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Notes On Left Distributive Groupoids

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A groupoid satisfying the identity $x \,.\, yz = xy \,.\, xz$ is said to be left distributive. In the present paper, some basic properties of these groupoids are proved.

Grupoid splňující identitu $x \,.\, yz = xy \,.\, xz$ se nazývá zleva distributivní. V článku se dokazují některé základní vlastnosti těchto grupoidů.

Группоид выпольняющий тождество $x \,.\, yz = xy \,.\, xz$ называется леводистрибутивным. В статье исследуются некоторые основные свойства этих групподов.

1. Introduction

A groupoid G is said to be

- idempotent if $aa = a$ for every $a \in G$,
- commutative if $ab = ba$ for all $a, b \in G$,
- left distributive (an LD-groupoid) if $a \,.\, bc = ab \,.\, ac$ for all $a, b, c \in G$,
- distributive if it is left distributive and $ab \,.\, c = ac \,.\, bc$ for all $a, b, c \in G$,
- medial if $ab \,.\, cd = ac \,.\, bd$ for all $a, b, c, d \in G$,
- a left unar if $ab = ac$ for all $a, b, c \in G$,
- a right unar if $ba = ca$ for all $a, b, c \in G$,
- left symmetric if $a \,.\, ab = b$ for all $a, b \in G$,
- right symmetric if $ba \,.\, a = b$ for all $a, b \in G$,
- semisymmetric if $a \,.\, ba = b$ for all $a, b \in G$.

Let G be a groupoid. For all $a, b \in G$, $L_a(b) = ab$ and $R_a(b) = ba$. We shall say that G is left (right) cancellative if $L_a\,(R_a)$ is injective for every $a \in G$. We shall say that G is left (right) divisible if $L_a\,(R_a)$ is surjective for every $a \in G$. A left (right) cancellative and left (right) divisible groupoid is called a left (right) quasigroup.

Let G be a groupoid. Define two equivalences p_G and q_G on G by $(a, b) \in p$ iff $L_a = L_b$ and $(c, d) \in q$ iff $R_c = R_d$. We shall say that G is left (right) regular if $q = \ker L_a$ $(p = \ker R_a)$ for every $a \in G$.

Let G be a groupoid and $a \in G$. Then Id G is the set of idempotents of G and $[a]_G$ the subgroupoid generated by a. A subgroupoid H is said to be left closed in G

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if ab, $a \in H$ implies $b \in H$. For a subgroupoid K, $[K]_G^{cl}$ is the least left closed subgroupoid containing K.

For every $n = 1, 2, \ldots$, define a left unar $Cycl(n)$ as follows: $Cycl(n) = \{1, 2, \ldots, n\}$, $ab = b + 1$ and $an = 1$ for all $a, b \in Cycl(n)$, $b \neq n$. Further, define a left unar $Cycl(\infty)$ by $Cycl(\infty) = \{1, 2, \ldots\}$, $ab = b + 1$.

1.1. Lemma. Let A and B be left unars. Suppose that A can be generated by one element and there exist surjective homomorphisms f of A onto B and g of B onto A. Then these unars are isomorphic.

Proof. Obvious.

1.2 Lemma. The following conditions are equivalent for a left unar A:

(i) Every subunar of A generated by one element is isomorphic to A.
(ii) A is isomorphic either to $Cycl(n)$ for some $n \geqq 1$ or to $Cycl(\infty)$.

Proof. Obvious.

1.3 Lemma. Let G be a simple left unar. Then exactly one of the following four assertions is true:

(i) G is isomorphic to $Cycl(1)$.
(ii) G is isomorphic to $Cycl(p)$ for a prime $p \geqq 2$.
(iii) G is a two-element semigroup of right zeros.
(iv) G is a two-element semigroup with zero multiplication.

Proof. Obvious.

2. Basic Properties Of Left Distributive Groupoids

2.1 Lemma. Let G be an LD-groupoid and $a \in G$. Then:

(i) L_a is an endomorphism of G and $a \,.\, aa = aa \,.\, aa$.
(ii) If R_{aa} is inejctive then $a = aa$.
(iii) If $a = aa$ then $L_a R_a = R_a L_a$.
(iv) If L_a is surjective and f is a transformation of G such that $L_a f = \mathrm{id}_G$ then $ab \,.\, c = a \,.\, b f(c)$ for all $b, c \in G$.
(v) If L_a is surjective then $(a, aa) \in p$.

Proof. All the assertions are easy observations ((ii) follows from (i) and (v) follows from (iv) for $b = a$).

2.2 Proposition. Let G be an LD-groupoid. Then:

(i) Id G is either empty or a left ideal of G.
(ii) q_G is a congruence of G.

(iii) q_G is right (left) cancellative, provided G is so.
(iv) $(a, aa) \in q$ for every $a \in G$ iff $GG \subseteq \mathrm{Id}\, G$.

Proof. (i) For $a \in G$ and $b \in \mathrm{Id}\, G$, $ab \,.\, ab = a \,.\, bb = ab$.
(ii) We have $q = \bigcap \ker L_a$, $a \in G$.
(iii) If G is left cancellative then $q = \mathrm{id}$. Suppose that G is right cancellative and $(ba, ca) \in q$. Then $db \,.\, da = d \,.\, ba = d \,.\, ca = dc \,.\, da$ and $db = dc$ for every $d \in G$.

