal in G, then XN/N is c-normal in G/N.*

(c) *If N is a solvable minimal normal subgroup of G and N possesses a maximal subgroup H which is c-normal in G, then N is a cyclic group of prime order.*

(d) *Suppose that $p \in \pi(G)$ is such that $(|G|, p-1) = 1$. If G_p possesses a maximal subgroup H which is c-normal in G, then the p-nilpotent residual $G(p)$ of G is a p-group.*

Proof. For the proof of (a), see [15]; for (b), see [9]. Now we prove (c) and (d).

(c) By definition, there is a normal subgroup K of G such that $G = HK$ and $H \cap K = H_G$. As N is a minimal normal subgroup of G, it follows that $H_G = 1$ and hence $G = HK$ with $H \cap K = 1$. Moreover, we have that $N = H(N \cap K)$ and $H \cap K$ is a normal subgroup of order p. Consequently, $N = N \cap K$ by the minimality of N.

(d) By definition, there is a normal subgroup K of G such that $G = HK$ and $H \cap K = H_G$. Then $G/H_G = H/H_G \cdot K/H_G$. Therefore $|K/H_G|_p = [G : H]_p = |G_p : H| = p$, i.e., the quotient group K/M_G possesses a cyclic Sylow subgroup of order p. By Lemma 2.3, K/H_G must be p-nilpotent. So K/H_G has a normal Hall p'-subgroup of G/H_G, which is also a normal Hall p'-subgroup of G/H_G. Consequently, G/H_G is p-nilpotent. Hence $G(p) \leqslant H_G$ is a p-group. □

The following Tate's theorem is used in the proof of our Theorem 3.1.

Lemma 2.5 ([14]). *If P is a Sylow p-subgroup of G and $N \trianglelefteq G$ is such that $P \cap N \leqslant \Phi(P)$, then N is p-nilpotent.*

The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. Its definition and important properties can be found in [7, X 13]. We would like to give the following basic facts we will use in our proof.

Lemma 2.6 ([7, X 13]). *Let G be a group and M a subgroup of G.*
(1) *If M is normal in G, then $F^*(M) \leqslant F^*(G)$;*
(2) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = soc(F(G)C_G(F(G))/F(G))$;*
(3) *$F^*(F^*(G)) = F^*(G) \geqslant F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*

3. Main results

Theorem 3.1. *Suppose that $p \in \pi(G)$ is such that $(|G|, p-1) = 1$. Let P be a Sylow p-subgroup of a group G. Assume that every member of $\mathscr{M}(P)$ is either c-normal or S-quasinormally embedded in G. Then G is p-nilpotent.*

Proof. Assume that the theorem is not true and let G be a counterexample of minimal order. Let $\mathscr{M}(P) = \{P_1, \ldots, P_m\}$. By hypothesis, each P_i is either c-normal or S-quasinormally embedded in G. Without loss of generality, let $1 \leqslant k \leqslant m$ such that $P_1, \ldots, P_k$ are c-normal in G and $P_{k+1}, \ldots, P_m$ are S-quasinormally embedded in G.

If P_i is c-normal in G, then by Lemma 2.4 (d), $G/(P_i)_G$ is p-nilpotent. If P_i is S-quasinormally embedded in G, by Lemma 2.2(c) there is a normal subgroup M_i of G such that P_i is a Sylow p-subgroup of M_i. Then we have $|G/M_i|_p = p$. By Lemma 2.3, $G/(M_i)_G$ is p-nilpotent.

