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STABILITY FOR A DIFFUSIVE DELAYED PREDATOR-PREY MODEL WITH MODIFIED LESLIE-GOWER AND HOLLING-TYPE II SCHEMES

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Abstract. A diffusive delayed predator-prey model with modified Leslie-Gower and Holling-type II schemes is considered. Local stability for each constant steady state is studied by analyzing the eigenvalues. Some simple and easily verifiable sufficient conditions for global stability are obtained by virtue of the stability of the related FDE and some monotonous iterative sequences. Numerical simulations and reasonable biological explanations are carried out to illustrate the main results and the justification of the model.

Keywords: delayed diffusive predator-prey model; modified Leslie-Gower scheme; Holling-type II scheme; persistence; stability; eigenvalue; monotonous iterative sequence

MSC 2010: 35K55, 35B25, 92C40

1. Introduction

In recent years considerable attention has been paid to the dynamic relationship between predators and their prey due to its universal existence and importance in population dynamics. The investigations on predator-prey models have been developed during these thirty years, and more realistic models have been derived in due to laboratory experiments and observations. In these models, more factors such as age-structure, seasonal effects, radio dependence, etc. have been taken into consideration (see [1], [3], [4], [5], [6], [7], [11] and the references therein). An important factor is the time delay (see [8], [13], [12], [1], [7] and the references therein). In [1], [7], Nindjin et al. considered a predator-prey model incorporating a modified version of the Leslie-Gower functional response as well as the Holling-type II functional

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response with delay:

$$\begin{aligned} \dot{x} &= x\Big(a_1 - bx - \frac{c_1 y}{x+k_1}\Big), \\ \dot{y} &= y\Big(a_2 - \frac{c_2 y(t-\tau)}{x(t-\tau)+k_2}\Big), \end{aligned} \tag{1.1}$$

where (1.1) is considered associated with the initial conditions $x(s) \geqslant 0, y(s) \geqslant 0$, $s \in [-\tau, 0]$. Here, a single discrete delay $\tau > 0$ is introduced as a negative feedback in the predator's density.

Note that the species always diffuse to areas of smaller population concentration in order to look for more food, so more and more mathematicians focus their attention on diffusive predator-prey systems. Let $U = U(t,x)$, $W = W(t,x)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ represent the density of the preys and the predators at time t and location x respectively, let D_1 and D_2 be the diffusion coefficients at x of the preys and the predators respectively. We plan to derive a reaction diffusion equation by Fick's Law. The Law says the predators always move from areas where population density is high to areas where it is lower, and it can be represented as $J(t,x) = -D_2 \nabla_x W(t,x)$, where J is the flux of the predators $W(t,x)$, and ∇_x is the gradient operator $\nabla_x = (\partial W/\partial x_1, \partial W/\partial x_1, \partial W/\partial x_2, \ldots, \partial W/\partial x_n)$. On the other hand, the reaction rate is $c_2 W(t-\tau)/U(t-\tau) + k_2$. Choose an arbitrary region O, the total population of the predators in O is $\int_O W(t,x)\,\mathrm{d}x$ and the rate of changes of the population W is $(\mathrm{d}/\mathrm{d}t)\int_O W(t,x)\,\mathrm{d}x$. The net growth of the population inside the region O is $\int_O W(a_2 - c_2 W(t-\tau)/(U(t-\tau)+k_2))\,\mathrm{d}x$ and the total out flux is

$$\int_{\partial O} J(t,x) \cdot \mathbf{n}(x)\,\mathrm{d}S, \tag{1.2}$$

where ∂O is the boundary of O and $\mathbf{n}(x)$ is the outer normal direction at x. Then the balance law implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_O W(t,x)\,\mathrm{d}x = -\int_{\partial O} J(t,x)\cdot \mathbf{n}(x)\,\mathrm{d}S + \int_O W\Big(a_2 - \frac{c_2 W(t-\tau)}{U(t-\tau)+k_2}\Big)\,\mathrm{d}x. \tag{1.3}$$

From the Divergence Theorem in multi-variable calculus, we have

$$\int_{\partial O} J(t,x)\cdot \mathbf{n}(x)\,\mathrm{d}S = \int_O \operatorname{div}(J(t,x))\,\mathrm{d}x. \tag{1.4}$$

Combining (1.2), (1.3) and (1.4), and interchanging the order of differentiation and integration, we obtain

$$\int_O \frac{\partial W(t,x)}{\partial t}\,\mathrm{d}x = \int_O \Big[\operatorname{div}(D_2 \nabla_x W(t,x)) + W\Big(a_2 - c_2 W(t-\tau)/U(t-\tau) + k_2\Big)\Big]\,\mathrm{d}x.$$

