Guoqiang Zhao; Lirong Yin

Wakamatsu tilting modules with finite injective dimension

*Czechoslovak Mathematical Journal*, Vol. 63 (2013), No. 4, 865–876

Persistent URL: `http://dml.cz/dmlcz/143602`

**Czechoslovak Mathematical Journal, 63 (138) (2013), 865–876**

# WAKAMATSU TILTING MODULES WITH FINITE INJECTIVE DIMENSION

Guoqiang Zhao, Lirong Yin, Hangzhou

(Received April 27, 2012)

*Abstract.* Let $R$ be a left Noetherian ring, $S$ a right Noetherian ring and ${}_R\omega$ a Wakamatsu tilting module with $S = \mathrm{End}({}_R\omega)$. We introduce the notion of the $\omega$-torsionfree dimension of finitely generated $R$-modules and give some criteria for computing it. For any $n \geqslant 0$, we prove that $\mathrm{l.id}_R(\omega) = \mathrm{r.id}_S(\omega) \leqslant n$ if and only if every finitely generated left $R$-module and every finitely generated right $S$-module have $\omega$-torsionfree dimension at most $n$, if and only if every finitely generated left $R$-module (or right $S$-module) has generalized Gorenstein dimension at most $n$. Then some examples and applications are given.

*Keywords*: Wakamatsu tilting module; $\omega$-$k$-torsionfree module; $\mathcal{X}$-resolution dimension; injective dimension; $\omega$-torsionless property

*MSC 2010*: 16E10, 16E30

## 1. Introduction

Let $R$ be a ring. We use $\mathrm{Mod}\, R$ (resp. $\mathrm{Mod}\, R^{\mathrm{op}}$) to denote the category of left (resp. right) $R$-modules, and use $\mathrm{mod}\, R$ (resp. $\mathrm{mod}\, R^{\mathrm{op}}$) to denote the category of finitely generated left (resp. right) $R$-modules. For a module $M$ in $\mathrm{Mod}\, R$ (resp. $\mathrm{Mod}\, S^{\mathrm{op}}$), we use $\mathrm{l.id}_R(M)$, $\mathrm{l.pd}_R(M)$ and $\mathrm{l.fd}_R(M)$ (resp. $\mathrm{r.id}_S(M)$, $\mathrm{r.pd}_S(M)$ and $\mathrm{r.fd}_S(M)$) to denote the injective dimension, projective dimension and flat dimension of ${}_RM$ (resp. $M_S$), respectively.

We define $\mathrm{gen}^*({}_RR) = \{X \in \mathrm{mod}\, R$; there exists an exact sequence $\ldots \to P_i \to \ldots \to P_1 \to P_0 \to X \to 0$ in $\mathrm{mod}\, R$ with $P_i$ projective for any $i \geqslant 0\}$ (see [15]). A module ${}_R\omega$ in $\mathrm{mod}\, R$ is called selforthogonal if $\mathrm{Ext}^i_R({}_R\omega, {}_R\omega) = 0$ for any $i \geqslant 1$.

The research was supported by National Natural Science Foundation of China (Grant Nos. 11126092, 11226062 and 11201220) and Science Research Foundation of Hangzhou Dianzi University (Grant No. KYS075610050).

**Definition 1.1** ([13]). A selforthogonal module ${}_R\omega$ in $\mathrm{gen}^*({}_RR)$ is called a *Wakamatsu tilting module* (sometimes it is also called a *generalized tilting module*) if there exists an exact sequence:

$$0 \to {}_RR \to \omega_0 \to \omega_1 \to \ldots \to \omega_i \to \ldots$$

such that: (1) $\omega_i \in \mathrm{add}_R\,\omega$ for any $i \geqslant 0$, where $\mathrm{add}_R\,\omega$ denotes the full subcategory of $\mathrm{mod}\,R$ consisting of all modules isomorphic to direct summands of finite sums of copies of ${}_R\omega$, and (2) after applying the functor $\mathrm{Hom}_R(-, {}_R\omega)$ the sequence is still exact.

Let $R$ and $S$ be any rings. Recall that a bimodule ${}_R\omega_S$ is called a *faithfully balanced bimodule* if the natural maps $R \to \mathrm{End}(\omega_S)$ and $S^{\mathrm{op}} \to \mathrm{End}({}_R\omega)$ are isomorphisms. By [15, Corollary 3.2], we have that ${}_R\omega_S$ is faithfully balanced and selforthogonal with ${}_R\omega \in \mathrm{gen}^*({}_RR)$ and $\omega_S \in \mathrm{gen}^*(S_S)$ if and only if ${}_R\omega$ is Wakamatsu tilting with $S = \mathrm{End}({}_R\omega)$ if and only if $\omega_S$ is Wakamatsu tilting with $R = \mathrm{End}(\omega_S)$.

In the following, we always assume that $R$ is a left Noetherian ring and $S$ is a right Noetherian ring (unless stated otherwise) and ${}_R\omega_S$ is a faithfully balanced selforthogonal bimodule.