Set
$$N = \left(\bigcap_{i=1}^{k}(P_i)_G\right) \cap \left(\bigcap_{i=k+1}^{d}(M_i)_G\right).$$

Then $N \trianglelefteq G$. We now claim that N is p-nilpotent. Consider the subgroup $P \cap N$. Recall that P_i is a Sylow p-subgroup of $(M_i)_G$. We have $P \cap (M_i)_G = P_i$, so

$$\begin{aligned} P \cap N &= P \cap \left(\bigcap_{i=1}^{k}(P_i)_G\right) \cap \left(\bigcap_{i=k+1}^{d}(M_i)_G\right) \\ &= \left(\bigcap_{i=1}^{k}(P_i)_G\right) \cap \left(\bigcap_{i=k+1}^{d}(P \cap (M_i)_G\right) \\ &= \left(\bigcap_{i=1}^{k}(P_i)_G\right) \cap \left(\bigcap_{i=k+1}^{d} P_i\right) \leqslant \bigcap_{i=1}^{d} P_i = \Phi(P). \end{aligned}$$

Thus we get $P \cap N \leqslant \Phi(P)$ and $N \trianglelefteq PN$. Applying Tate's theorem (Lemma 2.5) to the subgroup PN, we conclude that N is p-nilpotent.

Let U be the Hall p'-normal subgroup of N. Then U is normal in G and it follows that $U \leqslant O_{p'}(G)$. It is easy to see that $O_{p'}(G) = 1$ by the choice of G. Consequently, N is a normal p-subgroup of G. Thus $N \leqslant P \cap N = \Phi(P)$. It follows by [6, III, 3.3 Hilfssatz] that $N \leqslant \Phi(G)$.

Now, $G/\Phi(G)$ is p-nilpotent. As the class of all p-nilpotent groups is a saturated formation, we conclude that G is p-nilpotent, a contradiction. $\square$

Corollary 3.2. *Suppose that G is a group. If every member of $\mathscr{M}(G)$ is either c-normal or S-quasinormally embedded in G, then G has a Sylow tower of supersolvable type.*

Proof. Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. By hypothesis, every member of $\mathscr{M}(P)$ is either c-normal or S-quasinormally embedded in G. In particular, G satisfies the condition of Theorem 3.1, so G is p-nilpotent. Let U be the normal p-complement of G. By Lemmas 2.2 (b) and 2.4(b), U satisfies the hypothesis. It follows by induction that U, and hence G possess the Sylow town property of supersolvable type. $\square$

Theorem 3.3. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and let G be a group. Then the following statements are equivalent:*

(a) *G is in $\mathscr{F}$.*

(b) *There is a normal subgroup H such that $G/H \in \mathscr{F}$ and every member of $\mathscr{M}(H)$ is either c-normal or S-quasinormally embedded in G.*

Proof. (a) $\Rightarrow$ (b): Trivial by taking $H = 1$.

(b) $\Rightarrow$ (a): Let G satisfy (b). We have to show that G is in $\mathscr{F}$. Suppose that this is not true so that there exists a counterexample G with minimal order. The proof is divided into five steps.

(1) H is a q-group for some prime q.

By Lemmas 2.2 (a) and 2.4(a), H satisfies the conditions of Corollary 3.2, hence H possesses the Sylow town property of supersolvable type. Let q be the largest prime dividing $|H|$ and let Q be a Sylow q-subgroup of H. Then Q char H and $H \trianglelefteq G$, so $Q \trianglelefteq G$. By Lemmas 2.2(b) and 2.4(b) we see that $(G/Q, H/Q)$ satisfies the condition of the theorem. By the choice of G, G/Q belongs to $\mathscr{F}$. Thus we have $H = Q$, as desired.

(2) $\Phi(Q) = 1$.

Otherwise, by Lemmas 2.2(b) and 2.4(b), $(G/\Phi(Q), Q/\Phi(Q))$ satisfies the hypothesis. So $G/\Phi(Q)$ is an $\mathscr{F}$-group by the choice of G. Furthermore, $\Phi(Q) \leqslant \Phi(G)$ by

[6, III, 3.3 Hilfssatz], hence $G/\Phi(G)$ belongs to $\mathscr{F}$. As the formation $\mathscr{F}$ is saturated, it follows that G belongs to $\mathscr{F}$, contrary to the choice of G.

(3) Q is a minimal normal subgroup of G.