Since the choice of the region O is arbitrary, the differential equation

$$\frac{\partial W(t,x)}{\partial t} = D_2\Delta W + W\Big(a_2 - \frac{c_2 W(t-\tau)}{U(t-\tau)+k_2}\Big)$$

holds for any (t,x), where Δ is the Laplacian operator. Together with the similar arguments on the preys $U(t,x)$, it leads to the consideration of the reaction-diffusion model

$$\begin{aligned} \frac{\partial U}{\partial t} &= D_1\Delta U + U\Big(a_1 - bU - \frac{c_1 W}{U+k_1}\Big), \\ \frac{\partial W}{\partial t} &= D_2\Delta W + W\Big(a_2 - \frac{c_2 W(t-\tau)}{U(t-\tau)+k_2}\Big), \end{aligned} \tag{1.5}$$

which expresses the interaction of spatially distributed populations of predator W and prey U. If the predator and prey are confined to a fixed bounded domain Ω in $\mathbb{R}^n$ with smooth and impermeable boundary, that means the model is self-contained and has no population flux across the boundary $\partial\Omega$, then the homogeneous Neumann boundary condition is admissible. Doing the same variable changes as in [10], (1.5) becomes

$$\begin{aligned} &\frac{\partial u}{\partial t} = D\Delta u + u\Big(1 - u - \frac{\beta_1 w}{u+k_1}\Big), \quad t>0,\ x\in\Omega, \\ &\frac{\partial w}{\partial t} = \Delta w + \alpha w\Big(1 - \frac{\beta_2 w(t-\tau)}{u(t-\tau)+k_2}\Big), \quad t>0,\ x\in\Omega, \\ &\frac{\partial u}{\partial \mathbf{n}} = \frac{\partial w}{\partial \mathbf{n}} = 0, \quad t>0,\ x\in\partial\Omega, \end{aligned} \tag{1.6}$$

$$\begin{aligned} u(s,x) &= u_0(s,x) \geqslant 0,\ (s,x)\in[-\tau,0]\times\Omega, \\ w(s,x) &= w_0(s,x) \geqslant 0,\ (s,x)\in[-\tau,0]\times\Omega, \end{aligned}$$

where D, α, β_1, β_2, k_1, k_2 are positive constants. The initial data u_0, w_0 are continuous functions of s and x. By the method of upper and lower solutions we note from [8, Theorem 2.1] that (1.6) has a unique nonnegative global solution (u,w). In addition, if $u_0 \not\equiv 0$, $w_0 \not\equiv 0$, then the solution is positive, i.e., $u(t,x)>0$, $w(t,x)>0$ on $\overline{\Omega}$ for all $t>0$, by the maximum principle.

In view of the fact that the non-existence of the non-constant steady states can occur under some conditions (refer to [2, Theorem 3.1]), so the purpose of our work is to investigate the stability of the constant steady states. Our paper is organized as follows. In Section 2, we discuss the stability of the related FDE. The boundedness and persistence of (1.6) are considered in Section 3. The local stability and the global stability for the constant steady states are considered in Section 4. Numerical simulations are presented in Section 5 to illustrate some main results. Some conclusion and discussion are given in Section 5. The conclusion tells us the theorems in our paper agree with the natural rules of ecology.

2. Preliminary: stability for an FDE

In this section we give some stability results on an FDE in the form

$$\tag{2.1} \begin{gathered} \frac{\mathrm{d}z}{\mathrm{d}t} = \alpha z\Big(1 - \frac{1}{d} z(t-\tau)\Big), \quad t > 0, \\ z(s) \geqslant 0, \quad s \in [-\tau, 0),\ z(0) > 0, \end{gathered}$$

in order to study the global stabilities of the constant steady states in Section 4.

Lemma 2.1. *For the equation* (2.1), *the following statements are valid:*

(i) $z(t) > 0$ *for all* $t > 0$.

(ii) $\limsup\limits_{t\to+\infty} z(t) \leqslant d\mathrm{e}^{\alpha\tau}$, $\liminf\limits_{t\to+\infty} z(t) \geqslant d\mathrm{e}^{A\alpha\tau}$, $A = 1 - \mathrm{e}^{\alpha\tau}$.

Proof. Suppose that there is $T > 0$ such that $z(T) = 0$. Then there is $L > 0$ such that $\max\limits_{t\in[0,T]} z(t) \leqslant L$ and $\mathrm{d}z/\mathrm{d}t \geqslant \alpha z(1 - (1/d)L)$, which leads to $z(t) \geqslant z(0)\mathrm{e}^{\alpha(1-(1/d)L)t}$ for $0 < t < T$. It is a contradiction with $z(T) = 0$. Thus (i) is valid.