Huang in [9] posed the following two questions: (1) Do the injective dimensions of ${}_R\omega$ and $\omega_S$ coincide provided both of them are finite? (2) If one of the injective dimensions of ${}_R\omega$ and $\omega_S$ is finite, is the other also finite? The author showed that the answer to first question is always affirmative (see [9, Theorem 2.7]) and gave some partial answers to the question (2). He proved that if the injective dimension of $\omega_S$ is equal to $n$ and the $U$-limit dimension of each of the first $n-1$ terms is finite, then the injective dimension of ${}_R\omega$ is also equal to $n$. In addition, he proved that the left and right injective dimensions of ${}_R\omega$ and $\omega_S$ are identical if one of them is quasi-Gorenstein. Note that, for Artin algebras, the affirmative answer to the second question is equivalent to the validity of the Wakamtsu Tilting Conjecture (WTC). This conjecture states that every Wakamtsu tilting module with finite injective dimension is cotilting. Moreover, WTC implies the validity of the Gorenstein Symmetry Conjecture (GSC), which states that if one of the left and right self-injective dimensions of $R$ is finite than the other is also finite (see [4]). In a recent paper [10], Huang further gave some equivalent conditions that the injective dimension of $\omega_S$ is finite implies that of ${}_R\omega$ is also finite.

On the other hand, Huang and Tang showed in [12] that $\mathrm{l.id}_R(\omega) = \mathrm{r.id}_S(\omega) \leqslant n$ if and only if every module in $\mathrm{mod}\,R$ and every module in $\mathrm{mod}\,S^{\mathrm{op}}$ have finite generalized Gorenstein dimension at most $n$, where $n$ is a negative integer. So, it is natural to ask whether $\mathrm{l.id}_R(\omega) = \mathrm{r.id}_S(\omega) \leqslant n$ if and only if every module in $\mathrm{mod}\,R$ (or in $\mathrm{mod}\,S^{\mathrm{op}}$) has finite generalized Gorenstein dimension at most $n$. In this paper, to solve the above problem, we introduce the notion of the $\omega$-torsionfree dimension of finitely generated modules, which is "simpler" than that of the generalized Gorenstein dimension of finitely generated modules. Then we show that the answer to this question is always affirmative. As an application, we give some other equivalent conditions that the injective dimension of ${}_R\omega$ is finite implies that of $\omega_S$ is also finite. Then we give some examples to illustrate the main result and other applications are also given. Finally, we provide some equivalent descriptions when ${}^{\perp_n}{}_R\omega$ has the $\omega$-torsionless property and then extend the main result of [9, Theorem 2.7]. The question when ${}^{\perp}{}_R\omega$ has the $\omega$-torsionless property is also considered.

## 2. Preliminaries

For any $k \geqslant 1$, let ${}^{\perp_k}{}_R\omega = \{M \in \operatorname{mod} R;\ \operatorname{Ext}^i_R(M,\omega) = 0$ for any $1 \leqslant i \leqslant k\}$ (resp. ${}^{\perp_k}\omega_S = \{N \in \operatorname{mod} S^{\mathrm{op}};\ \operatorname{Ext}^i_{S^{\mathrm{op}}}(N,\omega) = 0$ for any $1 \leqslant i \leqslant k\}$) and ${}^{\perp}{}_R\omega = \bigcap\limits_{k\geqslant 1} {}^{\perp_k}{}_R\omega$ (resp. ${}^{\perp}\omega_S = \bigcap\limits_{k\geqslant 1} {}^{\perp_k}\omega_S$). We use $(-)^{\omega}$ to denote $\operatorname{Hom}(-,\omega)$. Suppose that $A \in \operatorname{mod} R$. Let $\sigma_A\colon A \to A^{\omega\omega}$ defined via $\sigma_A(x)(f) = f(x)$, for any $x \in A$ and $f \in A^{\omega}$, be the canonical evaluation homomorphism. Then, we call $A$ *$\omega$-torsionless* (or *$\omega$-reflexive*) if $\sigma_A$ is a monomorphism (an isomorphism, respectively).

Now let $P_1 \xrightarrow{f} P_0 \to A \to 0$ be a projective resolution of $A$ in $\operatorname{mod} R$. Then we have an exact sequence $0 \to A^{\omega} \to P_0^{\omega} \xrightarrow{f^{\omega}} P_1^{\omega} \to \operatorname{Coker} f^{\omega} \to 0$. For the sake of convenience, we denote $\operatorname{Coker} f^{\omega}$ by $\operatorname{Tr}_{\omega} A$. For a positive integer $k$, a module $A$ in $\operatorname{mod} R$ is called *$\omega$-$k$-torsionfree* if $\operatorname{Tr}_{\omega} A \in {}^{\perp_k}\omega_S$ and $A$ is called *$\omega$-$\infty$-torsionfree* if $A$ is $\omega$-$k$-torsionfree for all $k$. We know from [8] that the definition does not depend on the choice of the projective resolution of $A$. $A$ is called *$\omega$-$k$-syzygy* if there is an exact sequence $0 \to A \to X_0 \to X_1 \to \ldots \to X_{k-1}$ with all $X_i$ in $\operatorname{add}_R \omega$. We remark that a module is $\omega$-torsionless (resp. $\omega$-reflexive) if and only if it is $\omega$-1-torsionfree (resp. $\omega$-2-torsionfree) (see [8]).

