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POSITIVE SOLUTIONS FOR THIRD ORDER MULTI-POINT SINGULAR BOUNDARY VALUE PROBLEMS

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Abstract. We study a third order singular boundary value problem with multi-point boundary conditions. Sufficient conditions are obtained for the existence of positive solutions of the problem. Recent results in the literature are significantly extended and improved. Our analysis is mainly based on a nonlinear alternative of Leray-Schauder.

Keywords: positive solution, singular boundary value problem, multi-point boundary condition, nonlinear alternative of Leray-Schauder

MSC 2010: 34B15, 34B10, 34B16

1. Introduction

In this paper, we are concerned with the existence of positive solutions of the singular boundary value problem (BVP) consisting of the forced nonlinear third order differential equation

$$u''' = f(t,u) + e(t), \quad t \in (0,1), \tag{1.1}$$

and the multi-point boundary condition (BC)

$$u(0) = u'(p) = \int_q^1 w(s)u''(s)\,\mathrm{d}s = 0, \tag{1.2}$$

where $1/2 < p < q < 1$ are constants, $f\colon (0,1)\times(0,\infty) \to [0,\infty)$, $e\colon (0,1) \to \mathbb{R}$, and $w\colon [q,1] \to [0,\infty)$ are continuous functions, and $e \in L(0,1)$. Note that $f(t,x)$ may be singular at $t = 0, 1$, and $x = 0$ and $e(t)$ may be singular at $t = 0, 1$. By a *positive solution* of BVP (1.1), (1.2), we mean a function $u \in C^2[0,1] \cap C^3(0,1)$ such that $u(t)$ satisfies Eq. (1.1) and BC (1.2), and $u(t) > 0$ on $(0,1]$.

In recent years, the existence of positive solutions of singular higher order BVPs has been studied by many researchers. For a small sample of such work, we refer the reader to [1], [2], [6], [7], [9], [10], [11], [12], [13], [14], [15] and the references therein. In particular, Graef, Henderson, and Yang [6] studied BVP (1.1), (1.2) with $e(t) \equiv 0$ on $(0,1)$, i.e., the BVP consisting of the equation

$$u''' = f(t,u), \quad t \in (0,1), \tag{1.3}$$

and BC (1.2). Using a fixed point theorem for decreasing operators due to Gatica, Oliver, and Waltman [5], they proved the following result.

Proposition 1.1. *Assume that the following conditions hold:*

(C1) *w is nondecreasing and $w(t) > 0$ on $(q,1]$;*

(C2) *$f(t,x) > 0$ on $(0,1] \times (0,\infty)$, and for each fixed $t \in (0,1]$, $f(t,x)$ is decreasing in x;*

(C3) *$\lim_{x \to 0^+} f(t,x) = \infty$ uniformly on compact subsets of $(0,1]$;*

(C4) *$\lim_{x \to \infty} f(t,x) = 0$ uniformly on compact subsets of $(0,1]$;*

(C5) *$\int_0^1 f(s,\theta a(s))\,\mathrm{d}s < \infty$ for each $\theta > 0$, where*

$$a(t) = \frac{t(2p-t)}{p^2} \quad \textit{for } t \in [0,1]. \tag{1.4}$$

Then BVP (1.3), (1.2) has at least one positive solution.

We will establish some new existence criteria for BVP (1.1), (1.2). In our results, the forcing term $e(t)$ in Eq. (1.1) need not be positive, and we note that, even for the special case when $e(t) \equiv 0$ on $(0,1)$, our results are still extensions and improvements of Proposition 1.1. In particular, (C2) is relaxed considerably, (C3) is removed, and (C4) is replaced by a more applicable condition (see Theorem 3.1). Our proof will employ a nonlinear alternative of Leray-Schauder. Our approach involves examining a one-parameter family of nonsingular problems constructed from a sequence of nonsingular perturbations of f. For each of these nonsingular problems, we will apply the nonlinear alternative of Leray-Schauder to obtain the existence of solutions. From this sequence of solutions, we will extract a subsequence that converges to a positive solution of some associated problem, and from here, we can deduce the existence of a positive solution of BVP (1.1), (1.2). This type of technique has been successfully used in obtaining positive solutions for several classes of singular BVPs, see, for example, [1], [2], [3], [4]. Our proofs are partly motivated by these works.