2.3 Lemma. Let G be an LD-groupoid.

(i) If $(a, aa) \in p$ for every $a \in G$ then the mapping $a \to aa$ is an endomorphism of G.
(ii) If G is left cancellative then $(a, aa) \in p$ iff $aa \,.\, a = aa$.
(iii) If the mapping $a \to aa$ is injective then $aa \,.\, a = aa$ for every $a \in G$.

Proof. (i) We have $aa \,.\, bb = a \,.\, bb = ab \,.\, ab$.
(ii) Let $aa = aa \,.\, a$. Then $aa \,.\, ab = (aa \,.\, a)(aa.\, b) = (aa)(aa \,.\, b)$.
(iii) We have $aa \,.\, aa = (aa \,.\, a)(aa \,.\, a)$.

2.4 Proposition. Let G be an LD-groupoid. Then p_G is a congruence of G, provided at least one of the following four conditions is satisfied:

(1) G is left divisible.
(2) G is left cancellative and $aa = aa \,.\, a$ for every $a \in G$.
(3) G is right regular.
(4) G is medial and $GG = G$.

Proof. (1) and (3). Let $a, b, c, d \in G$ and $(a, b) \in p$. Then $ca \,.\, cd = c \,.\, ad = c \,.\, bd = cb \,.\, cd$ and the rest is clear.
(2) Let $a, b, c, d \in G$ and $(a, b) \in p$. Then $(c \,.\, ac)(ca \,.\, d) = (ca \,.\, cc)(ca \,.\, d) = (ca)(cc \,.\, d) = ca \,.\, cd = c \,.\, ad = c \,.\, bd = (c \,.\, bc)(cb \,.\, d) = (c \,.\, ac)(cb \,.\, d)$, since $c \,.\, ac = c \,.\, bc$ and $cc \,.\, d = cd$ by 2.3(ii).
(4) Let $a, b, c, d, e \in G$ and $(a, b) \in p$. Then $ca \,.\, de = cd \,.\, ae = cd \,.\, be = cb \,.\, de$.

2.5 Proposition. Let G be an LD-groupoid. Then $(a, aa) \in p$ for every $a \in G$, provided at least one of the following six conditions is satisfied:

(1) G is left divisible.
(2) G is left cancellative and $aa = aa \,.\, a$ for every $a \in G$.
(3) G is right regular.
(4) G is medial and $GG = G$.
(5) The mapping $a \to aa$ is a surjective endomorphism of G.
(6) The mapping $a \to aa$ is an injective endomorphism of G.

Proof. (1) is proved in 2.1(v), (2) in 2.3(ii) and (3) follows from 2.1(i).
(4) We have $a \,.\, bc = ab \,.\, ac = aa \,.\, bc$ for all $a, b, c \in G$.

(5) and (6). Put $f(a) = aa$. Then $a\,f(b) = a \,.\, bb = ab \,.\, ab = aa \,.\, bb = aa \,.\, f(b)$ and the rest is clear, provided f is surjective. If f is injective then $f(ab) = f(a) \,.$ $.\, f(b) = f(a) \,.\, bb = f(a)\, b \,.\, f(a)\, b = f(f(a)\, b)$ yields the result.

2.6 Theorem. Let G be an LD-groupoid satisfying at least one of the conditions (1), (2), (3) and (4) from 2.4. Then:

(i) p_G is a congruence of G and G/p is an idempotent LD-groupoid.
(ii) Every block of p_G is a subgroupoid of G and a left unar.
(iii) For every $a \in G$, $[a]_G$ is a left unar.
(iv) If G is right divisible then the left unars $[a]$ and $[b]$ are isomorphic for all $a, b \in G$.
(v) If G is right divisible and left cancellative then any two blocks of p are isomorphic left unars.

Proof. (i), (ii) and (iii). See 2.4 and 2.5.

(iv) and (v). Let $a, b \in G$. There are $c, d \in G$ with $ca = b$ and $db = a$. Hence $L_c(A) = B$, $L_d(B) = A$, where $A = [a]$ and $B = [b]$, and we can use 1.1 and 1.2. Finally, let P and Q be blocks of p. There are $a, b \in G$ with $aP \subseteq Q$, $bQ \subseteq P$ and the rest is clear.

2.7 Corollary. Let G be a right divisible LD-groupoid satisfying at least one of the four conditions from 2.4. Then there exists $n \in \{1, 2, \ldots, \infty\}$ such that $[a]_G$ is isomorphic to $Cycl(n)$ for every $a \in G$.

2.8 Proposition. An LD-groupoid G is idempotent, provided at least one of the following two conditions is satisfied:

(i) G is right cancellative.
(ii) G is right divisible and Id G is non-empty.

Proof. Use 2.1(ii) and 2.2(i).

2.9 Proposition. Let G be an LD-groupoid. Then p_G is left (right) cancellative, provided G is so.

Proof Let G be left cancellative, $(ca, cb) \in p$ and $d \in G$. Then $c \,.\, ad = ca \,.\, cd = cb \,.\, cd = c \,.\, bd$ and $ad = bd$.

2.10 Proposition. Let G be a left cancellative LD-groupoid such that $aa = aa \,.\, a$ for every $a \in G$. Then there exists a groupoid H with the following properties:

(i) G is a subgroupoid of H and $H = [G]^{\mathrm{cl}}_G$.
(ii) H is an LD-groupoid and a left quasigroup.
(iii) G and H generate the same groupoid variety.
(iv) H is idempotent iff G is.