Put ${}_R\omega_S = {}_RR_R$. Then, in this case, the notions of $\omega$-$k$-torsionfree modules and $\omega$-$k$-syzygy modules are just the $k$-torsionfree modules and $k$-syzygy modules, respectively (see [1] for the definitions of $k$-torsionfree modules and $k$-syzygy modules). We use $\mathcal{T}^k_{\omega}(R)$ (resp. $\mathcal{T}_{\omega}(R)$) to denote the full subcategory of $\operatorname{mod} R$ consisting of $\omega$-$k$-torsionfree modules (resp. $\omega$-$\infty$-torsionfree modules) and $\Omega^k_{\omega}(R)$ to denote the full subcategory of $\operatorname{mod} R$ consisting of $\omega$-$k$-syzygy modules.

**Lemma 2.1** ([8, Theorem 1]). *Let $M \in \operatorname{mod} R$ and $k$ be a positive integer. Then the following statements are equivalent.*

(1) *$M$ is an $\omega$-$k$-torsionfree module.*

(2) *There is an exact sequence* $0 \to M \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_k} X_k$ *such that each* $\operatorname{Im} f_i \to X_i$ *is a left* $\operatorname{add}_R \omega$*-approximation of* $\operatorname{Im} f_i$, $1 \leqslant i \leqslant k$.

**Proposition 2.2.** *For any* $k \geqslant 1$, *a module in* $\operatorname{mod} R$ *is* $\omega$-$k$*-torsionfree if and only if it is an* $\omega$-1*-syzygy of an* $\omega$-$(k-1)$*-torsionfree module* $A$ *in* $\operatorname{mod} R$ *with* $A \in {}^{\perp_1}{}_R\omega$. *In particular, a module in* $\operatorname{mod} R$ *is* $\omega$-$\infty$*-torsionfree if and only if it is an* $\omega$-1*-syzygy of an* $\omega$-$\infty$*-torsionfree module* $A$ *in* $\operatorname{mod} R$ *with* $A \in {}^{\perp_1}{}_R\omega$.

Proof. This is an immediate consequence of Lemma 2.1. □

Recall from [3] that a module $M$ in $\operatorname{mod} R$ is said to have *generalized Gorenstein dimension zero* (with respect to $\omega$), denoted by $G\text{-}\dim_\omega(M) = 0$, if the following conditions hold: (1) $M$ is $\omega$-reflexive, and (2) $M \in {}^{\perp}{}_R\omega$ and $M^\omega \in {}^{\perp}\omega_S$. We use $\mathcal{G}_\omega(R)$ to denote the full subcategory of $\operatorname{mod} R$ consisting of the modules with generalized Gorenstein dimension zero.

**Lemma 2.3.** *For any* $M \in \operatorname{mod} R$, *the following statements are equivalent.*

(1) $G\text{-}\dim_\omega(M) = 0$.
(2) $M \in {}^{\perp}{}_R\omega$ *and* $\operatorname{Tr}_\omega M \in {}^{\perp}\omega_S$.

Proof. Note that, for any $M \in \operatorname{mod} R$, we have an exact sequence $0 \to M^\omega \to P_0^\omega \to P_1^\omega \to \operatorname{Tr}_\omega M \to 0$. So $\operatorname{Ext}_S^i(M^\omega, \omega) = \operatorname{Ext}_S^{i+2}(\operatorname{Tr}_\omega M, \omega)$ for any $i \geqslant 1$. Then it is easy to see that the assertion holds by [12, Lemma 2.1]. □

**Definition 2.4** ([3])**.** For any $n \geqslant 0$, $M$ in $\operatorname{mod} R$ is said to have *generalized Gorenstein dimension at most* $n$ (with respect to $\omega$), denoted by $G\text{-}\dim_\omega(M) \leqslant n$, if there is an exact sequence $0 \to M_n \to \ldots \to M_1 \to M_0 \to M \to 0$ in $\operatorname{mod} R$ with $G\text{-}\dim_\omega(M_i) = 0$ for any $0 \leqslant i \leqslant n$.

## 3. Injective dimensions of ${}_R\omega$ and $\omega_S$

Let $\mathcal{X}$ be a full subcategory of $\operatorname{mod} R$ and $M$ a module in $\operatorname{mod} R$. If there exists an exact sequence $\ldots \to X_n \to \ldots \to X_1 \to X_0 \to M \to 0$ in $\operatorname{mod} R$ with each $X_i \in \mathcal{X}$ for any $i \geqslant 0$, then we define the *$\mathcal{X}$-resolution dimension* of $M$, denoted by $\mathcal{X}\text{-res.dim}_R(M)$, as $\inf\{n;$ there exists an exact sequence $0 \to X_n \to \ldots \to X_1 \to X_0 \to M \to 0$ in $\operatorname{mod} R$ with each $X_i \in \mathcal{X}$ for any $0 \leqslant i \leqslant n\}$. We set $\mathcal{X}\text{-res.dim}_R(M)$ to be infinity if there does not exist such an integer (see [2]). We call $\mathcal{T}_\omega\text{-res.dim}_R(M)$ the $\omega$-torsionfree dimension of $M$ and ${}^{\perp}\omega\text{-res.dim}_R(M)$ the $\omega$-left orthogonal dimension of $M$.

**Lemma 3.1.** *Let* $0 \to M_1 \xrightarrow{f} M_2 \to M_3 \to 0$ *be an exact sequence in* $\operatorname{mod} R$. *Then we have exact sequences* $0 \to M_3^\omega \to M_2^\omega \to M_1^\omega \to \operatorname{Coker} f^\omega \to 0$ *and* $0 \to \operatorname{Coker} f^\omega \to \mathrm{Tr}_\omega M_3 \to \mathrm{Tr}_\omega M_2 \to \mathrm{Tr}_\omega M_1 \to 0$ *in* $\operatorname{mod} S^{\mathrm{op}}$.