To demonstrate the applicability of our general existence results, we also derive some sufficient conditions for the existence of positive solutions of the BVP consisting of the equation

$$u''' = c(t)u^{-\alpha} + \mu d(t)u^{\beta} + e(t),\ t \in (0,1), \tag{1.5}$$

and BC (1.2), where $\alpha \geqslant 0$ and $\beta \geqslant 0$ are constants, c, d, and e are continuous functions on $[0,1]$ with $c(t) > 0$ on $(0,1)$ and $d(t) \geqslant 0$ on $(0,1)$, and $\mu > 0$ is a parameter (see Corollary 3.1).

The rest of this paper is organized as follows. In Section 2, we present some preliminary lemmas, and the main results together with their proofs are given in Section 3.

2. Preliminary results

Recall that if $I \subseteq \mathbb{R}$ is an interval, then the characteristic function χ on I is given by

$$\chi_I(t) = \begin{cases} 1, & t \in I, \\ 0, & t \notin I. \end{cases}$$

From [8], the Green function $G(t,s)$ for the BVP consisting of the equation

$$u'''(t) = 0, \quad t \in (0,1),$$

and BC (1.2) is given by

$$\begin{aligned} G(t,s) = &-t(p-s)\chi_{[0,p]}(s) + \frac{(t-s)^2}{2}\chi_{[0,t]}(s) \\ &+ \frac{t(2p-t)}{2}W(s)\chi_{[q,1]}(s) + \frac{t(2p-t)}{2}\chi_{[0,q]}(s), \end{aligned}$$

where

$$W(s) = \left(\int_q^1 w(v)\,\mathrm{d}v\right)^{-1} \int_s^1 w(v)dv,\ s \in [q,1].$$

Our first lemma provides a lower bound for the function $G(t,s)$ defined above.

Lemma 2.1. *$G(t,s) \geqslant a(t)b(s)$ for $(t,s) \in [0,1]\times[0,1]$, where $a(t)$ is defined by* (1.4) *and*

$$b(s) = \min\Big\{\frac{p^2}{2}\left(W(s)\chi_{[q,1]}(s) + \chi_{[0,q]}(s)\right),\ \frac{p^2s^2}{4}\Big\}. \tag{2.1}$$

Proof. We will prove the lemma by discussing three different cases.

Case 1. $s \geqslant p$. In this case, we have

$$\begin{aligned}
G(t,s) &= \frac{(t-s)^2}{2}\chi_{[0,t]}(s) + \frac{t(2p-t)}{2}W(s)\chi_{[q,1]}(s) + \frac{t(2p-t)}{2}\chi_{[0,q]}(s)\\
&\geqslant \frac{t(2p-t)}{2}\left(W(s)\chi_{[q,1]}(s) + \chi_{[0,q]}(s)\right)\\
&= \frac{t(2p-t)}{p^2}\Big(\frac{p^2}{2}\left(W(s)\chi_{[q,1]}(s) + \chi_{[0,q]}(s)\right)\Big)\\
&= a(t)\Big(\frac{p^2}{2}\left(W(s)\chi_{[q,1]}(s) + \chi_{[0,q]}(s)\right)\Big).
\end{aligned}$$

Case 2. $s \leqslant p$ and $s \geqslant t$. Here we have

$$\begin{aligned}
G(t,s) &= \frac{t(2s-t)}{2} = \frac{t(2p-t)}{p^2}\frac{p^2(2s-t)}{2(2p-t)}\\
&\geqslant \frac{t(2p-t)}{p^2}\frac{ps}{4} \geqslant \frac{t(2p-t)}{p^2}\frac{p^2s^2}{4} = a(t)\frac{p^2s^2}{4}.
\end{aligned}$$

Case 3. $s \leqslant p$ and $s \leqslant t$. In this case, we have

$$G(t,s) = \frac{s^2}{2} \geqslant \frac{t(2p-t)}{p^2}\frac{p^2s^2}{4} = a(t)\frac{p^2s^2}{4}.$$

Combining the above cases yields that $G(t,s) \geqslant a(t)b(s)$ for $(t,s) \in [0,1]\times[0,1]$. This completes the proof of the lemma. $\square$

The following result, which gives information about functions that satisfy BC (1.2), is taken from [8, Lemmas 2.2–2.3].

Lemma 2.2. *Assume* (C1) *holds. Let $z \in C^2[0,1]\cap C^3(0,1)$ satisfy $z'''(t) \geqslant 0$ on $(0,1)$ and $z(0) = z'(p) = \int_q^1 w(s)z''(s)\,\mathrm{d}s = 0$. Then:*

(i) $|z(p)| = \max\limits_{t\in[0,1]}|z(t)|$;

(ii) $z(t) \geqslant a(t)\max\limits_{t\in[0,1]}|z(t)| \geqslant 0$ *for* $t \in [0,1]$.