(v) $p_G = p_H \mid G$.
(vi) $p_H = \mathrm{id}$ iff $p_G = \mathrm{id}$.
(vii) H is right (left) cancellative (divisible), provided G is so.
(viii) H is simple, provided G is.

Proof. By 2.4 and 2.5, p_G is a congruence of G and $(a, aa) \in p_G$ for each $a \in G$. Now, let $a \in G$. Consider the subgroupoid $K = aG$ of G. Then $K \subseteq G$, K is isomorphic to G and $K = aa \,.\, G$. The rest is clear.

2.11 Corollary. The following conditions are equivalent for an LD-groupoid G:

(i) G can be imbedded into an LD-groupoid H such that H is a left quasigroup.
(ii) G is left cancellative and $aa = aa \,.\, a$ for each $a \in G$.

2.12 Proposition. Let G be an LD-groupoid. Define a relation r on G by $(a, b) \in r$ iff there are $n \geqq 1$ and $a_1, \ldots, a_n \in G$ such that $a_1(\ldots(a_n a)) = a_1(\ldots(a_n b))$. Then r is the least left cancellative congruence of G. Moreover, if $(aa \,.\, a, aa) \in r$ for some $a \in G$ then $bb = bb \,.\, b$ for some $b \in G$. Similarly, if $(cc, c) \in r$ for some $c \in G$ then $\mathrm{Id}\, G$ is non-empty.

Proof. Easy.

2.13 Proposition. Let G be a finite LD-groupoid. Then there exists at least one element $a \in G$ with $aa = aa \,.\, a$.

Proof. Consider the congruence r defined in 2.12. Then G/r is a left quasigroup, and so $(aa \,.\, a, aa) \in r$ for every $a \in G$.

2.14 Proposition. Let G be a left cancellative LD-groupoid. Put $A = \{a \in G;\ aa \,.\, a = aa\}$ and $B = \{b \in G;\ bb \,.\, b \neq bb\}$. Then:

(i) $G = A \cup B$ and $A \cap B = \emptyset$.
(ii) A is either empty or a left ideal of G.
(iii) B is either empty or a left ideal of G.
(iv) $r = (A \times A) \cup (B \times B)$ is a left cancellative congruence.
(v) If $r \neq G \times G$ then G/r is a two-element semigroup of right zeros.

Proof. Easy.

3. Examples Of Left Distributive Groupoids

3.1 Proposition. Let G be a left unar and let f be the transformation of G such that $ab = f(b)$ for all $a, b \in G$. Then:

(i) G is a medial LD-groupoid and G is regular.
(ii) G is distributive iff $f^2 = f$.

(iii) Id G is empty iff $f(a) \neq a$ for every $a \in G$.
(iv) If Id G is an ideal then $f^2 = f$.
(v) $p = G \times G$ and $q = \ker f$.
(vi) G is left cancellative (divisible) iff f is injective (surjective).

Proof. Obvious.

3.2 Example. The left unar $Cycl(2)$ is an LD-groupoid without idempotents. Moreover, this groupoid is a left quasigroup, it is medial, regular and left symmetric and it is not distributive.

3.3 Proposition. Let G be a groupoid such that $G = A \cup B$, where A is the set of left units of G and $B = \{a \in G;\ ab = ac \in \text{Id } G \text{ for all } b, c \in G\}$. Then:

(i) G is an LD-groupoid.
(ii) G is distributive iff either G is a right unar or G is idempotent and contains at most one left zero.
(iii) G is idempotent iff every element from B is a left zero.
(iv) Id G is an ideal iff either $B = G$ or G is idempotent.
(v) p_G is a congruence of G.
(vi) The mapping $x \to xx$ is an endomorphism of G iff either G contains just one left unit e and $aa = e$ for every $a \in G$ or $aa \in B$ for every $a \in B$.
(vii) $(x, xx) \in p$ for every $x \in G$ iff $aa \in B$ for every $a \in B$.

Proof. (i) Let $a, b, c \in G$. If $a \in A$ then $a \,.\, bc = bc = ab \,.\, ac$. If $a \in B$ then there is an $e \in \text{Id } G$ such that $ax = e$ for each $x \in G$ and we have $a \,.\, bc = e = ee = ab \,.\, ac$.

(ii) Suppose that G is distributive. If $B = G$ then G is a right unar. Let $B \neq G$ and $e \in A$. We have $a = ea = ee \,.\, a = ea \,.\, ea = aa$ for each $a \in G$, and so G is idempotent. Moreover, if $z \in G$ is a left zero then $z = za = ez \,.\, a = ea \,.\, za = az$ for evera $a \in G$ and z is a zero. The rest is clear.
(iii) and (iv). These assertion are easy.
(v) Let $(a, b) \in p$ and $c \in G$. Then either $c \in A$ and $ca = a$, $cb = b$ or $c \in B$ and $ca = cb$.
(vi) Suppose that $x \to xx$ is an endomorphism. Let $e = aa \in B$ for some $a \in B$. For each $f \in A$, $f = ef = aa \,.\, f = aa \,.\, ff = af \,.\, af = ee = e$. Moreover, for every $b \in B$, $bb = e \,.\, bb = aa \,.\, bb = ab \,.\, ab = ee = e$.
(vii) This is evident.

3.4 Example. Consider the following groupoid $L(1)$:

$L(1)$	0	1
0	1	1
1	0	1

This groupoid is an LD-groupoid, it is not distributive and the set $\{1\}$ of idempotents is not an ideal. Moreover, $(a, aa) \notin p = \text{id}$ for $a = 0$ and the mapping $x \to xx$ is an endomorphism of $L(1)$.