Let N be a minimal normal subgroup of G contained in Q. Clearly the quotient group $(G/N, Q/N)$ satisfies the condition, so $G/N \in \mathscr{F}$. As $\mathscr{F}$ is a formation, N must be the unique minimal normal subgroup of G which is contained in Q. If $N \leqslant \Phi(G)$, as the formation $\mathscr{F}$ is saturated, G is in $\mathscr{F}$. So $N \nsubseteq \Phi(G)$ and there is a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Thus $Q = N(Q \cap M)$. In view of $G = QM$ and Q is normal abelian in G, we know that $Q \cap M$ is normal in G. If $Q \cap M > 1$, let N_1 be a minimal normal subgroup of G such that $N_1 \leqslant Q \cap M$, hence $N_1 \leqslant Q$ and $N \neq N_1$, this is a contradiction. Hence $Q \cap M = 1$, which implies $Q = N$.

(4) Every $Q_i \in \mathscr{M}(Q)$ is S-quasinormally embedded in G.

Assume that there is a Q_i in $\mathscr{M}(Q)$ such that Q_i is c-normal in G. By definition, there is a normal subgroup K_i of G such that $G = Q_iK_i$ and $Q_i \cap K_i = (Q_i)_G$ is a normal subgroup of G. By (3), $Q_i \cap K_i = 1$ or Q. If $Q_i \cap K_i = Q$, then $Q_i = Q$, a contradiction. If $Q_i \cap K_i = 1$, then $Q = Q_i(Q \cap K_i)$. But then $Q \cap K_i$ is a normal subgroup of order q of G. So $Q = Q \cap K_i$ by (3). As the formation $\mathscr{F}$ contains all supersolvable groups, G is in $\mathscr{F}$, contrary to the choice of G.

(5) The final contradiction.

Let G_q be a Sylow q-subgroup of G. Then $Q \leqslant O_q(G) \leqslant G_q$ and $1 \neq Q \cap Z(G_q)$. Thus we can find a subgroup X of order q of $Q \cap Z(G_q)$. Let $\{Q_1, \ldots, Q_m\}$ be the subset of $\mathscr{M}(Q)$ satisfying $X \leqslant Q_i$. Now, every Q_i is S-quasinormally embedded in G, that is, there exists an S-quasinormal subgroup M_i of G such that Q_i is a Sylow q-subgroup of M_i. Then $Q_i = Q \cap M_i$. In particular, Q_i is the intersection of the two S-quasinormal subgroups. By Lemma 2.1(e), Q_i is S-quasinormal in G. Again applying Lemma 2.1(e), we obtain that $\bigcap_{i=1}^{m} (Q_i)$ is S-quasinormal in G. It is clear that $\bigcap_{i=1}^{m} (Q_i) = X$ by the definition of $\{Q_1, \ldots, Q_m\}$, so X is S-quasinormal in G. By Lemma 2.1(f), $O^q(G) \leqslant N_G(X)$. On the other hand, G_q centralizes X. Consequently, X is normal in G. But then we have $Q = X$ by (3), and we get $G \in \mathscr{F}$, which is the final contradiction. □

From Theorem 3.3 the following corollaries are immediate.

Corollary 3.4 ([1]). *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and let G be a group and H a normal subgroup such that $G/H \in \mathscr{F}$. Suppose that every member of $\mathscr{M}(H)$ is S-quasinormally embedded in G. Then G is in $\mathscr{F}$.*

Corollary 3.5 ([16]). *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and let G be a group and H a normal subgroup such that $G/H \in \mathscr{F}$. Suppose that every member of $\mathscr{M}(H)$ is c-normal in G. Then G is in $\mathscr{F}$.*

Remark. The following example indicates that our theorem covers the results of Asaad and Heliel [1] and Wei [16] result's properly.

Example 3.6. $G = \langle a, b, c\colon a^5 = b^4 = c^5 = 1, b^{-1}ab = a^2, [a, c] = [b, c] = 1\rangle$. This group is supersolvable with order $2^2.5^2$. Sylow 2-subgroup $T = \langle b\rangle$ is of order 4, $\langle b^2\rangle$ is a maximal subgroup of T, it is S-quasinormally embedded in G, but not c-normal. All maximal subgroups of Sylow 5-subgroup are c-normal, but not all are S-quasinormally embedded in G, in fact, the subgroup $\langle u\rangle$ $(u = ac)$ is not S-quasinormally embedded in G.