We also need the following version of the well known nonlinear alternative of Leray-Schauder. We refer the reader to [1, Theorem 1.2.3] for this result.

Lemma 2.3. *Let K be a convex subset of a normed linear space X, and let Ω be a bounded open subset with $\tilde{p} \in \Omega$. Then every compact map $N\colon \overline{\Omega} \to K$ has at least one of the following two properties:*

(i) *N has at least one fixed point;*

(ii) *there is $u \in \partial\Omega$ and $\lambda \in (0,1)$ such that $u = (1-\lambda)\tilde{p} + \lambda N u$.*

Throughout this paper, let the Banach space $C[0,1]$ be equipped with the norm $\|u\| = \max\limits_{t\in[0,1]} |u(t)|$, and for the function $e(t)$ given in Eq. (1.1), we define

$$\gamma(t) = \int_0^1 G(t,s)e(s)\,\mathrm{d}s, \quad t \in [0,1].$$

Since $e \in L(0,1)$, we have that $\gamma \in C^2[0,1] \cap C^3(0,1)$ and moreover γ is the unique solution of the BVP consisting of the equation

$$u''' = e(t), \quad t \in (0,1),$$

and BC (1.2). For convenience, we also introduce the notations

$$\gamma_* = \min_{t\in[0,1]} \gamma(t) \quad \text{and} \quad \gamma^* = \max_{t\in[0,1]} \gamma(t).$$

Observe that $\gamma_* \leqslant 0$ since $\gamma(0) = 0$.

3. Main results

We first state our existence results.

Theorem 3.1. *Assume that* (C1) *and the following conditions hold:*

(H1) *there exist continuous, nonnegative functions $g(x)$, $h(x)$, and $\varphi(t)$ such that*

$$f(t,x) \leqslant \varphi(t)(g(x)+h(x)) \quad \textit{for } (t,x) \in (0,1)\times(0,\infty),$$
$$\int_0^1 \varphi(s)g(a(s))\,\mathrm{d}s < \infty,$$

$g(x) > 0$ is nonincreasing on $(0,\infty)$, and $h(x)/g(x)$ is nondecreasing on $(0,\infty)$;

(H2) *there exists $\delta > 0$ such that*

$$g(xy) \leqslant \delta g(x)g(y) \quad \textit{for all } x,y \in (0,\infty);$$

(H3) *for each $r > 0$, there exists a continuous nonnegative function $\psi_r(t)$ such that*

$$f(t,x) \geqslant \psi_r(t) \quad \textit{for } (t,x) \in (0,1)\times(0,r]$$

and

$$0 < \int_0^1 b(s)\psi_r(s)\,\mathrm{d}s < \infty,$$

where $b(s)$ is defined by (2.1)*;*

(H4) *there exists $R > 0$ such that*

$$R > \delta g(R)\left(1 + \frac{h(R+\gamma^*)}{g(R+\gamma^*)}\right)\int_0^1 G(p,s)\varphi(s)g(a(s))\,\mathrm{d}s$$

with δ given in (H2)*.*

If $\gamma_* = 0$, then BVP (1.1), (1.2) *has at least one positive solution $y(t)$ satisfying $y(t) > \gamma(t)$ on $(0,1]$ and $0 < \|y-\gamma\| < R$.*

As a consequence of Theorem 3.1, we have the following corollary.

Corollary 3.1. *Assume that* (C1) *holds, $0 \leqslant \alpha < 1$, and $\gamma_* = 0$.*

(i) *If $\beta < 1$, then BVP* (1.5), (1.2) *has at least one positive solution for each $\mu \in (0,\infty)$.*

(ii) *If $\beta \geqslant 1$, then there exists $\bar{\mu} > 0$ such that BVP* (1.5), (1.2) *has at least one positive solution for each $\mu \in (0,\bar{\mu})$.*

In the remainder of this section, we will prove Theorem 3.1 and Corollary 3.1.

P r o o f o f T h e o r e m 3.1. Define an operator $T\colon C[0,1] \to C[0,1]$ by

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s)+\gamma(s))\,\mathrm{d}s.$$

It is easy to verify that T is compact and that finding a fixed point of T is equivalent to finding a solution of the BVP consisting of the equation

$$u''' = f(t,u+\gamma), \quad t \in (0,1), \tag{3.1}$$

and BC (1.2).