3.5 Example. Consider the following groupoid $L(6)$:

$L(6)$	0	1	2
0	0	1	2
1	1	1	1
2	0	0	0

This groupoid is a simple LD-groupoid, p is a congruence and $x \to xx$ is not an endomorphism (see 3.3).

3.6 Proposition. Let G be an LD-groupoid and $0 \notin G$. Define a groupoid $H(*)$ as follows: $H = G \cup \{0\}$, $a * b = ab$, $a * 0 = 0 * 0 = 0$, $0 * a = a$ for all $a, b \in G$. Then:

(i) $H(*)$ is an LD-groupoid.
(ii) $H(*)$ is distributive iff G is an idempotent distributive groupoid satisfying the identites $x = yx \cdot x$ and $xy = y \cdot xy$.
(iii) $p_{H(*)}$ is a congruence of $H(*)$ iff p_G is a congruence of G and the set of left units of G is either empty or a left ideal of G.
(iv) The map $a \to a * a$ is an endomorphism of $H(*)$ iff $b \to bb$ is an endomorphism of G.
(v) $(a, a * a) \in p$ for every $a \in H$ iff $(b, bb) \in p$ for every $b \in G$.

Proof. Easy.

3.7 Example. Consider the following groupoids:

$L(2)$	0	1	2
0	0	0	0
1	1	1	1
2	1	0	2

$L(3)$	0	1	2
0	0	0	0
1	1	1	1
2	0	1	2

$L(4)$	0	1	2
0	0	0	2
1	0	1	2
2	0	1	2

$L(5)$	0	1	2
0	0	1	2
1	0	1	1
2	0	2	2

One may check easily that these are pair-wise non-isomorphic LD-groupoids which are idempotent and not distributive. Moreover, p is not a congruence of $L(4)$.

4. Non-Distributive Idempotent Left Distributive Groupoids With At Most Three Elements

4.1 Proposition. (i) Every idempotent LD-groupoid containing at most two elements is distributive.

(ii) The groupoids $Cycl(2)$ and $L(1)$ are two-element non-distributive LD-groupoids. Moreover, these groupoids are not isomorphic.

(iii) Every non-distributive two-element LD-groupoid is isomorphic to one of the groupoids $Cycl(2)$ and $L(1)$.

Proof. Easy.

4.2 Lemma. Let G be a three-element LD-groupoid such that Id G is non-empty and G contains no left and no right zero. Then G is distributive.

Proof. Let $G = \{a, b, c\}$. Since Id G is a left ideal and G contains no right zero, Id G has at least two elements, say a and b. Let us distinguish the following situations:

(i) Id $G = \{a, b\}$. We can assume that $cc = a$. Further, $ab, ba, ca, cb \in \{a, b\}$ and $a = a \,.\, cc = ac \,.\, ac$, $ac \in \{a, c\}$. First, let $ac = a$. Then $ab = b$, since a is not a left zero. Moreover, $a \,.\, cb = ac \,.\, ab = a \,.\, ab = ab = b$, $cb = b$ and b is a right zero, a contradiction. Hence $ac = c$ and $ca = c \,.\, cc = cc \,.\, cc = aa = a$, and so $ba = b$. On the other hand, $bc = b \,.\, ac = ba \,.\, bc = b \,.\, bc$, $bc = c$ and $b = ba = b \,.\, cc = bc \,.\, bc = cc = a$, a contradiction.

(ii) G is idempotent. Since a is not a left zero, we can assume that $ac \neq a$.

(ii1) Let $ac = b$. Then $a \,.\, ca = ac \,.\, a = ba$ and $cb = c \,.\, ac = ca \,.\, c$. If $ca = a$ then $a = a \,.\, ca = ba$, a is a right zero, a contradiction. If $ca = b$ then $ab = a \,.\, ca = ba$, $cb = ca \,.\, c = bc$ and G is commutative. If $ca = c$ then $cb = ca \,.\, c = cc = c$, c is a left zero, a contradiction.

(ii2) Let $ac = c$. Since c is not a right zero, $bc \neq c$.

(ii2a) Let $bc = a$. Then $a = a \,.\, bc = ab \,.\, ac = ab \,.\, c$, and so $ab = b$. Further, $a = bc = b \,.\, ac = ba \,.\, bc = ba \,.\, a$, $b = b \,.\, ab = ba \,.\, b$ and $cb \neq b$. Thus $ba = a$. If $cb = c$ then $ca = b$ (since a is not a right zero and c is not a left zero) and $c = cb = c \,.\, ab = ca \,.\, cb = bc = a$, a contradiction. If $cb = a$ then $a = ba = b \,.\, cb = bc \,.\, b = ab = b$, a contradiction.

(ii2b) Let $bc = b$. If $ba = a$ then $b = bc = b \,.\, ac = ba \,.\, bc = ab$, $b \,.\, ca = bc \,.\, ba = ba = a$ and $ca = a$, a contradiction. Thus $ba = c$ and $b = bc = b \,.\, ac = ba \,.\, bc = cb$, $ab = a \,.\, bc = ab \,.\, ac = ab \,.\, c$ and $ab = c$. From this, $ca = ab \,.\, a = a \,.\, ba = ac = c$ and G is commutative.