Theorem 3.7. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and let G be a group. Then the following two statements are equivalent:*

(a) $G \in \mathscr{F}$.

(b) *There exists a normal solvable subgroup H of G such that $G/H \in \mathscr{F}$ and every member of $\mathscr{M}(F(H))$ is either c-normal or S-quasinormally embedded in G.*

P r o o f. (a) $\Rightarrow$ (b): Consider $H = 1$.

(b) $\Rightarrow$ (a): Assume that G satisfies (b). We want to show that G belongs to $\mathscr{F}$. As H is assumed to be solvable, we have that $F(H) > 1$; otherwise $H = 1$, the trivial case.

(1) $\Phi(H) = 1$ and hence $F(H)$ is abelian.

We know that $F(H)$ is the largest normal nilpotent subgroup of H, it follows that $\Phi(H) \leqslant F(H)$ and $\Phi(H) \trianglelefteq G$. Put $N = \Phi(H)$ We claim that $(G/N, F(H/N))$ satisfies the condition. For this purpose, let L/N be the Fitting subgroup of H/N. As $N = \Phi(H) \leqslant \Phi(G)$ by [6, III, 3.3 Hilfssatz], L/N is a nilpotent normal subgroup of H/N. By [6, III, 3.7, Satz], L is nilpotent, so $L \leqslant F(H)$ and it follows that $F(H/N) = F(H)/N$. Thus every Sylow subgroup of $F(H/N)$ possesses the form PN/N where P is a Sylow subgroup of H and $\mathscr{M}(PN/N) = \{P_iN/N | P_i \in \mathscr{M}(P)\}$. By hypothesis, P_i is either c-normal or S-quasinormally embedded in G. It follows by Lemmas 2.2(b) and 2.4(b) that P_iN/N is either c-normal or S-quasinormally embedded in G/N. Consequently, G/N satisfies the condition. If $N > 1$, the induction implies that the theorem holds for $(G/N, F(H/N))$, so G/N belongs to $\mathscr{F}$. As $\mathscr{F}$ is saturated and $N \leqslant \Phi(G)$, we can conclude that $G \in \mathscr{F}$, as desired. Therefore we may assume that $N = 1$. Hence $F(H)$ is a direct product of normal subgroups of prime order.

(2) Every minimal normal subgroup of G contained in $F(H)$ is cyclic.

Let N be any minimal normal subgroup of G which is contained in $F(H)$. Then $N \leqslant O_p(G)$ for some prime p. Let G_p be a Sylow p-subgroup of G. Then $N \cap Z(G_p) > 1$. So we can find a subgroup X of order p such that $X \leqslant N \cap Z(G_p)$. Let P be a Sylow p-subgroup of $F(H)$ and let $\{P_1, \ldots, P_m\}$ be the set of maximal subgroups P_i of P satisfying $X \leqslant P_i$ $(m \geqslant 1)$. If P_i is S-quasinormally embedded in G, then there is a S-quasinormal subgroup M_i such that P_i is a Sylow p-subgroup of M_i. Then $P_i = P \cap M_i$ and hence P_i is S-quasinormal in G because the intersection of two S-quasinormal subgroups is also S-quasinormal (see Lemma 2.1(e)). Suppose that P_i is not S-quasinormally embedded in G. By hypothesis, P_i is c-normal in G. By definition, there is a normal subgroup K_i of G such that $G = P_iK_i$ and $P_i \cap K_i \leqslant (P_i)_G$, the normal core of P_i in G. Write K^* for the subgroup K_iX. Thus $(K^*)^G = (K^*)^{P_iK_i} = K^*$, that is, K^* is normal in G. As P is abelian by conclusion (1), we see that $P_i \cap K^*$ is normal in G. So $P_i \cap K^* \leqslant (P_i)_G$ and hence $X \leqslant (P_i)_G$. Now

