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Hybrid Variational Principles and Their Use in the Finite Element Method

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The first biharmonic problem is formulated variationally such that no boundary conditions and relaxed continuity requirements are imposed upon the "test" functions. The application of that to nonconform finite element method is indicated.

1. Plate Problem

In what follows, Ω is a bounded domain of the plane. Let $H^2(\Omega)$ and $H_0^2(\Omega)$ be the closure of $\mathscr{E}(\Omega)$ and $\mathscr{D}(\Omega)$ (standard notation-see e.g. [1]) respectively in the norm $\| \cdot \|_{2,\Omega} = (\sum_{|\alpha|=2} \|D^\alpha \cdot \|^2_{L_2(\Omega)} + \| \cdot \|^2_{L_2(\Omega)})^{1/2}$. Let us consider the following

Problem 1: Let $f \in L_2(\Omega)$ and $u_0 \in H^2(\Omega)$ are given functions. It is necessary to find a function $u \in H^2(\Omega)$ such that

$$\Delta^2 u = f \text{ on } \Omega \text{ in the sense of distributions} \tag{1}$$

$$u - u_0 \in H_0^2(\Omega) . \tag{2}$$

Using the classical variational formulation of Problem 1 in a finite element approach, we must impose very strong continuity and boundary conditions upon "trial functions". It is difficult to fulfil them. Now I should like to give another variational formulation of Problem 1, imposing relaxed continuity requirements and no boundary conditions upon "test functions".

2. Hybrid Variational Formulation

Definition 1: Let $\Omega_h = \{\Omega_{ih}\}_{i=1}^{m(h)}$ be a division of Ω for all fixed $h \in (0,1)$. This means that $\overline{\Omega} = \bigcup_{i=1}^{m(h)} \overline{\Omega}_{ih}$ and $\Omega_{ih} \cap \Omega_{jh} = \emptyset$ for all $i \neq j$ where $i, j \in (1, \ldots, m(h))$, $h \in (0,1)$.

Definition 2: Let $H^2(\Omega_h) = \prod_{i=1}^{m(h)} H^2(\Omega_{ih})$ (Cartesian product of spaces) with the norm $\| \cdot \|_{2,\Omega_h}$ which is induced by the norms of the spaces $H^2(\Omega_{ih})$ as usual.

Definition 3: Let $\overset{\circ}{\mathscr{H}}{}^{-2}(\Omega_h) = \{F;\ F \in (H^2(\Omega_h))'$ where $(H^2(\Omega_h))'$ is the space dual to $H^2(\Omega_h)$, if $\varphi \in H_0^2(\Omega)$ then $F(\varphi) = 0$. The norm in the space $\overset{\circ}{\mathscr{H}}{}^{-2}(\Omega_h)$ is defined as the usual supreme norm: $F \in \overset{\circ}{\mathscr{H}}{}^{-2}(\Omega_h)$, $\|F\|_{-2,\Omega_h} = \sup\limits_{\|\varphi\|_{2,\Omega_h} = 1} |F(\varphi)|$.

Problem 2: Let $f \in L_2(\Omega)$ and $u_0 \in H^2(\Omega)$ are given functions. Let h be fixed, $h \in (0,1)$ and let Ω_h be corresponding division of Ω. It is necessary to find $\{u_h, F_h\} \in H^2(\Omega_h) \times \overset{\circ}{\mathscr{H}}{}^{-2}(\Omega_h)$ such that

$$a_h(u_h, \varphi) + F_h(\varphi) + F(u_h) = \int f\varphi \, dx_1 dx_2 + F(u_0)$$

for all $\{\varphi, F\} \in H^2(\Omega_h) \times \overset{\circ}{\mathscr{H}}{}^{-2}(\Omega_h)$ where $a_h(u_h, \varphi) =$

$$= \sum_{i=1}^{m(h)} \int_{\Omega_{ih}} \left\{ \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 \varphi}{\partial x_1^2} + 2(1-\sigma) \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} + \sigma \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 \varphi}{\partial x_1^2} + \right.$$

$$\left. + \sigma \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 \varphi}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 \varphi}{\partial x_2^2} \right\} dx_1 \, dx_2 \ ; \qquad \sigma \in (0,1). \tag{3}$$

This idea was proposed by many authors and proceeds from physical approach — see [2]. What might be new one in this paper is the pure mathematical conception and assertion about uniqueness (Theorem 1). Theorems 2 and 3 are giving the physical interpretation.