4.3 Lemma. Let G be a three-element idempotent LD-groupoid containing a zero element. Then G is distributive.

Proof. Suppose that $G = \{a, b, c\}$ and a is a zero element of G. Let G be not distributive. It is easy to check that then we have either $cb = a$ and $bc \in \{b, c\}$ or

$bc = a$ and $cb \in \{b, c\}$. In the first case, $c . bc = cb . c = ac = a$, and therefore $bc = b$ and $b = bc . b = b . cb = ba = a$, a contradiction. In the second case, $a = c . bc = cb . c$, $cb = b$ and $b = b . cb = bc . b = ab = a$, a contradiction.

4.4 Lemma. Let G be a three-element idempotent LD-groupoid containing at least two left zeros. Then G is either distributive or isomorphic to one of the groupoids $L(2)$, $L(3)$.

Proof. Let $G = \{a, b, c\}$ and let the elements a and b be left zeros. Suppose that G is neither distributive nor isomorphic to $L(2)$. Then either $ca = a$, $cb = b$ and G is isomorphic to $L(3)$ or $ca = a$, $cb = c$ or $ca = c$, $cb = b$. If $ca = a$, $cb = c$ then $c = cb = c . ba = cb . ca = ca = a$, a contradiction. If $ca = c$, $cb = b$ then $c = ca = c . ab = ca . cb = cb = b$, a contradiction.

4.5 Lemma. Let G be a three-element idempotent LD-groupoid containing just one left zero and no right zero. Then G is distributive.

Proof. Let $G = \{a, b, c\}$ and let a be the only left zero of G. Since a is not a right zero, we can assume that $ca \neq a$.

(i) Let $ca = b$. If $ba = a$ then $b = b . ca = bc . ba = bc . a$, and hence $bc = c$ and $b = ca = c . ac = ca . c = bc = c$, a contradiction. Consequently, $ba \in \{b, c\}$.

(i1) Let $ba = b$. Then $b = ba = b . ac = ba . bc = b . bc$, and so $bc = a$, since b is not a left zero. Finally, $b = ba = b . ca = bc . ba = ab = a$, a contradiction.

(i2) Let $ba = c$. Then $c = ba = b . ab = ba . b = cb$ and $c . bc = cb . c = c$ yields $bc \in \{b, c\}$. If $bc = b$ then G is distributive. If $bc = c$ then $b = b . ca = cc = c$, a contradiction.

(ii) Let $ca = c$. Then $cb \in \{a, b\}$.

(ii1) Let $cb = a$. We have $c . ba = cb . ca = ac = a$, hence $ba = b$ and $bc \in \{a, c\}$. But $\{b, c\}$ is not a subgroupoid of G, and so $bc = a$ and $c = ca = c . bc = cb . c = ac = a$, a contradiction.

(ii2) Let $cb = b$. If $bc = b$ then $c . ba = cb . ca = bc = b$, $ba = b$ and b is a left zero, a contradiction. Hence $bc = c$ (since $\{b, c\}$ is a subgroupoid), $c . ab = ca . cb = cb = b$, $ab = b$ and $b = ba = b . ac = ba . bc = bc = c$, a contradiction.

4.6 Lemma. Let G be a three-element idempotent LD-groupoid containing a right zero and no left zero. Then G is either distributive or isomorphic to one of the groupoids $L(4)$, $L(5)$.

Proof. Put $G = \{a, b, c\}$ and let a be a right zero. We can assume that $ac \neq a$.

(i) Let $ac = b$. Then $cb = c . ac = ca . c = ac = b$. If $bc = a$ then $b = bb = b . ac = ba . bc = aa = a$, a contradiction. If $bc = b$ then $ab = a . bc =$

$= ab \,.\, ac = ab \,.\, b$ and either $ab = a$ and $b = b \,.\, ac = ba \,.\, bc = ab = a$, a contradiction, or $ab = b$ and G is distributive. If $bc = c$ then either $ab = a$ and $b = a \,.\, bc = ab \,.\, ac = ab = a$, a contradiction, or $ab \in \{b, c\}$ and G is distributive.

(ii) Let $ac = c$. Then $bc = b \,.\, ac = ba \,.\, bc = a \,.\, bc = ab \,.\, ac = ab \,.\, c$ and $a \,.\, cb = ac \,.\, ab = c \,.\, ab$.

(ii1) Let $ab = a$. Then $bc = c$ and $a \,.\, cb = a$, $cb \in \{a, b\}$. If $cb = b$ then G is isomorphic to $L(4)$. If $cb = a$ then G is distributive.

(ii2) Let $ab = b$. If $cb = a$ then $c \,.\, bc = cb \,.\, c = ac = c$, and so $bc = c$ and G is distributive. If $cb = b$ and $bc = a$ then G is distributive. If $cb = b$ and $bc = b$ then G is isomorphic to $L(4)$. If $cb = b$ and $bc = c$ then G is distributive. If $cb = c$ and $bc = a$ then $a = c \,.\, bc = cb \,.\, c = cc = c$, a contradiction. If $cb = c$ and $bc = b$ then G is isormophic to $L(5)$. If $cb = c$ and $bc = c$ then G is isomorphic to $L(4)$.

(ii3) Let $ab = c$. Then $bc = ab \,.\, c = cc = c$ and $a \,.\, cb = c \,.\, ab = cc = c$. Consequently, $cb \in \{b, c\}$. In both cases, G is distributive.

4.7 Proposition. (i) The groupoids $L(2)$, $L(3)$, $L(4)$ and $L(5)$ are pair-wise non-isomorphic non-distributive idempotent LD-groupoids.