$$X \leqslant \left(\bigcap_{i=1}^{l}(P_i)\right) \cap \left(\bigcap_{i=l+1}^{m}((P_i)_G)\right) \leqslant \bigcap_{i=1}^{m}(P_i) = X,$$

where $P_1, \ldots, P_l$ are all S-quasinormal in G and all $(P_i)_G$ are normal in G. The inclusion gives $X = \left(\bigcap_{i=1}^{l}(P_i)\right) \cap \left(\bigcap_{i=l+1}^{m}(P_i)_G\right)$. In particular, X is the intersection of some S-quasinormal subgroups. Again applying Lemma 2.1(e), we conclude that X is S-quasinormal in G. Thus $O^p(G) \leqslant N_G(X)$ by Lemma 2.1(f). Note that G_p centralizes X and $G = O^p(G)G_p$, hence it follows that X is normal in G. As N is a minimal normal subgroup of G, we have $N = X$ and (2) holds.

(3) The conclusion.

It is well-known that $F(H)$ is the product of minimal subgroups X_i which are normal in G. By conclusion (2), all X_i are of prime order. Denote by $\mathscr{S}$ the set of all subgroups X_i. Then for each $X \in \mathscr{S}$ we have $C_H(X) = H \cap C_G(X) \trianglelefteq G$ and $H/C_H(X)$ is cyclic. Also, by hypothesis, $G/H \in \mathscr{F}$ and $\mathscr{F}$ contains $\mathscr{U}$. Hence $G/C_H(X) \in \mathscr{F}$ for all $X \in \mathscr{S}$. Because

$$C_H(F(H)) = \bigcap_{X \in \mathscr{S}} C_H(X)$$

and $\mathscr{F}$ is a formation, we get $G/C_H(F(H)) \in \mathscr{F}$. On the other hand, since H is solvable, it follows that $C_H(F(H)) \leqslant F(H)$ by [6, III, 4.2 Satz]. This yields that $G/F(H) = G/C_H(F(H))$. Thus $G/F(H) \in \mathscr{F}$. Applying Theorem 3.3, we get that G belongs to $\mathscr{F}$, completing the proof. $\square$

Corollary 3.8. *Let G be a group. If there exists a normal solvable subgroup H of G such that G/H is supersolvable and every member of $\mathscr{M}(F(H))$ is either c-normal or S-quasinormally embedded in G, then G is supersolvable.*

Next we want to delete the solvability of H in the assumption of Theorem 3.7 by replacing $F(H)$ by $F^*(H)$, the generalized Fitting subgroup of H. First we generalize Corollary 3.8 as follows.

Theorem 3.9. *Suppose that G is a group with a normal subgroup H such that G/H is supersolvable. If every member of $\mathscr{M}(F^*(H))$ is either c-normal or S-quasinormally embedded in G, then G is supersolvable.*

Proof. Suppose that the theorem is false and let G be a counter-example of smallest order; then we have:

(1) Every proper normal subgroup of G containing $F^*(H)$ is supersolvable.

If N is a proper normal subgroup of G containing $F^*(H)$, we have that $N/N \cap H \cong NH/H$ is supersolvable. By Lemma 2.6(c), $F^*(H) = F^*(F^*(H)) \leqslant F^*(H \cap N) \leqslant F^*(H)$, so $F^*(H \cap N) = F^*(H)$. Then every member of $\mathscr{M}(F^*(H \cap N))$ (i.e., of $\mathscr{M}(F^*(H))$) is either c-normal or S-quasinormally embedded in G, thus in N by Lemmas 2.1 (a) and 2.4(a). So $N, N \cap H$ satisfy the hypotheses of the theorem, and the minimal choice of G implies that N is supersolvable.

(2) $H = G$ and $F^*(G) = F(G) < G$.