P r o o f. Let $Q_1 \to P_1 \to M_1 \to 0$ and $Q_3 \to P_3 \to M_3 \to 0$ be projective resolutions of $M_1$ and $M_3$ in $\operatorname{mod} R$, respectively. We get an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Q_1 \arrow[r] \arrow[d] & Q_2 \arrow[r] \arrow[d] & Q_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & P_2 \arrow[r] \arrow[d] & P_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with $P_2 = P_1 \oplus P_3$ and $Q_2 = Q_1 \oplus Q_3$. Applying the functor $\mathrm{Hom}_R(-, \omega)$, we obtain the exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & M_3^\omega \arrow[r] \arrow[d] & M_2^\omega \arrow[r] \arrow[d] & M_1^\omega \arrow[d] & \\
0 \arrow[r] & P_3^\omega \arrow[r] \arrow[d] & P_2^\omega \arrow[r] \arrow[d] & P_1^\omega \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q_3^\omega \arrow[r] \arrow[d] & Q_2^\omega \arrow[r] \arrow[d] & Q_1^\omega \arrow[r] \arrow[d] & 0 \\
 & Tr_\omega M_3 \arrow[r] \arrow[d] & Tr_\omega M_2 \arrow[r] \arrow[d] & Tr_\omega M_1 \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By the snake lemma, we have an exact sequence $0 \to M_3^\omega \to M_2^\omega \xrightarrow{f^\omega} M_1^\omega \to \mathrm{Tr}_\omega M_3 \to \mathrm{Tr}_\omega M_2 \to \mathrm{Tr}_\omega M_1 \to 0$ in $\operatorname{mod} S^{\mathrm{op}}$. We are done. $\square$

The following result gives some criteria for computing $\omega$-torsionfree dimension.

**Proposition 3.2.** *Let* $M \in \operatorname{mod} R$ *and* $n \geqslant 0$. *Then the following statements are equivalent.*

(1) $\mathcal{T}_\omega$-res.dim$_R(M) \leqslant n$.

(2) *There is an exact sequence* $0 \to M \to H \to T \to 0$ *in* $\operatorname{mod} R$ *with* $\mathrm{add}_R\,\omega$-res.dim$_R(H) \leqslant n$ *and* $T \in \mathcal{T}_\omega(R) \cap {}^{\perp_1}{}_R\omega$.

(3) *There is an exact sequence* $0 \to H' \to T' \to M \to 0$ *in* $\operatorname{mod} R$ *with* $T' \in \mathcal{T}_\omega(R)$ *and* $\operatorname{add}_R \omega$-res.dim$_R(H') \leqslant n-1$.

P r o o f. (1) $\Rightarrow$ (2) Suppose that $\mathcal{T}_\omega$-res.dim$_R(M) \leqslant n$, we proceed by induction on $n$. If $n \leqslant 1$, then there is an exact sequence $0 \to T_1 \to T_0 \to M \to 0$ in $\operatorname{mod} R$ with both $T_0$ and $T_1$ in $\mathcal{T}_\omega(R)$. By Proposition 2.2, there is an exact sequence $0 \to T_1 \to \omega_1 \to A_1 \to 0$ in $\operatorname{mod} R$ with $\omega_1 \in \operatorname{add}_R \omega$ and $A_1 \in \mathcal{T}_\omega(R) \cap {}^{\perp_1}{}_R\omega$. Consider the following push-out diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & T_1 \arrow[r] \arrow[d] & T_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \omega_1 \arrow[r] \arrow[d] & T_0' \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & A_1 \arrow[r, equal] \arrow[d] & A_1 \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Because $A_1 \in {}^{\perp_1}{}_R\omega$, we have an exact sequence $0 \to \operatorname{Tr}_\omega A_1 \to \operatorname{Tr}_\omega T_0' \to \operatorname{Tr}_\omega T_0 \to 0$ by Lemma 3.1 and the exactness of the middle column. Note that both $A_1$ and $T_0$ are in $\mathcal{T}_\omega(R)$, thus $\operatorname{Tr}_\omega T_0' \in {}^{\perp}\omega_S$, and hence $T_0' \in \mathcal{T}_\omega(R)$. Thus there is an exact sequence $0 \to T_0' \to \omega_0 \to A_0 \to 0$ in $\operatorname{mod} R$ with $\omega_0 \in \operatorname{add}_R \omega$ and $A_0 \in \mathcal{T}_\omega(R) \cap {}^{\perp_1}{}_R\omega$ again by Proposition 2.2. So we get the following push-out diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \omega_1 \arrow[r] \arrow[d, equal] & T_0' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \omega_1 \arrow[r] & \omega_0 \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 0 \\
 & & A_0 \arrow[r, equal] \arrow[d] & A_0 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

It is clear that the third column is the desired sequence.