(ii) Every non-distributive three-element idempotent LD-groupoid is isomorphic to one of the groupoids $L(2)$, $L(3)$, $L(4)$ and $L(5)$.

Proof. Use 3.7, 4.2, 4.3, 4.4, 4.5 and 4.6.

5. Simple Left Distributive Groupoids

Let G be an LD-groupoid. Denote by $A(G)$ the set of all $a \in G$ such that the translation L_a is injective and by $B(G)$ the set of all $a \in G$ such that $ab = aa$ for every $b \in G$. Further, let $C(G) = \{a \in B(G);\ aa \in A(G)\}$ and $D(G) = \{a \in G;\ aa, a \in \in B(G)\}$.

5.1 Lemma. Let G be an LD-groupoid and $a \in B(G)$. Then there is an idempotent $e = e(a) \in G$ such that $aa = e = ab$ for every $b \in G$. If $a \in D(G)$ then $e \in \in D(G)$ and $eb = e$.

Proof. Easy.

5.2 Lemma. Let G be a non-trivial simple LD-groupoid. Then:

(i) $G = A(G) \cup B(G)$ and $A(G) \cap B(G) = \emptyset$.

(ii) $B(G) = C(G) \cup D(G)$ and $C(G) \cap D(G) = \emptyset$.

Proof. Let $a \in G$. Then $r = \ker L_a$ is a congruence of G, and hence either $r = \mathrm{id}$ and $a \in A(G)$ or $r = G \times G$ and $a \in B(G)$. The rest is clear.

5.3 Lemma. Let G be a non-trivial simple LD-groupoid. Then:

(i) $A(G)$ is either empty or a subgroupoid of G.
(ii) $D(G)$ is either empty or a right ideal of G.
(iii) $A(G)\,B(G) \subseteq B(G)$, $A(G)\,C(G) \subseteq C(G)$ and $A(G)\,D(G) \subseteq D(G)$.

Proof. (i) Let $a, b \in A(G)$, $c, d \in G$, $c \neq d$. Then $ab \,.\, ac = a \,.\, bc \neq a \,.\, bd = ab \,.\, ad$. By 5.3(i), $ab \in A(G)$.

(ii) If $a \in D(G)$ and $b \in G$ then $ab = e(a) \in D(G)$.
(iii) Let $a \in A(G)$, $c \in C(G)$ and $d \in D(G)$. For every $b \in G$, $ac \,.\, ab = a \,.\, cb = a\, e(c)$ and $ad \,.\, ab = ae(d)$. Since $a, e(c) \in A(G)$, $ae(c) \in A(G)$ by (i) and $ac \in C(G)$. Finally, $ae(d) \,.\, ab = a \,.\, e(d)b = ae(d)$, and so $ae(d) \in D(G)$ and $ad \in D(G)$.

5.4 Lemma. Let G be a simple LD-groupoid containing at least three elements. Then either $A(G) = G$ or $D(G) = G$ or card $A(G) =$ card $C(G) =$ card $D(G) = 1$.

Proof. Put $r = (A(G) \times A(G)) \cup (C(G) \times C(G)) \cup (D(G) \times D(G))$. Then r is an equivalence and we are going to show that r is a congruence. Let $a, b, c \in G$ and $(a, b) \in r$. If $c \in A(G)$ then $(ca, cb) \in r$ by 5.3(i), (iii). If $c \in B(G)$ then $ca = cb$, and so $(ca, cb) \in r$. If $a, b, c \in A(G)$ then $ac, bc \in A(G)$ by 5.3(i) and we have $(ac, bc) \in r$. If $a, b \in D(G)$ then $ac, bc \in D(G)$ by 5.3(ii) and $(ac, bc) \in r$. If $a, b \in C(G)$ then $ac = e(a)$, $bc = e(b)$, $e(a), e(b) \in A(G)$, and hence $(ac, bc) \in r$. If $a, b \in A(G)$ and $c \in C(G)$ ($c \in D(G)$) then $ac, bc \in C(G)$ ($ac, bc \in D(G)$) by 5.3(iii), and therefore $(ac, bc) \in r$. We have proved that r is a congruence of G. First, suppose $r = G \times G$. Then either $A(G) = G$ or $C(G) = G$ or $D(G) = G$. If $C(G) = G$ then $A(G) = \emptyset$, a contradiction. Finally, let $r \neq G \times G$. Then $r = \mathrm{id}$ and card $A(G) =$ card $C(G) =$ card $D(G) = 1$, since G contains at least three elements.

5.5 Example. Consider the following groupoids:

$D(1)$	0
0	0

$D(2)$	0	1
0	0	0
1	0	0

$D(3)$	0	1
0	0	0
1	0	1

$D(4)$	0	1
0	0	0
1	1	1

$D(5)$	0	1
0	0	1
1	0	1

It is easy to check that these groupoids are pair-wise non-isomorphic simple distributive groupoids.

5.6 Theorem. Let G be a simple LD-groupoid. Then exactly one of the following three assertions is true:

(i) G is isomorphic to one of the groupoids $D(1)$, $D(2)$, $D(3)$, $D(4)$, $D(5)$, $L(1)$, $Cycl(2)$.

(ii) G is isomorphic to $L(6)$.
(iii) G is a left cancellative groupoid containing at least three elements.