If $H < G$, then H is supersolvable by (1). In particular, H is solvable, so G is solvable and $F^*(H) = F(H)$, hence G is supersolvable by Corollary 3.8, a contradiction.

If $F^*(G) = G$, then G is supersolvable by applying Theorem 3.3 for the special case $\mathscr{F} = \mathscr{U}$, a contradiction. Thus $F^*(G) < G$, it is supersolvable by (1), so $F^*(G) = F(G)$ by Lemma 2.6(c).

(3) For any Sylow p-subgroup P of $F(G)$, $G = PO^p(G)$.

Otherwise, $PO^p(G)$ is a proper normal subgroup of G. Obviously $F(G) \leqslant PO^p(G)$, so $PO^p(G)$ is supersolvable by (1), thus $O^p(G)$ is supersolvable. Since $G/O^p(G)$ is a p-group, G is solvable. Now G is supersolvable by Corollary 3.8, a contradiction.

(4) The final contradiction.

For any maximal subgroup P_1 of P, P_1 is either c-normal or S-quasinormally embedded in G by hypotheses. If P_1 is S-quasinormally embedded, then there exists an S-quasinormal subgroup K of G such that P_1 is a Sylow p-subgroup of K. Hence $P_1 = P \cap K$. Noticing that P_1 is the intersection of two S-quasinormal subgroups of G, we have that P_1 is S-quasinormal in G by Lemma 2.1(e). Consequently, $N_G(P_1) \geqslant O^p(G)$ by Lemma 2.1(f). Obviously, P normalizes P_1, so P_1 is normal in G by (3).

Therefore P_1 is c-normal in G. We have proved that every member of $\mathscr{M}(F^*(G))$ is c-normal in G. Now applying [17, Theorem 3.1] we get G is supersolvable, the final contradiction. $\square$

Theorem 3.10. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups, and suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If every member of $\mathscr{M}(F^*(H))$ is either c-normal or S-quasinormally embedded in G, then $G \in \mathscr{F}$.*

Proof. By hypotheses every member of $\mathscr{M}(F^*(H))$ is either c-normal or S-quasinormally embedded in G, thus in H by Lemmas 2.1 (a) and 2.4(a). Hence H is supersolvable by Theorem 3.9. In particular, H is solvable and so $F^*(H) = F(H)$. Therefore $G \in \mathscr{F}$ by Theorem 3.7, as desired. $\square$

The following corollaries are immediate from Theorem 3.10.

Theorem 3.11 ([11]). *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and let G be a group. Then $G \in \mathscr{F}$ if and only if there exists a normal subgroup H such that $G/H \in \mathscr{F}$ and every member of $\mathscr{M}(F^*(H))$ is S-quasinormally embedded in G.*

Theorem 3.12 ([17]). *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and let G be a group. Then $G \in \mathscr{F}$ if and only if there exists a normal subgroup H such that $G/H \in \mathscr{F}$ and every member of $\mathscr{M}(F^*(H))$ is c-normal in G.*

4. Dual results

Many authors also considered how the properties of minimal subgroups of G influence the structure of G. Here we mention two results of this kind.

Theorem 4.1 ([17, Theorem 3.2]). *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups, and suppose that G is a group. If every cyclic subgroup of prime order or order 4 of $F^*(G^{\mathscr{F}})$ is c-normal in G, where $G^{\mathscr{F}}$ is the $\mathscr{F}$-residual subgroup of G, then $G \in \mathscr{F}$.*

Theorem 4.2 ([11, Theorem 3.4]). *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups, and suppose that G is a group. If every cyclic subgroup of prime order or order 4 of $F^*(G^{\mathscr{F}})$ is S-quasinormally embedded in G, then $G \in \mathscr{F}$.*

Now we can unify Theorems 4.1 and 4.2 to get

Theorem 4.3. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If every cyclic subgroup of any Sylow subgroups of $F^*(H)$ of prime order or order 4 is either c-normal or S-quasinormally embedded in G, then $G \in \mathscr{F}$.*