Now assume $n > 1$, then there is an exact sequence $0 \to K_1 \to T_0 \to M \to 0$ with $T_0 \in \mathcal{T}_\omega(R)$ and $\mathcal{T}_\omega$-res.dim$_R(K_1) \leqslant n-1$. By induction hypothesis, there is an exact sequence $0 \to K_1 \to H_1 \to A_1 \to 0$ with $\operatorname{add}_R \omega$-res.dim$_R(H_1) \leqslant n-1$

and $A_1 \in \mathcal{T}_\omega(R) \cap {}^{\perp_1}{}_R\omega$. By the foregoing proof, there exist exact sequences $0 \to H_1 \to \omega_0 \to H \to 0$ and $0 \to M \to H \to A_0 \to 0$, where $\omega_0 \in \operatorname{add}_R \omega$ and $A_0 \in \mathcal{T}_\omega(R) \cap {}^{\perp_1}{}_R\omega$. It is easy to see that $0 \to M \to H \to A_0 \to 0$ is the required sequence.

(2) $\Rightarrow$ (3) By (2), there is an exact sequence:

$$0 \to M \to H \to T \to 0$$

in $\operatorname{mod} R$ with $\operatorname{add}_R \omega$-res.dim$_R(H) \leqslant n$ and $T \in \mathcal{T}_\omega(R) \cap {}^{\perp_1}{}_R\omega$. So there exists an exact sequence $0 \to H' \to \omega_0 \to H \to 0$ with $\omega_0 \in \operatorname{add}_R \omega$ and $\operatorname{add}_R \omega$-res.dim$_R(H') \leqslant n-1$. Consider the following pull-back diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & H' \arrow[r, equal] \arrow[d] & H' \arrow[d] & & \\
0 \arrow[r] & T' \arrow[r] \arrow[d] & \omega_0 \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & T \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $T \in \mathcal{T}_\omega(R) \cap {}^{\perp_1}{}_R\omega$, it is easy to see that $T' \in \mathcal{T}_\omega(R)$ by Proposition 2.2. Then the first column $0 \to H' \to T' \to M \to 0$ is as desired.

(3) $\Rightarrow$ (1) is trivial. $\square$

**Lemma 3.3** ([11, Lemma 17.2.4]). $\operatorname{r.id}_S(\omega) = \sup\{\operatorname{l.fd}_S(\operatorname{Hom}_R(\omega, E));$ *$E$ is injective in* $\operatorname{Mod} R\}$. *Moreover,* $\operatorname{r.id}_S(\omega) = \operatorname{l.fd}_S(\operatorname{Hom}_R(\omega, Q))$ *for any injective cogenerator $Q$ for* $\operatorname{Mod} R$.

The following result is crucial in proving the main result.

**Theorem 3.4.** *For any $n \geqslant 0$, if every module in* $\operatorname{mod} R$ *has $\omega$-torsionfree dimension at most $n$, then* $\operatorname{r.id}_S(\omega) \leqslant n$.

Proof. Let $E$ be an injective module in $\operatorname{Mod} R$. Then by [14, Exercise 2.32], $E = \varinjlim_{i \in I} M_i$, where $\{M_i;\ i \in I\}$ is the set of all finitely generated submodules of $E$ and $I$ is a directed index set. By Proposition 3.2, for any $i \in I$, there is an exact sequence $0 \to M_i \xrightarrow{f_i} H_i$ in $\operatorname{mod} R$ with $\operatorname{add}_R \omega$-res.dim$_R(H_i) \leqslant n$.

For each $i, j \in I$, because $I$ is directed, there exists $k \in I$ with $i \leqslant k$ and $j \leqslant k$. Set $H = \bigoplus_{k\in I} H_k$. For any $i \leqslant j$, we have the following commutative diagram:

$$\begin{array}{ccccccc}
M_i & \xrightarrow{\varphi_k^i} & M_k & \xrightarrow{f_k} & H_k & \xrightarrow{\lambda_k} & H \\
\downarrow{\scriptstyle \varphi_j^i} & & \| & & \| & & \| \\
M_j & \xrightarrow{\varphi_k^j} & M_k & \xrightarrow{f_k} & H_k & \xrightarrow{\lambda_k} & H
\end{array}$$

where $\varphi_j^i\colon M_i \to M_j$ and $\lambda_k\colon H_k \to H$ are the embedding homomorphisms. It is clear that $H$ is a constant direct system over index set $I$. So by [14, Theorem 2.18], the sequence $0 \to E \to \varinjlim_{i\in I} H$ is exact. Thus we get an exact sequence

$$0 \to \operatorname{Hom}_R(\omega, E) \to \operatorname{Hom}_R\Bigl(\omega, \varinjlim_{i\in I} H\Bigr)$$

which is split. Since ${}_R\omega$ is finitely generated, $\operatorname{Hom}_R\Bigl(\omega, \varinjlim_{i\in I} H\Bigr) \cong \varinjlim_{i\in I} \operatorname{Hom}_R(\omega, H) \cong \varinjlim_{i\in I} \bigoplus_{k\in I} \operatorname{Hom}_R(\omega, H_k)$ by [6, Lemma 1.2.5]. Because $\operatorname{add}_R\omega\text{-res.dim}_R(H_k) \leqslant n$, $\operatorname{l.pd}_S(\operatorname{Hom}_R(\omega, H_k)) \leqslant n$. Therefore $\operatorname{l.fd}_S\Bigl(\varinjlim_{i\in I} \bigoplus_{k\in I} \operatorname{Hom}_R(\omega, H_k)\Bigr) \leqslant n$ since the functor Tor commutes with $\varinjlim_{i\in I}$ by [14, Theorem 8.11]. It follows this inequality $\operatorname{l.fd}_S(\operatorname{Hom}_R(\omega, E)) \leqslant n$ and hence $\operatorname{r.id}_S(\omega) \leqslant n$ by Lemma 3.3. □