Proof. If G contains at most two elements then (i) is true. Suppose that G contains at least three elements. If $A(G) = G$ then G is left cancellative. Let $D(G) = G$. Then there is a mapping $e : G \to \mathrm{Id}\, G$ such that $ab = e(a)$ and $e(e(a)) = e(a)$ for all $a, b \in G$. Since $\ker e$ is a congruence of G, either $\ker e = \mathrm{id}$ and e is injective or $\ker e = G \times G$. If e is injective then $a = e(a)$ for every $a \in G$, G is a semigroup of left zeros and consequently G contains at most two elements, a contradiction. If $\ker e = G \times G$ then $e(a) = e(b)$ for all $a, b \in G$, G is a semigroup with zero multiplication, and hence G contains at most two elements, a contradiction. Finally, let $A(G) \neq G \neq D(G)$. By 5.4 and 5.2, $\mathrm{card}\, G = 3$ and $\mathrm{card}\, A(G) = \mathrm{card}\, C(G) = \mathrm{card}\, D(G) = 1$. Assume $G = \{a, b, c\}$, $A(G) = \{a\}$, $C(G) = \{c\}$ and $D(G) = \{b\}$. Then $aa = a$, $ab = b$, $ac = c$, $ba = bb = b$, $ca = cb = cc = a$ (use 5.3) and G is isomorphic to $L(6)$.

5.7 Theorem. (i) The groupoids $D(1)$, $D(2)$, $D(3)$, $D(4)$, $D(5)$, $L(1)$, $L(6)$, $Cycl(p)$, $p \geqq 2$ a prime, are pair-wise non-isomorphic simle LD-groupoids.

(ii) Every finite simple LD-groupoid G is either isomorphic to one of the groupoids from (i) or it is an idempotent left quasigroup with $p_G = \mathrm{id}_G$.

Proof. Let G be a finite simple LD-groupoid. With respect to 5.6, we can assume that G is left cancellative. Then G is a left quasigroup and by 2.4, p is a congruence of G. If $p = \mathrm{id}$ then G is idempotent by 2.5. If $p = G \times G$ then G is a left unar and we can use 1.3.

5.8 Proposition. Let G be a simple LD-groupoid such that the mapping $a \to aa$ is an endomorphism of G. Then G is either isomorphic to one of the groupoids $D(1)$, $D(2)$, $D(3)$, $D(4)$, $D(5)$, $L(1)$, $Cycl(p)$, $p \geqq 2$ a prime, or it is idempotent and left cancellative.

Proof. Taking into account 5.6 and 3.5, we can assume that G is left cancellative. Put $f(a) = aa$. Then $\ker f$ is a congruence of G. First, let $\ker f = G \times G$. Then $aa = bb$ for all $a, b \in G$ and $aa = aa \,.\, aa = a \,.\, aa$ implies $a = aa$. Now, let $\ker f = \mathrm{id}$. Then f is an injective endomorphism and $(a, aa) \in p$ for every $a \in G$ by 2.5(6). By 2.3(i) and 2.4(ii), p is a congruence of G and we can proceed similarly as in the proof of 5.7.

5.9 Proposition. Let G be an infinite simple left cancellative LD-groupoid. Then either G is idempotent or $aa \,.\, a \neq aa$ for every $a \in G$.

Proof. Apply 2.14 and 5.8.

6. Group Constructions Of Left Distributive Groupoids

6.1 Proposition. Let f be an endomorphism of a group G. Let $x \in C(G)$ (= the centre of G) be such that $f(x) = x$. Put $g(a) = a f(a^{-1})$ and $a * b = g(a)xf(b)$ for all $a, b \in G$. Then:

(i) $G(*)$ is a regular LD-groupoid.
(ii) $G(*)$ is distributive iff $x = 1$ and $fg(a) fg(b) = fg(b) fg(a)$ for all $a, b \in G$.
(iii) $G(*)$ is medial iff $fg(a) fg(b) = fg(b) f g(a)$ for all $a, b \in G$.
(iv) $G(*)$ is idempotent iff $x = 1$.
(v) $G(*)$ is left (right) cancellative (divisible) iff f (g) is injective (surjective).

Proof. Easy.

6.2 Example. Let $G(+)$ be a quasicyclic 2-group. There is an elefent $0 \neq x \in G$ with $2x = 0$. Put $a * b = 2a - b + x$ for all $a, b \in G$. Then $G(*)$ is a regular divisible LD-groupoid containing no idempotents.

6.3 Proposition. Let f be an endomorphism of a group G and $K = \{x \in G; f(x) = x\}$. Put $a * b = a f(ba^{-1})$ for all $a, b \varepsilon G$. Then:

(i) $G(*)$ is an idempotent LD-groupoid.
(ii) $G(*)$ is distributive iff it is medial iff $f(G') \subseteq K$ and $f(G)$ is nilpotent of class at most 2; these conditions are equivalent to $f(G') \subseteq K \cap C(f(G))$.
(iii) If f is either injective or surjective then $G(*)$ is distributive iff $G' \subseteq K$ and G is nilpotent of class at most 2.
(iv) $G(*)$ is left symmetric iff $f^2 = \mathrm{id}$ and $a f(a) \in C(G)$ for every $a \in G$.
(v) $G(*)$ is right symmetric iff $f(a^2) = f^2(a)$ for every $a \in G$ and the group $f^2(G)$ is commutative.
(vi) $G(*)$ is semisymmetric iff $f(a) = a f^2(a) = f^2(a)\, a$ and $f^2(aba^{-1}b^{-1}) = a^{-1}b^{-1}ab$ for all $a, b \in G$.
(vii) $G(*)$ is commutative iff $f(a^2) = a$ for every $a \in G$ and G is commutative.
(viii) $G(*)$ is symmetric iff G is commutative, $a^3 = 1$ and $f(a) = a^2$ for every $a \in G$.
(ix) $G(*)$ is left regular and $q = \ker f$.
(x) $G(*)$ is left cancellative (divisible) iff f is injective (surjective).
(xi) p is a congruence of $G(*)$ and $(a, b) \in p$ iff $a^{-1}b$ is contained in $K \cap C(f(G))$.
(xii) $G(*)$ is right regular iff, for all $a, b \in G$, $a f(b) = f(ba)$ implies $a \in K \cap C(f(G))$; in this case, $K \subseteq C(f(G))$.
(xiii) $G(*)$ is right cancellative iff, for all $a, b \in G$, $a f(b) = f(ba)$ implies $a = 1$; in this case, $K = 1$.
(xiv) $G(*)$ is right divisible iff the mapping $a \to a f(a^{-1})$ is surjective; in this case, $G(*)$ is a right quasigroup iff this mapping is a permutation.

Proof. Only the assertion (ii) needs a proof. First, assume that $G(*)$ is distributive. Then $f(b^{-1}ac^{-1}ba^{-1}c) = f^2(cb^{-1}ac^{-1}ba^{-1})$ for all $a, b, c \in G$. Setting $c = 1$, we get $f(b^{-1}aba^{-1}) = f^2(b^{-1}aba^{-1})$ and the inclusion $f(G') \subseteq K$ is evident. Consequently, $f(b^{-1}ac^{-1}ba^{-1}c) = f^2(cb^{-1}ac^{-1}ba^{-1}) = f^2(cb^{-1}c^{-1}bb^{-1}cac^{-1}ba^{-1}) = f(cb^{-1}c^{-1}bb^{-1}cac^{-1}ba^{-1}) = f(cb^{-1}ac^{-1}ba^{-1})$ for all $a, b, c \in G$. For $b = 1$, $f(ac^{-1}a^{-1}c) = f(cac^{-1}a^{-1})$, and so conjugated elements commute in the group $f(G)$. Further, $f(c^{-1}ba^{-1}cab^{-1}) = f(a^{-1}bcb^{-1}ac^{-1}) = f(c^{-1}a^{-1}bcb^{-1}a)$, $f(ba^{-1}cab^{-1}) = f(a^{-1}bcb^{-1}a)$, $f(b^{-1}aba^{-1}c) = f(cb^{-1}aba^{-1})$ and $f(G') \subseteq C(f(G))$. Now, conversely, suppose that $f(G') \subseteq K \cap C(f(G))$. Then $f(a^{-1}cb^{-1} . ac^{-1}b) = f^2(a^{-1}cb^{-1}ac^{-1}b) \in C(f(G))$, and so $f(a^{-1}cb^{-1}ac^{-1}b\, f(d)) = f^2(da^{-1} cb^{-1}ac^{-1}b)$ for all $a, b, c, d \in G$. The rest is clear.

6.4 Example. As it is well known, there exists a non-trivial torsionfree group G such that any two elements $a \neq 1 \neq b$ are conjugated in G. Put $H = G \setminus \{1\}$ and $a * b = aba^{-1}$ for all $a, b \in H$. Then $H(*)$ is a divisible idempotent LD-groupoid and $H(*)$ is a left quasigroup. On the other hand, $p = \mathrm{id} = q$ and $H(*)$ is not right regular.

6.5 Proposition. Let f be an endomorphism of a group G. Put $a * b = a f(b^{-1}a)$ for all $a, b \in G$. Then:

(i) $G(*)$ is an idempotent groupoid.
(ii) $G(*)$ is an LD-groupoid iff $f(a)f^2(a^{-1})f^2(b) = f^2(b)f^2(a^{-1})f(a)$ for all $a, b \in G$.
(iii) If f is either injective or surjective then $G(*)$ is an LD-groupoid iff $a^{-1}f(a) \in C(G)$ for every $a \in G$.
(iv) $G(*)$ is right distributive iff $f(abca^{-1}c f(b)) = f(c f(b) f(a) bc f(a^{-1}))$ for all $a, b, c \in G$.
(v) $G(*)$ is left symmetric iff $f = \mathrm{id}$.
(vi) $G(*)$ is right symmetric iff $f(a^2)f^2(a) = 1$ for every $a \in G$.
(vii) $G(*)$ is semisymmetric iff $a f(a) f^2(a) = 1$ and $a f(a) = f(a)\, a$ for every $a \in G$.
(viii) $G(*)$ is commutative iff $a f(a^2) = 1$ for every $a \in G$.
(ix) $G(*)$ is symmetric iff $f = \mathrm{id}$ and $a^3 = 1$ for every $a \in G$.
(x) $G(*)$ is left regular and $q = \ker f$.
(xi) $G(*)$ is left cancellative (divisible) iff f is injective (surjective).
(xii) $(a, b) \in p$ iff $a^{-1}b = f(ab^{-1}) \in C(f(G))$.
(xiii) p is a congruence of $G(*)$ iff, for all $a, b \in G$, $a^{-1}b = f(ab^{-1}) \in C(f(G))$ implies $f(a^2) = f(b^2)$.

Proof. Easy.

6.6 Corollary. Let G be a group and $a * b = ab^{-1}a$ for all $a, b \in G$. Then $G(*)$ is a left symmetric LD-groupoid.

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