P r o o f. Since $G/H \in \mathscr{F}$, we have that $G^{\mathscr{F}}$, the $\mathscr{F}$-residual subgroup of G, is contained in H. Hence, for any cyclic subgroup $\langle x\rangle$ of $F^*(G^{\mathscr{F}}) \leqslant F^*(H)$ of prime order or order 4, $\langle x\rangle$ is either c-normal or S-quasinormally embedded in G. If $\langle x\rangle$ is c-normal in G, then there exists a normal subgroup K of G such that $G = \langle x\rangle K$ and $\langle x\rangle \cap K = \langle x\rangle_G$. Hence G/K is cyclic, then $G/K \in \mathscr{F}$ by the hypotheses. Therefore $G^{\mathscr{F}} \leqslant K$. This implies that $\langle x\rangle \leqslant K$, so $\langle x\rangle = \langle x\rangle \cap K = \langle x\rangle_G$ is a normal subgroup of G. Obviously, $\langle x\rangle$ is S-quasinormally embedded in G. Hence we have proved that every cyclic subgroup of prime order or order 4 of $F^*(G^{\mathscr{F}})$ is S-quasinormally embedded in G. Applying Theorem 4.2, we have $G \in \mathscr{F}$, as desired. □

Acknowledgment. The authors thank the referee for his or her useful remarks.

References

[1] *M. Asaad and A. A. Heliel*: On S-quasinormal embedded subgroups of finite groups. J. Pure App. Algebra *165* (2001), 129–135.

[2] *A. Ballester-Bolinches and M. C. Pedraza-Aguilera*: Sufficient conditions for supersolvability of finite groups. J. Pure App. Algebra *127* (1998), 113–118.

[3] *W. E. Deskins*: On quasinormal subgroups of finite groups. Math. Z. *82* (1963), 125–132.

[4] *O. H. Kegel*: Sylow Gruppen und subnormalteiler endlicher Gruppen. Math. Z. *78* (1962), 205–221.

[5] *X. Y. Guo and K. P. Shum*: On c-normal maximal and minimal subgroups of Sylow p-subgroups of finite groups. Arch. Math. *80* (2003), 561–569.

[6] *B. Huppert*: Endliche Gruppen I. Springer-Verlag, Berlin-Heidelberg-New York, 1967.

[7] *B. Huppert and N. Blackburn*: Finite Groups III. Springer-Verlag, Berlin, New York, 1982.

[8] *D. Li and X. Guo*: The influence of c-normality of subgroups on structure of finite groups. Comm. Algebra *26* (1998), 1913–1922.

[9] *D. Li and X. Guo*: The influence of c-normality of subgroups on structure of finite groups II. J. Pure App. Algebra *150* (2000), 53–60.

[10] *Shirong Li and Xuanli He*: On normally embedded subgroups of prime power order in finite groups. Comm. Algebra *36* (2008), 2333–2340.

[11] *Yangming Li and Yanming Wang*: On π-quasinormally embedded subgroups of finite group. J. Algebra *281* (2004), 109–123.

[12] *P. Schmid*: Subgroups permutable with all Sylow subgroups. J. Algebra *207* (1998), 285–293.

[13] *S. Srinivasan*: Two sufficient conditions for supersolvability of finite groups. Israel J. Math. (1980), 210–214.

[14] *J. Tate*: Nilpotent quotient groups. Topology *3* (1964), 109–111.

[15] *Yanming Wang*: c-normality of groups and its properties. J. Algebra *180* (1996), 954–965.

[16] *H. Wei*: On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups. Comm. Algebra *29* (2001), 2193–2200.
[17] *H. Wei, Y. Wang and Y. Li*: On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups II. Comm. Algebra *31* (2003), 4807–4816.

Authors' addresses: Shirong Li, Department of Mathematics, Guangxi University, Nanning, Guangxi, 530 004, China, e-mail: shirong@gxu.edu.cn; Yangming Li, Department of Mathematics, Guangdong College of Education, Guangzhou, 510 310, China, e-mail: liyangming@gdei.edu.cn.