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A Note On Subdirectly Irreducible Groupoids

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Every groupoid containing a zero element is isomorphic to a factorgroupoid of a subdirectly irreducible groupoid.

Если группоид G является факторгруппоидом подпрямо неразложимого группоида, то пересечение всех идеалов в G непусто. Наоборот, всякий группоид с нулем изоморфен такому факторгруппоиду.

Každý grupoid s nulovým prvkem je izomorfní faktorovému grupoidu nějakého subdirektně nerozložitelného grupoidu.

1. Ideals

Let G be a groupoid. We put $MN = \{xy \mid x \in M,\ y \in N\}$ for any subsets M and N of G. A non-empty subset I of G is said to be an ideal of G if $GI \subseteq I$ and $IG \subseteq I$. We denote by $I(G)$ the set of all ideals of G. This set is not empty, since G is an ideal of G. Farther, we put $\mathrm{Int}\,(G) = \bigcap I,\ I \in I(G)$.

1.1 Proposition. Let G be a groupoid. Then:

(i) The intersection of a system of ideals of G is either empty or an ideal.
(ii) If $\mathrm{Int}(G)$ is non-empty then it is the smallest ideal of G.
(iii) The union of a non-empty system of ideals is an ideal.
(iv) If I, J are ideals of G then $IJ \subseteq I \cap J$ and $I \cap J$ is an ideal.
(v) The intersection of a finite non-empty system of ideals of G is an ideal.
(vi) $I(G)$ is a distributive lattice with respect to inclusion of ideals. Moreover, if $\mathrm{Int}(G)$ is non-empty then $I(G)$ is a complete lattice.
(vii) Any subset of G containing GG is an ideal of G.

Proof. Easy.

An element 0 of a groupoid G is said to be a zero element if $0x = 0 = x0$ for every $x \in G$. Obviously, G contains at most one zero element.

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1.2 Lemma. Let G be a groupoid, $0 \in G$ and $I = \{0\}$. Then I is an ideal of G iff 0 is a zero element of G. In this case, $I = \mathrm{Int}(G)$.

Proof. Obvious.

1.3 Lemma. Let I be an ideal of a groupoid G. Put $r = (I \times I) \cup \mathrm{id}_G$. Then r is a congruence of G and I/r is a zero element of the groupoid G/r.

Proof. Easy.

1.4 Lemma. Let f be a homomorphism of a groupoid G onto a groupoid H. Then $f(\mathrm{Int}(G)) \subseteq \mathrm{Int}(H)$. Moreover, if $\mathrm{Int}(G)$ is non-empty, then $f(\mathrm{Int}(G)) = \mathrm{Int}(H)$.

Proof. Easy.

2. Subdirectly Irreducible Groupoids

2.1 Proposition. Let G be a subdirectly irreducible groupoid. Then $\mathrm{Int}(G)$ is non-empty.

Proof. We can assume that G is non-trivial. Let s be the smallest non-trivial congruence of G. There are $a, b \in G$ such that $a \neq b$ and $(a, b) \in s$. Let I be an ideal of G. Consider the congruence r defined in 1.3. If $r \neq \mathrm{id}_G$ then $s \subseteq r$, $(a, b) \in r$ and $a, b \in I$. Suppose $r = \mathrm{id}_G$. Then $I = \{0\}$ is a one-element set, 0 is a zero element of G and $\mathrm{Int}(G) = I$ by 1.2. Thus we can assume that G contains no zero element. Then $a \in \mathrm{Int}(G)$.

2.2 Corollary. Let G be a homomorphic image of a subdirectly irreducible groupoid. Then $\mathrm{Int}(G)$ is non-empty.

Let G be a subdirectly irreducible groupoid. If G is non-trivial then we denote by s_G the least non-trivial congruence of G. If G is trivial then we put $s_G = \mathrm{id}_G$.

A groupoid G is said to be faithful if $a = b$, whenever $a, b \in G$ and either $ax = bx$ for every x or $xa = xb$ for every $x \in G$.

2.3 Proposition. Let G be a groupoid containing a zero element. Then there exists a faithful subdirectly irreducible groupoid H such that G is isomorphic to H/s_H.

Proof. We can assume that G is non-trivial. Let 0 be the zero element of G and $K = G \setminus \{0\}$. One may see easily that there exists a groupoid L with the following properties: L is simple, L is faithful, L contains at least three elements, $L \cap G = \emptyset$, there is an injective mapping $f : K \times K \to L$ and there are elements $a, b \in L$ such that $a \neq b$ and $a \neq cc \neq b$ for every $c \in \mathrm{Im}\, f$. (For example, we can take a sufficiently large simple idempotent commutative groupoid.) Put $H = K \cup L$ and define a partial operation $*$ on H as follows: $x * y = xy$ if $x, y \in K$ and $xy \in K$; $x * y = a$ if $x, y \in K$ and $xy = 0$; $x * f(x, y) = a = f(x, y) * x$ and $y * f(x, y) = b = f(x, y) * y$ if $x, y \in K$ and $x \neq y$; $x * y = xy$ if $x, y \in L$. It is easy to check that this partial operation can be extended to a complete operation $*$ defined on H and satisfying the following conditions:

(i) $x * y \in L$, $y * x \in L$ for all $x \in H$ and $y \in L$.
(ii) $x * y \neq y * y \neq y * x$ for all $x \in K$ and $y \in L$.