We observe that, to prove the theorem, it suffices to show that BVP (3.1), (1.2) has a positive solution $u \in C^2[0,1] \cap C^3(0,1)$ satisfying $0 < \|u\| < R$. In fact, if this is true, we let $y(t) = u(t) + \gamma(t)$. Then, $y \in C^2[0,1] \cap C^3(0,1)$, $y(t) > \gamma(t)$ on $(0,1]$, $0 < \|y-\gamma\| < R$,

$$y''' = u''' + \gamma''' = f(t,u+\gamma) + e(t) = f(t,y) + e(t), \quad t \in (0,1),$$

and $y(t)$ satisfies (1.2). Thus, $y(t)$ is a positive solution of BVP (1.1), (1.2) with the required properties.

From (H4), there exists $n_0 \in \mathbb{N}$ such that

$$R > \frac{1}{n_0} + \delta g(R)\left(1 + \frac{h(R+\gamma^*)}{g(R+\gamma^*)}\right)\int_0^1 G(p,s)\varphi(s)g(a(s))\,\mathrm{d}s. \tag{3.2}$$

Let $\mathbb{N}_0 = \{n_0, n_0+1, \ldots\}$. For any fixed $n \in \mathbb{N}_0$, consider the family of problems

$$\begin{gathered} u''' = \lambda f_n(t, u+\gamma),\ t \in (0,1), \\ u(0) = \frac{1}{n},\ u'(p) = \int_q^1 w(s)u''(s)\,\mathrm{d}s = 0, \end{gathered} \tag{3.3}$$

where $\lambda \in (0,1)$ and $f_n(t,x) = f(t, \max\{x, 1/n\})$ for $(t,x) \in (0,1) \times \mathbb{R}$.

Clearly, solving BVP (3.3), (3.4) is equivalent to finding a solution of the integral equation

$$u(t) = \frac{1}{n} + \lambda\int_0^1 G(t,s)f_n(s, u(s)+\gamma(s))\,\mathrm{d}s = \frac{1}{n} + \lambda T_n u(t), \tag{3.5}$$

where

$$T_n u(t) = \int_0^1 G(t,s)f_n(s, u(s)+\gamma(s))\,\mathrm{d}s.$$

We claim that

$$\text{for any } \lambda \in (0,1),\ \text{any solution } u(t) \text{ of (3.5) satisfies } \|u\| \neq R. \tag{3.6}$$

If this is not the case, assume that $u(t)$ is a solution of (3.5) for some $\lambda \in (0,1)$ with $\|u\| = R$. Since $\lambda T_n u(t) \geqslant 0$ on $[0,1]$, we have $u(t) \geqslant 1/n$, which in turn implies that $u(t) + \gamma(t) \geqslant u(t) + \gamma_* = u(t) \geqslant 1/n$. Then, (3.5) becomes

$$u(t) = \frac{1}{n} + \lambda\int_0^1 G(t,s)f(s, u(s)+\gamma(s))\,\mathrm{d}s.$$

By Lemma 2.2, we see that $u(p) = \|u\|$ and $u(t) \geqslant a(t)\|u\|$ on $[0,1]$. From (H1) and (H2), it follows that

$$\begin{aligned} R = \|u\| = u(p) &= \frac{1}{n} + \lambda\int_0^1 G(p,s)f(s, u(s)+\gamma(s))\,\mathrm{d}s \\ &\leqslant \frac{1}{n_0} + \int_0^1 G(p,s)\varphi(s)g(u(s)+\gamma(s))\left(1 + \frac{h(u(s)+\gamma(s))}{g(u(s)+\gamma(s))}\right)\mathrm{d}s \\ &\leqslant \frac{1}{n_0} + \left(1 + \frac{h(\|u\|+\gamma^*)}{g(\|u\|+\gamma^*)}\right)\int_0^1 G(p,s)\varphi(s)g(a(s)\|u\| + \gamma_*)\,\mathrm{d}s \\ &= \frac{1}{n_0} + \left(1 + \frac{h(R+\gamma^*)}{g(R+\gamma^*)}\right)\int_0^1 G(p,s)\varphi(s)g(a(s)R)\,\mathrm{d}s \\ &\leqslant \frac{1}{n_0} + \delta g(R)\left(1 + \frac{h(R+\gamma^*)}{g(R+\gamma^*)}\right)\int_0^1 G(p,s)\varphi(s)g(a(s))\,\mathrm{d}s, \end{aligned}$$

which contradicts (3.2). Hence, (3.6) holds.

