D. Pavlica
Morse-Sard theorem for delta-convex curves


Persistent URL: [http://dml.cz/dmlcz/140622](http://dml.cz/dmlcz/140622)

**Terms of use:**

© Institute of Mathematics AS CR, 2008

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* [http://dml.cz](http://dml.cz)
MORSE-SARD THEOREM FOR DELTA-CONVEX CURVES

D. Pavlica, Praha

(Received December 20, 2006)

Abstract. Let $f : I \to X$ be a delta-convex mapping, where $I \subset \mathbb{R}$ is an open interval and $X$ a Banach space. Let $C_f$ be the set of critical points of $f$. We prove that $f(C_f)$ has zero $1/2$-dimensional Hausdorff measure.

Keywords: Morse-Sard theorem, delta-convex mapping

MSC 2010: 26A51

Let $Z$ and $X$ be Banach spaces, $U \subset Z$ an open convex set and $f : U \to X$ a mapping. We say that $f$ is a delta-convex mapping (d.c. mapping) if there exists a continuous convex function $h$ on $U$ such that $y^* \circ f + h$ is a continuous convex function for each $y^* \in Y^*$, $\|y^*\| = 1$. We say that $f : U \to X$ is locally d.c. if for each $x \in U$ there exists an open convex $U'$ such that $x \in U' \subset U$ and $f|_{U'}$ is d.c.

This notion of d.c. mappings between Banach spaces (see [7]) generalizes Hartman’s [3] notion of d.c. mappings between Euclidean spaces. Note that in this case it is easy to see that $F$ is d.c. if and only if all its components are d.c. (i.e., they are differences of two convex functions).

For $f : U \to X$ we denote $C_f := \{x \in U : f'(x) = 0\}$.

A special case of [2, Theorem 3.4.3] says that for a mapping $f : \mathbb{R}^m \to X$ of class $C^2$, where $X$ is a normed vector space, the set $f(C_f)$ has zero $(m/2)$-dimensional Hausdorff measure.

We will generalize this result in the case $m = 1$ showing that it is sufficient to suppose that $f$ is d.c. on $I$ (equivalently: $f$ is continuous and $f'_+$ is locally of bounded variation on $I$).

The work was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503.
A similar generalization of the above mentioned result on $C^2$ mappings holds for $m = 2$ as is shown (by a completely different method) in [6] where it is proved that $f(C_f)$ has zero 1-dimensional Hausdorff measure for any d.c. mapping $f : \mathbb{R}^2 \to X$.

Whether $f(C_f)$ has zero $(m/2)$-dimensional Hausdorff measure for each d.c. mapping $f : \mathbb{R}^m \to X$ for $m \geq 3$ remains open even for $X$ Euclidean space.

We denote $\alpha$-dimensional Hausdorff measure (on a metric space $X$) by $\mathcal{H}^\alpha$ and for each $Y \subset X$ we put (see [5])

$$\mathcal{H}^\alpha_\infty(Y) = \frac{\omega_\alpha}{2^\alpha} \cdot \inf \left\{ \sum_{i=1}^\infty \text{diam}^\alpha(M_i) : Y \subset \bigcup_{i=1}^\infty M_i \right\},$$

where $\omega_\alpha = (\Gamma(1/2))^{\alpha} \cdot (\Gamma(\alpha/2 + 1))^{-1}$.

For an open interval $I$, a Banach space $X$, $g : I \to X$ and $x \in I$, we denote

$$\text{md}(g, x) := \lim_{r \to 0} \frac{\|g(x + r) - g(x)\|}{|r|}.$$

If $g$ is Lipschitz, then $\text{md}(g, x)$ exists a.e. on $I$. This fact is a special case of Kirchheim’s theorem [4, Theorem 2] on a.e. metric differentiability of Lipschitz mappings (from $\mathbb{R}^n$ to $X$). In a standard way we obtain the following more general fact.

**Lemma 1.** Let $I$ be an open interval, $X$ a Banach space, and let $g : I \to X$ have bounded variation on $I$. Then $\text{md}(g, x)$ exists almost everywhere on $I$.

**Proof.** We may suppose $I = \mathbb{R}$. Denote $s(x) = \sqrt[0]{x}$, $x \in \mathbb{R}$. By [2, 2.5.16.] there exists a Lipschitz mapping $H : \mathbb{R} \to X$ such that $g = H \circ s$. By [4, Theorem 2], $\text{md}(H, x)$ exists a.e. on $\mathbb{R}$. Now, changing in the obvious way the last argument of [2, 2.9.22.], we obtain our assertion. \(\square\)

**Theorem 2.** Let $X$ be a Banach space, $I \subset \mathbb{R}$ an open interval and $f : I \to X$ a locally d.c. mapping. Let $C := \{x \in I : f'(x) = 0\}$. Then $\mathcal{H}^{1/2}(f(C)) = 0$.