**Lemma 3.5** ([10, Proposition 3.1]). *For a non-negative integer $n$, $\operatorname{l.id}_R(\omega) \leqslant n$ if and only if ${}^{\perp}\omega\text{-res.dim}_R(M) \leqslant n$ for any $M \in \operatorname{mod} R$.*

**Theorem 3.6.** *For any $n \geqslant 0$, the following statements are equivalent.*

(1) $\operatorname{l.id}_R(\omega) = \operatorname{r.id}_S(\omega) \leqslant n$.
(2) *Every module in $\operatorname{mod} R$ and every module in $\operatorname{mod} S^{\mathrm{op}}$ have $\omega$-left orthogonal dimension at most $n$.*
(3) *Every module in $\operatorname{mod} R$ and every module in $\operatorname{mod} S^{\mathrm{op}}$ have $\omega$-torsionfree dimension at most $n$.*
(4) *Every module in $\operatorname{mod} R$ has generalized Gorenstein dimension at most $n$.*
(5) *Every module in $\operatorname{mod} S^{\mathrm{op}}$ has generalized Gorenstein dimension at most $n$.*

Proof. (1) ⇔ (2) follows from Lemma 3.5 and its symmetric version.

(3) ⇒ (1) follows from Theorem 3.4 and its symmetric version.

(1) ⇒ (4) + (5) follows from [12, Theorem 3.5].

(4) ⇒ (1) Let $M$ be any module in $\operatorname{mod} R$. By hypothesis, $G\text{-}\dim_\omega(M) \leqslant n$ and hence $\mathcal{T}_\omega\text{-res.dim}_R(M) \leqslant n$ by Lemma 2.3. Thus $\operatorname{r.id}_S(\omega) \leqslant n$ from Theorem 3.4.

On the other hand, because ${}^{\perp}\omega$-res.dim$_R(M) \leqslant G$-dim$_\omega(M) \leqslant n$, l.id$_R(\omega) \leqslant n$ by Lemma 3.5.

Symmetrically, we get (5) $\Rightarrow$ (1).

(4) + (5) $\Rightarrow$ (3) Because $\mathcal{T}_\omega$-res.dim$_R(M) \leqslant G$-dim$_\omega(M)$ and $\mathcal{T}_\omega$-res.dim$_{S^{\rm op}}(N) \leqslant G$-dim$_\omega(N)$ for any $M \in \operatorname{mod} R$ and $N \in \operatorname{mod} S^{\rm op}$, the assertion follows. $\square$

Now, we construct a Wakamatsu tilting module and give an example to illustrate the main result.

**Example 3.7.** Assume $R$ is a Gorenstein Artin algebra with gl.dim$(R) = \infty$. Let $C = \oplus I_j$, where $I_j$ are all the indecomposable and nonisomorphic direct summands of modules appeared in the minimal injective resolution of $R$. Then $C$ is a Wakamatsu tilting module. In this case, every finitely generated $R$-module has generalized Gorenstein dimension zero. On the other hand, the class of finitely generated $R$-modules in $\operatorname{add} C$ is just the class of all finitely generated injective $R$-modules. However, it is clear that there exists an $R$-module which is not projective and injective.

**Remark 3.8.** It is easy to see that every projective $R$-module and $R$-module in add$_R C$ are in $\mathcal{G}_C(R)$. The above example also gives a "nontrivial" example of modules having generalized Gorenstein dimension zero.

As an application, we give some other equivalent conditions that the injective dimension of ${}_R\omega$ is finite implies that of $\omega_S$ is also finite.

**Proposition 3.9.** *Let $R$ be a left Noetherian ring, $S$ a right Noetherian ring and ${}_R\omega$ a Wakamatsu tilting module with $S = \operatorname{End}({}_R\omega)$. If the injective dimension of ${}_R\omega$ is finite, then the following statements are equivalent for a nonnegative integer $n$.*

(1) *The injective dimension of $\omega_S$ is at most $n$.*

(2) $\mathcal{T}_\omega$-res.dim$_R(M) \leqslant n$ *for any* $M \in \operatorname{mod} R$.

(3) *For any $M \in \operatorname{mod} R$, there is an exact sequence $0 \to M \to H \to T \to 0$ in* $\operatorname{mod} R$ *with* add$_R\omega$-res.dim$_R(H) \leqslant n$ *and* $T \in \mathcal{T}_\omega(R) \cap {}^{\perp_1}{}_R\omega$.

(4) *For any $M \in \operatorname{mod} R$, there is an exact sequence $0 \to H' \to T' \to M \to 0$ in* $\operatorname{mod} R$ *with* $T' \in \mathcal{T}_\omega(R)$ *and* add$_R\omega$-res.dim$_R(H') \leqslant n - 1$.

Proof. (1) $\Rightarrow$ (2) follows from Theorem 3.6 and [9, Theorem 2.7].

(2) $\Rightarrow$ (1) by Theorem 3.4.