Theorem 1: For each $f \in L_2(\Omega)$ and $u_0 \in H^2(\Omega)$ there exists one and only one solution $\{u_h, F_h\}$ of the Problem 2. Further it holds

$$\|u_h\|_{2,\Omega_h}^2 + \|F_h\|_{-2,\Omega_h}^2 \leqq C(\|f\|_{L_2(\Omega)}^2 + \|u_0\|_{2,\Omega}^2) \tag{4}$$

where C is independent of f and u_0 and h too. Finally $u_h \in H^2(\Omega)$ and u_h solves Problem 1.

Theorem 2: Suppose that $u \in H^3(\Omega)$, where u is the solution of Problem 1. Let $\{u_h, F_h\}$ be the solution of the corresponding Problem 2 (according to Theorem 1 is $u_h = u$ in the space $H^2(\Omega)$). It holds:

$$F_h(\varphi) = \sum_{i=1}^{m(h)} \int_{\partial\Omega_{ih}} Mu \frac{\partial \varphi}{\partial \nu} \, d\sigma + F_h^T(\varphi) \tag{5}$$

where

$$F_h^T(\varphi) \in \overset{\circ}{\mathscr{H}}{}^{-1}(\Omega_h) \text{ (it means that } \{F_h^T \in (H^1(\Omega_h))', \text{ if } \varphi \in H_0^1(\Omega) \text{ then } F_h^T(\varphi) = 0\}) \tag{6}$$

$$Mu = -\sigma\Delta u - (1-\sigma)\left(\frac{\partial^2 u}{\partial x_1^2} \nu_1^2 + 2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \nu_1 \nu_2 + \frac{\partial^2 u}{\partial x_2^2} \nu_2^2\right). \tag{7}$$

Theorem 3: Suppose that $u \in H^4(\Omega)$. Then it holds:

$$F_h(\varphi) = \sum_{i=1}^{m(h)} \int_{\partial\Omega_{ih}} Tu\, \varphi\, \mathrm{d}\sigma + \sum_{i=1}^{m(h)} \int_{\partial\Omega_{ih}} Mu \frac{\partial\varphi}{\partial\nu} \mathrm{d}\sigma + \sum_{i=1}^{m(h)} \int_{\partial\Omega_{ih}} Ru \frac{\partial\varphi}{\partial\tau} \mathrm{d}\sigma \qquad (8)$$

where

$$Tu = \frac{\partial}{\partial\nu} \Delta u\, ; \quad Ru = (1 - \sigma) \left(\frac{\partial^2 u}{\partial x_1^2} \nu_1\nu_2 - \frac{\partial^2 u}{\partial x_1 \partial x_2} (\nu_1^2 - \nu_2^2) - \frac{\partial^2 u}{\partial x_2^2} \nu_1\nu_2 \right) \quad (9)$$

and Mu is defined by (7).

3. Approximation

Definition 4: Let S_h be a finite dimensional subspace of $\{H^2(\Omega_h) \times \overset{\circ}{\mathscr{H}}{}^{-2}(\Omega_h)\}$. Then $\{u_h^*, F_h^*\} \in S_h$ is an approximate solution of Problem 1 if and only if

$$a_h(u_h^*, \varphi) + F_h^*(\varphi) + F(u_h^*) = \int_\Omega f\varphi \,\mathrm{d}x_1 \mathrm{d}x_2 + F(u_0) \qquad (10)$$

for all $\{\varphi, F\} \in S_h$.

We can use such finite element trial functions φ which are non-conform according to the classical variational principle but they are conform in accordance with presented principle. On the other hand we must approximate F_h — it means moments and effective forces. We can study the convergence of the finite element method of this particular kind, using the "conform" methodics.

References

[1] NEČAS, J.: Les méthodes directes en théorie des equations elliptiques. Masson, Paris (1967).

[2] PRAGER, W.: Variational Principles for Elastic Plates with Relaxed Continuity Requirements. Int. J. Solids Structures, 4, 837 (1968).