Note that $1/n \leqslant 1/n_0 < R$ and (3.5) can be rewritten as

$$u(t) = (1-\lambda)\frac{1}{n} + \lambda N_n u(t),$$

where $N_n u(t) = T_n u(t) + 1/n$. Then, by Lemma 2.3, N_n has at least one fixed point u_n in $\Omega = \{u \in C[0,1] \colon \|u\| < R\}$. Thus, for any $n \in \mathbb{N}_0$, we have proved that the BVP consisting of the equation

$$u''' = f_n(t, u+\gamma), \quad t \in (0,1),$$

and BC (3.4) has a solution $u_n(t)$ with $\|u_n\| < R$. Since $u_n(t) + \gamma(t) \geqslant u_n(t) + \gamma_* = u_n(t) \geqslant 1/n$, we see that $u_n(t)$ is actually a solution of BVP (3.1), (3.4) and so

$$u_n(t) = \frac{1}{n} + \int_0^1 G(t,s) f(s, u_n(s) + \gamma(s))\, \mathrm{d}s. \tag{3.7}$$

From (H3), we see that there exists a continuous nonnegative function $\psi_{R+\gamma^*}(t)$ such that

$$f(t, u_n(t) + \gamma(t)) \geqslant \psi_{R+\gamma^*}(t) \quad \text{on } (0,1).$$

Let $l = \int_0^1 b(s)\psi_{R+\gamma^*}(s)\, \mathrm{d}s$. Then, again from (H3), $l > 0$. From Lemma 2.1 and (3.7), we obtain

$$u_n(t) \geqslant a(t) \int_0^1 b(s)\psi_{R+\gamma^*}(s)\, \mathrm{d}s = la(t) \quad \text{on } [0,1]. \tag{3.8}$$

Clearly, the sequence $\{u_n(t)\}_{n\in\mathbb{N}_0}$ is uniformly bounded. In what follows, we show that it is also equicontinuous on $[0,1]$. Note that

$$u_n'(t) = \int_0^1 G_t(t,s) f(s, u_n(s) + \gamma(s))\, \mathrm{d}s.$$

From (H1), (H2), and (3.8), we have

$$\begin{aligned}
|u_n'(t)| &\leqslant \int_0^1 |G_t(t,s)|\varphi(s) g(u_n(s)+\gamma(s))\Big(1 + \frac{h(u_n(s)+\gamma(s))}{g(u_n(s)+\gamma(s))}\Big)\, \mathrm{d}s \\
&\leqslant \Big(1 + \frac{h(\|u_n\|+\gamma^*)}{g(\|u_n\|+\gamma^*)}\Big) \int_0^1 |G_t(t,s)|\varphi(s) g(la(s)+\gamma_*)\, \mathrm{d}s \\
&\leqslant \delta g(l)\Big(1 + \frac{h(R+\gamma^*)}{g(R+\gamma^*)}\Big) \int_0^1 |G_t(t,s)|\varphi(s) g(a(s))\, \mathrm{d}s \\
&\leqslant \delta g(l)\Big(1 + \frac{h(R+\gamma^*)}{g(R+\gamma^*)}\Big) \sup_{t\in[0,1]} \int_0^1 |G_t(t,s)|\varphi(s) g(a(s))\, \mathrm{d}s \\
&=: M < \infty
\end{aligned}$$

for $t \in [0,1]$. Therefore, $\{u_n(t)\}_{n\in\mathbb{N}_0}$ is equicontinuous on $[0,1]$.

Now, by the Arzelà-Ascoli theorem, $\{u_n(t)\}_{n\in\mathbb{N}_0}$ has a subsequence, which converges uniformly to a function $u \in C[0,1]$. For simplicity, we still denote it by $\{u_n(t)\}_{n\in\mathbb{N}_0}$. Note that $u_n(t)$ satisfies (3.7), (3.8), and

$$|f(s,u_n(s)+\gamma(s))| \leqslant \delta g(l)\left(1+\frac{h(R+\gamma^*)}{g(R+\gamma^*)}\right)\varphi(s)g(a(s)) \in L(0,1).$$