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) by Proposition 3.2. $\square$

Recall that a ring $R$ is called $n$-Gorenstein, if $R$ is two-sided Noetherian and l.id$_R(R) = $ r.id$_R(R) \leqslant n$. By specializing Theorem 3.6 to the case ${}_R\omega = {}_R R$, we obtain the main result proved by Hoshino in [7].

**Corollary 3.10** ([7, Theorem]). *The following statements are equivalent:*

(1) *$R$ is $n$-Gorenstein.*
(2) *Every module in* $\operatorname{mod} R$ *has Gorenstein dimension at most $n$.*

Recall from [10] that a full subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is said to have the *$\omega$-torsionless property* if every module in $\mathcal{X}$ is $\omega$-torsionless.

**Proposition 3.11.** *For any $n \geqslant 1$, the following statements are equivalent.*

(1) *${}^{\perp_n}{}_R\omega \subseteq \mathcal{T}^1_\omega(R)$, i.e., ${}^{\perp_n}{}_R\omega$ has the $\omega$-torsionless property.*
(2) *${}^{\perp_n}{}_R\omega \subseteq \mathcal{T}_\omega(R)$.*
(3) *${}^{\perp_n}\omega_S = {}^{\perp}\omega_S$.*
(4) *Every module in ${}^{\perp_n}{}_R\omega$ has $\omega$-torsionfree dimension at most $n$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follows from [12, Lemma 3.3] and its proof. (2) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (2) Suppose that $M \in {}^{\perp_n}{}_R\omega$. Then $\mathcal{T}_\omega$-res.dim$_R(M) \leqslant n$ by assumption. By Proposition 3.2, there is an exact sequence $0 \to H' \to T' \to M \to 0$ in $\operatorname{mod} R$ with $T' \in \mathcal{T}_\omega(R)$ and add$_R\omega$-res.dim$_R(H') \leqslant n-1$. Because $M \in {}^{\perp_n}{}_R\omega$, the above short exact sequence splits, which implies that $M \in \mathcal{T}_\omega(R)$. $\square$

From the above Proposition 3.11, it is clear that if r.id$_S(\omega) \leqslant n$, then ${}^{\perp_n}{}_R\omega$ has the $\omega$-torsionless property. The following result extends [9, Theorem 2.7], which states that l.id$_R(\omega) =$ r.id$_S(\omega)$ provided both of them are finite.

**Corollary 3.12.** *If $n = \min\{t;\ {}^{\perp_t}{}_R\omega$ has the $\omega$-torsionless property$\}$ and $m = \min\{r;\ {}^{\perp_r}\omega_S$ has the $\omega$-torsionless property$\}$, then $n = m$.*

Proof. We may assume that $n \leqslant m$. Because ${}^{\perp_n}{}_R\omega$ has the $\omega$-torsionless property, ${}^{\perp_n}\omega_S = {}^{\perp}\omega_S$ by Proposition 3.11. Note that ${}^{\perp}\omega_S \subseteq {}^{\perp_m}\omega_S$ and ${}^{\perp_m}\omega_S$ has the $\omega$-torsionless property, so ${}^{\perp_n}\omega_S$ has the $\omega$-torsionless property. Thus $n \geqslant m$ by the minimality of $m$. We are done. $\square$

From [10, Proposition 2.3], the fact that ${}^{\perp}{}_R\omega$ has the $\omega$-torsionless property is equivalent to the condition that ${}^{\perp}{}_R\omega = \mathcal{G}_\omega(R)$. Since $\mathcal{G}_\omega(R) = {}^{\perp}{}_R\omega \cap \mathcal{T}_\omega(R)$ by Lemma 2.3, it is interesting to consider the following question:

**Question.** When $\mathcal{T}_\omega(R) = \mathcal{G}_\omega(R)$?

In the case of ${}_R\omega_S = {}_R R_R$, we have the following result.

**Theorem 3.13.** *${}^{\perp}R_R$ has the $R$-torsionless property if and only if $\mathcal{T}_R(R) = \mathcal{G}_R(R)$.*

We first prove the following lemma.

**Lemma 3.14.** *The following statements are equivalent.*

(1) ${}^{\perp}R_R \subseteq \mathcal{T}_R^1(R^{\rm op})$, *i.e.,* ${}^{\perp}R_R$ *has the $R$-torsionless property.*
(2) ${}^{\perp}R_R \subseteq \mathcal{T}_R(R^{\rm op})$.
(3) $\mathcal{T}_R(R) \subseteq {}^{\perp}{}_RR$.

Proof. (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2) Assume that $M \in {}^{\perp}R_R$. Then $M$ is $R$-torsionless by (1). So, by the symmetric version of Proposition 2.2, we have an exact sequence $0 \to M \to P_0 \to M_1 \to 0$ in $\operatorname{mod} R^{\rm op}$ with $P_0$ projective and $M_1 \in {}^{\perp_1}R_R$, which yields that $M_1 \in {}^{\perp}R_R$. Then $M_1$ is $R$-torsionless by (1), and again by the symmetric version of Proposition 2.2, we have an exact sequence $0 \to M_1 \to P_1 \to M_2 \to 0$ in $\operatorname{mod} R^{\rm op}$ with $P_1$ projective and $M_2 \in {}^{\perp_1}R_R$, which implies that $M_2 \in {}^{\perp}R_R$. Repeating this procedure, we get an exact sequence:

$$0 \to M \to P_0 \to P_1 \to \ldots \to P_i \to \ldots$$

in $\operatorname{mod} R^{\rm op}$ with $P_i$ projective and $\operatorname{Im}(P_i \to P_{i+1}) \in {}^{\perp}R_R$, which implies that $M \in \mathcal{T}_R(R^{\rm op})$ by Lemma 2.1.