Now, put $s = (L \times L) \cup \mathrm{id}_H$. Since L is an ideal, s is a congruence of $H(*)$. Moreover, $s \neq \mathrm{id}_H$ and $H(*)/s$ is isomorphic to G. Let $r \neq \mathrm{id}_H$ be a congruence of $H(*)$. We are going to show that $s \subseteq r$. There are $c, d \in H$ such that $c \neq d$ and $(c, d) \in r$. The following cases can arise:

(1) $c, d \in L$. Then $r \mid L \neq \mathrm{id}_L$, $L \times L \subseteq r$ and $s \subseteq r$, since L is simple.
(2) $c \in K$, $d \in L$. Then $(c * d, d * d) \in r$, $c * d$, $d * d \in L$ and $c * d \neq d * d$. Now, we can proceed similarly as in (1).
(3) $c \in L$, $d \in K$. Dual to (2).
(4) $c, d \in K$. Then $a = c * f(c, d)$, $b = d * f(c, d)$, $(a, b) \in r$ and (1) may be applied.

We have proved that $H(*)$ is subdirectly irreducible. It remains to show that this groupoid is faithful. For, let $x, y \in H$, $x \neq y$. If $x, y \in K$ then we have $x * f(x, y) \neq y * f(x, y)$ and $f(x, y) * x \neq f(x, y) * y$. If $x \in K$, $y \in L$ then $x * y \neq y * y \neq y * x$. Similarly, if $x \in L$, $y \in K$. Finally, if $x, y \in L$ then $x * u \neq y * u$ and $v * x \neq v * y$ for some $u, v \in L$, since L is faithful.

2.4 Proposition. Let G be a subdirectly irreducible groupoid containing two elements a, b such that $a \neq b$, $a = aa$ and $(a, b) \in s_G$. Then there exists a subdirectly irreducible groupoid H such that G is isomorphic to H/s_H.

Proof. Let α be an element not belonging to G and $H = G \cup \{\alpha\}$. Define an operation $*$ on H as follows: $x * y = xy$ if $x, y \in G$; $x * \alpha = xa$ and $\alpha * x = ax$ for every $a \neq x \in G$; $a * \alpha = \alpha = \alpha * a$; $\alpha * \alpha = a$. Put $s = \{(a, \alpha), (\alpha, a)\} \cup \mathrm{id}_H$. It is easy to see that $s \neq \mathrm{id}_H$ is a congruence of $H(*)$ and $H(*)/s$ is isomorphic to G. Now, let $r \neq \mathrm{id}_H$ be a congruence of $H(*)$. There are $c, d \in H$ with $c \neq d$ and $(c, d) \in r$. We can assume that $c \in G$. The following cases can arise:

(i) $d \in G$. Then $r \mid G \neq \mathrm{id}_G$, $(a, b) \in r$ and $(a, \alpha) \in r$, since $\alpha = \alpha * a$, $\alpha * b = ab$, $(\alpha, ab) \in r$ and $(a, ab) \in r$.
(ii) $d = \alpha$. We can assume that $c \neq a$. Then $(a, \alpha) \in r$, since $ac = \alpha * c$, $a = \alpha * \alpha$, $(ac, a) \in r$, $ac = a * c$, $\alpha = a * \alpha$, $(ac, \alpha) \in r$.

2.5 Proposition. Let G be a groupoid satisfying at least one of the following conditions:

(i) G has a zero element.
(ii) G is subdirectly irreducible and idempotent.
(iii) G is subdirectly irreducible and there exist $a, b \in G$ such that $a \neq b$ and $ac = bc$, $ca = cb$ for every $c \in G$.
(iv) G is simple and contains at least one idempotent element.
(v) G is simple and $ab = cd$ for some $a, b, c, d \in G$ with $(a, b) \neq (c, d)$.
(vi) G is simple and finite.
(vii) G is a quasigroup and every congruence of G is either left or right cancellative.

(viii) G is a finite quasigroup.

(ix) G is a group.

Then there exists a subdirectly irreducible groupoid H such that G is isomorphic to H/s_H.

Proof. See 2.3, 2.4 and [1, Theorem 4.11].

Reference

[1] KEPKA T.: On a class of subdirectly irreducible groupoids, Acta Univ. Carolinae, Math. Phys. *22/1* (1981), 17.