If we let $n\to\infty$ in (3.7) and (3.8), then, by the Lebesgue Dominated Convergence Theorem, we obtain

$$u(t)=\int_0^1 G(t,s)f(s,u(s)+\gamma(s))\,\mathrm{d}s \tag{3.9}$$

and

$$u(t)\geqslant la(t)>0 \quad \text{on } (0,1],$$

i.e., $u(t)$ is a positive solution of BVP (3.1), (1.2). Moreover, from (3.9), we deduce that $u\in C^2[0,1]\cap C^3(0,1)$. Since $\|u_n\|<R$ and $u=\lim\limits_{n\to\infty}u_n$, we have $\|u\|\leqslant R$. By an argument similar to the one used to show that (3.6) holds, we see that $\|u\|<R$. Thus, $0<\|u\|<R$. This completes the proof of the theorem. □

Proof of Corollary 3.1. We will apply Theorem 3.1. To this end, let $f(t,x)=c(t)x^{-\alpha}+\mu d(t)x^{\beta}$, $g(x)=x^{-\alpha}$, $h(x)=\mu x^{\beta}$, and $\varphi(t)=\max\{c(t),\ d(t)\}$. Then, (H1), (H2) with $\delta=1$, and (H3) with $\psi_r(t)=r^{-\alpha}c(t)$ hold. Let

$$A=\int_0^1 G(p,s)\varphi(s)(a(s))^{-\alpha}\,\mathrm{d}s.$$

Since $0\leqslant\alpha<1$, we have $0<A<\infty$ and (H4) becomes

$$\mu<\frac{R^{\alpha+1}-A}{A(R+\gamma^*)^{\alpha+\beta}} \quad \text{for some } R>0.$$

Hence, BVP (1.5), (1.2) has at lease one positive solution for

$$0<\mu<\bar{\mu}:=\sup_{R>0}\frac{R^{\alpha+1}-A}{A(R+\gamma^*)^{\alpha+\beta}}.$$

Note that $\bar{\mu}=\infty$ if $\beta<1$ and $\bar{\mu}<\infty$ if $\beta\geqslant 1$. This completes the proof of the corollary. □

References

[1] *R. P. Agarwal and D. O'Regan*: Singular Differential and Integral Equations with Applications. Kluwer Academic Publishers, Boston, 2003.

[2] *R. P. Agarwal and D. O'Regan*: Positive solutions for $(p, n-p)$ conjugate boundary value problems. J. Differential Equations *150* (1998), 462–473.

[3] *R. P. Agarwal and D. O'Regan*: Singular boundary value problems for superlinear second order ordinary and delay differential equations. J. Differential Equations *130* (1996), 333–355.

[4] *J. Chu, P. J. Torres and M. Zhang*: Periodic solutions of second order non-autonomous singular dynamical systems. J. Differential Equations *239* (2007), 196–211.

[5] *J. A. Gatica, V. Oliver and P. Waltman*: Singular nonlinear boundary value problems for second order differential equations. J. Differential Equations *79* (1989), 62–78.

[6] *J. R. Graef, J. Henderson and B. Yang*: Positive solutions to a singular third order nonlocal boundary value problem. Indian J. Math. *50* (2008), 317–330.

[7] *J. R. Graef, J. Henderson and B. Yang*: Existence of positive solutions of a higher order nonlocal singular boundary value problem. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., *16, Supplement S1* (2009), 147–152.

[8] *J. R. Graef and B. Yang*: Positive solutions of a third order nonlocal boundary value problem. Discrete Contin. Dyn. Syst. Ser. S *1* (2008), 89–97.

[9] *P. W. Eloe and J. Henderson*: Singular nonlinear $(k, n-k)$ conjugate boundary value problems. J. Differential Equations *133* (1997), 136–151.

[10] *P. W. Eloe and J. Henderson*: Singular nonlinear boundary value problems for higher order ordinary differential equations. Nonlinear Anal. *17* (1991), 1–10.

[11] *M. Maroun*: Positive solutions to an N^{th} order right focal boundary value problem. Electron. J. Qual. Theory Diff. Equ. *2007*, 17 (electronic).

[12] *M. Maroun*: Positive solutions to an third-order right focal boundary value problem. Comm. Appl. Nonlinear Anal. *12* (2005), 71–82.

[13] *L. Kong and Q. Kong*: Positive solutions of higher-order boundary value problems. Proc. Edinburgh Math. Soc. *48* (2005), 445–464.

[14] *I. Rachůnková and S. Staněk*: Sturm-Liouville and focal higher order BVPs with singularities in phase variables. Georgian Math. J. *10* (2003), 165–191.

[15] *D. O'Regan*: Existence of solutions to third order boundary value problems. Proc. Royal Irish Acad. Sect. A *90* (1990), 173–189.

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