(2) $\Rightarrow$ (3) Let $P_1 \to P_0 \to A \to 0$ be a projective resolution of $A$ in $\operatorname{mod} R$. Then we have an exact sequence $0 \to (\operatorname{Tr} A)^R \to P_1^{RR} \to P_0^{RR} \to \operatorname{Tr}\operatorname{Tr} A \to 0$. Thus $A$ and $\operatorname{Tr}\operatorname{Tr} A$ are projectively equivalent. Assume $A \in \mathcal{T}_R(R)$, $\operatorname{Tr} A \in {}^{\perp}R_R \subseteq \mathcal{T}_R(R^{\rm op})$. So $\operatorname{Tr}\operatorname{Tr} A \in {}^{\perp}{}_RR$, and hence $A \in {}^{\perp}{}_RR$ since $A$ and $\operatorname{Tr}\operatorname{Tr} A$ are projectively equivalent.

Similarly, (3) $\Rightarrow$ (2) holds true. $\square$

Proof of Theorem 3.13. ($\Rightarrow$) If ${}^{\perp}R_R \subseteq \mathcal{T}_R^1(R^{\rm op})$, $\mathcal{T}_R(R) \subseteq {}^{\perp}{}_RR$ by Lemma 3.14. Thus $\mathcal{T}_R(R) = {}^{\perp}{}_RR \cap \mathcal{T}_R(R) = \mathcal{G}_R(R)$.

($\Leftarrow$) If $\mathcal{T}_R(R) = \mathcal{G}_R(R)$, then $\mathcal{T}_R(R) \subseteq {}^{\perp}{}_RR$. The assertion follows from Lemma 3.14 again. $\square$

**Acknowledgement.** The authors thank the referee for the helpful suggestions. The authors also thank Dr. Chonghui Huang for his useful comments.

*References*


[1] *M. Auslander, M. Bridger*: Stable Module Theory. Memoirs of the American Mathematical Society 94, AMS, Providence, 1969.

[2] *M. Auslander, R.-O. Buchweitz*: The homological theory of maximal Cohen-Macaulay approximations. Mém. Soc. Math. Fr., Nouv. Sér. *38* (1989), 5–37.

[3] *M. Auslander, I. Reiten*: Cohen-Macaulay and Gorenstein Artin algebras. Representation Theory of Finite Groups and Finite-Dimensional Algebras (G. O. Michler et al., eds.). Proc. Conf., Bielefeld/Ger. 1991, Prog. Math. 95, Birkhäuser, Basel, 1991, pp. 221–245.

[4] *A. Beligiannis, I. Reiten*: Homological and Homotopical Aspects of Torsion Theories. Memoirs of the American Mathematical Society 883, AMS, Providence, 2007.

[5] *D. Bennis, N. Mahdou*: Global Gorenstein dimensions. Proc. Am. Math. Soc. *138* (2010), 461–465.

[6] *R. Göbel, J. Trlifaj*: Approximations and Endomorphism Algebras of Modules. De Gruyter Expositions in Mathematics 41, Walter de Gruyter, Berlin, 2006.

[7] *M. Hoshino*: Algebras of finite self-injective dimension. Proc. Am. Math. Soc. *112* (1991), 619–622.

[8] *Z. Huang*: $\omega$-$k$-torsionfree modules and $\omega$-left approximation dimension. Sci. China, Ser. A *44* (2001), 184–192.

[9] *Z. Huang*: Generalized tilting modules with finite injective dimension. J. Algebra *311* (2007), 619–634.

[10] *Z. Huang*: Selforthogonal modules with finite injective dimension III. Algebr. Represent. Theory *12* (2009), 371–384.

[11] *Z. Huang*: Wakamatsu tilting modules, $U$-dominant dimension, and $k$-Gorenstein modules. Abelian Groups, Rings, Modules, and Homological Algebra (P. Goeters et al., eds.). Lecture Notes in Pure and Applied Mathematics 249. Selected papers of a conference on the occasion of Edgar Earle Enochs' 72nd birthday, Auburn, AL, USA, September 9–11, 2004, Chapman & Hall/CRC, 2006, pp. 183–202.

[12] *Z. Huang, G. Tang*: Self-orthogonal modules over coherent rings. J. Pure Appl. Algebra *161* (2001), 167–176.

[13] *F. Mantese, I. Reiten*: Wakamatsu tilting modules. J. Algebra *278* (2004), 532–552.

[14] *J. J. Rotman*: An Introduction to Homological Algebra. Pure and Applied Mathematics 85, Academic Press, New York, 1979.

[15] *T. Wakamatsu*: Tilting modules and Auslander's Gorenstein property. J. Algebra *275* (2004), 3–39.



*Authors' addresses*: Guoqiang Zhao, College of Science, Hangzhou Dianzi University, Hangzhou, 310018, P. R. China, e-mail: gqzhao@hdu.edu.cn; Lirong Yin, College of Qianjiang, Hangzhou Normal University, Hangzhou, 310018, P. R. China, e-mail: ylr2005@163.com.