**Proof.** Note that $f$ is continuous on $I$ (see [7, Proposition 1.10.]). By [7, Theorem 2.3], $f'_+$ exists and has locally bounded variation on $I$. Consider an arbitrary interval $[a, b] \subset I$. It is clearly sufficient to prove $\mathcal{H}^{1/2}(f(C_1)) = 0$, where $C_1 := C \cap (a, b)$.

Let $N_1$ be the set of all isolated points of $C_1$ and $N_2 := \{x \in C_1 : \text{md}(f'_+, x) \text{ does not exist}\}$. Since $N_1$ is countable, $\mathcal{H}^{1/2}(f(N_1)) = 0$.

To prove $\mathcal{H}^{1/2}(f(N_2)) = 0$, consider an arbitrary $\varepsilon > 0$. By Lemma 1, we find a countable disjoint system of open intervals $\{(a_i, b_i) : i \in J\}$ such that

$$N_2 \subset \bigcup_{i \in J} (a_i, b_i) \subset (a, b), \quad \sum_{i \in J} (b_i - a_i) < \varepsilon \quad \text{and} \quad (a_i, b_i) \cap N_2 \neq \emptyset, \ i \in J.$$
Clearly \( \|f'_+(x)\| \leq \bigvee_{a_i} f'_+ \) for each \( i \in J \) and \( x \in (a_i, b_i) \). Using the continuity of \( f \) and [1, Chap. I, par. 2, Proposition 3], we obtain

\[
\text{diam}(f((a_i, b_i))) \leq (b_i - a_i) \cdot \bigvee_{a_i} f'_+.
\]

Therefore, using the Cauchy-Schwartz inequality, we obtain

\[
H^{1/2}_\infty(f(N_2)) \leq \frac{\omega^{1/2}}{2^{1/2}} \sum_{i \in J} \left( (b_i - a_i) \cdot \bigvee_{a_i} f'_+ \right)^{1/2}
\]
\[
\leq \frac{\omega^{1/2}}{2^{1/2}} \left( \sum_{i \in J} (b_i - a_i) \right)^{1/2} \left( \sum_{i \in J} \bigvee_{a_i} f'_+ \right)^{1/2}
\leq \frac{\omega^{1/2}}{2^{1/2}} \varepsilon^{1/2} \left( \frac{b}{a} \right)^{1/2}.
\]

Since \( \varepsilon > 0 \) is arbitrary, we have \( H^{1/2}_\infty(f(N_2)) = 0 \); consequently (see [5, Lemma 4.6.]) we obtain \( H^{1/2}(f(N_2)) = 0 \).

To complete the proof, it is sufficient to prove \( H^{1/2}(f(C_2)) = 0 \), where \( C_2 = C_1 \setminus (N_1 \cup N_2) \). Let \( \varepsilon > 0 \) be arbitrary. Clearly \( \text{md}(f'_+, x) = 0 \) for each \( x \in C_2 \).

Therefore, for each \( x \in C_2 \) we can choose \( \delta_x > 0 \) such that \([x - \delta_x, x + \delta_x] \subset (a, b)\) and \( \|f'_+(y)\| \leq \varepsilon |y - x| \) for each \( y \in [x - \delta_x, x + \delta_x] \). Using the continuity of \( f \) and [1, Chap. I, par. 2, Proposition 3], we obtain

\[
\text{diam}(f([x - \delta_x, x + \delta_x])) \leq 2\varepsilon (\delta_x)^2.
\]

Besicovitch’s Covering Theorem (see [5]) easily implies that we can choose a countable set \( A \subset C_2 \) such that

\[
C_2 \subset \bigcup_{x \in A} [x - \delta_x, x + \delta_x] \quad \text{and} \quad \sum_{x \in A} 2\delta_x \leq c(b - a),
\]

where \( c \) is an absolute constant (not depending on \( \varepsilon \)). Since \( \varepsilon > 0 \) is arbitrary,

\[
f(C_2) \subset \bigcup_{x \in A} f([x - \delta_x, x + \delta_x]),
\]

and

\[
\sum_{x \in A} (\text{diam}(f([x - \delta_x, x + \delta_x])))^{1/2} \leq \sum_{x \in A} \sqrt{2\varepsilon} \delta_x \leq \sqrt{2\varepsilon} c(b - a),
\]

we have \( H^{1/2}_\infty(f(C_2)) = 0 \), hence \( H^{1/2}(f(C_2)) = 0 \).

Remark 3. Since each \( C^2 \)-function on \( I \) is a locally d.c. function (see [7]), [2, 3.4.4.] implies that the conclusion of Theorem 2 does not hold with \( H^\alpha \) (\( \alpha < 1/2 \)) in general.
References


Author’s address: **D. Pavlica**, Institute of Mathematics, Academy of Sciences of the Czech Republic, Zitná 25, 115 67 Praha 1, Czech Republic, e-mail: pavlica@math.cas.cz.