Integrating (2.1) over $[0, \tau]$, we obtain

$$\ln z(t) - \ln z(t-\tau) = \int_0^\tau \alpha\Big(1 - \frac{1}{d} z(t-\tau-s)\Big)\,\mathrm{d}s \leqslant \alpha\tau, \quad t > \tau,$$

hence there is $z(t) \leqslant z(t-\tau)\mathrm{e}^{\alpha\tau}$, i.e. $z(t-\tau) \geqslant z(t)\mathrm{e}^{-\alpha\tau}$. Substituting from (2.1), we obtain

$$\frac{\mathrm{d}z}{\mathrm{d}t} \leqslant \alpha z\Big(1 - \frac{1}{d} z\mathrm{e}^{-\alpha\tau}\Big), \quad t > \tau,$$

which leads to $\limsup\limits_{t\to+\infty} z(t) \leqslant d\mathrm{e}^{\alpha\tau}$.

Consequently, we have $T_1 > \tau$ such that $z(t-\tau) \leqslant d(\mathrm{e}^{\alpha\tau} + \varepsilon)$ for $t > T_1$, then $1 - (1/d)z(t-\tau) \geqslant 1 - (\mathrm{e}^{\alpha\tau} + \varepsilon)$. Substituting from (2.1), we also obtain

$$\frac{\mathrm{d}z}{\mathrm{d}t} \geqslant \alpha z(1 - \mathrm{e}^{\alpha\tau} - \varepsilon) = (A - \varepsilon)\alpha z, \quad t > T_1,$$

hence $\ln z(t) - \ln z(t-\tau) \geqslant (A-\varepsilon)\alpha\tau$ for $t > T_1$, then $z(t-\tau) \leqslant z(t)\mathrm{e}^{-(A-\varepsilon)\alpha\tau}$ follows. Substituting from (2.1) again, we have

$$\frac{\mathrm{d}z}{\mathrm{d}t} \geqslant \alpha z\Big(1 - \frac{1}{d} z \mathrm{e}^{-(A-\varepsilon)\alpha\tau}\Big), \quad t > T_1,$$

which yields $\liminf\limits_{t\to+\infty} z(t) \geqslant d\mathrm{e}^{(A-\varepsilon)\alpha\tau}$. Let $\varepsilon \to 0$, then $\liminf\limits_{t\to+\infty} z(t) \geqslant d\mathrm{e}^{A\alpha\tau}$. □

It is readily seen that (2.1) has two steady states $z = 0$ and $z = d$. Obviously, $z = 0$ is unstable, hence we discuss the global stability of $z = d$ by means of the V function.

Lemma 2.2. *If* $1 - \frac{3}{2}\alpha\tau\mathrm{e}^{\alpha\tau} > 0$, *then the solution of* (2.1) *satisfies* $\lim\limits_{t\to+\infty} z(t) = d$.

P r o o f. Let $z(t)$ be a solution of (2.1). Since $z(t) > 0$ for $t > 0$, we define $W(t) = \ln(z(t)/d)$ for $t > \tau$. Thus (2.1) can be written as

$$\dot{W}(t) = \alpha(1 - \mathrm{e}^{W(t-\tau)}), \quad t > \tau. \tag{2.2}$$

Define $V_1(t) = \int_0^{W(t)} (\mathrm{e}^u - 1)\,\mathrm{d}u$, then the derivative of V_1 along the solution of (2.2) is

$$\dot{V}_1(t)|_{(2.2)} = (\mathrm{e}^{W(t)} - 1)\dot{W}(t) = \alpha(\mathrm{e}^{W(t)} - 1)(1 - \mathrm{e}^{W(t-\tau)}). \tag{2.3}$$

Note that

$$\mathrm{e}^{W(t)} - \mathrm{e}^{W(t-\tau)} = \int_{t-\tau}^{t} \alpha\mathrm{e}^{W(s)}(1 - \mathrm{e}^{W(s-\tau)})\,\mathrm{d}s,$$

substitute from (2.3), then

$$\begin{aligned} \dot{V}_1(t) &= \alpha(\mathrm{e}^{W(t)} - 1)(1 - \mathrm{e}^{W(t)} + \mathrm{e}^{W(t)} - \mathrm{e}^{W(t-\tau)}) \\ &= -\alpha(\mathrm{e}^{W(t)} - 1)^2 + \alpha(\mathrm{e}^{W(t)} - 1)\int_{t-\tau}^{t} \alpha\mathrm{e}^{W(s)}(1 - \mathrm{e}^{W(s-\tau)})\,\mathrm{d}s \\ &\leqslant -\alpha(\mathrm{e}^{W(t)} - 1)^2 + \frac{\alpha^2}{2}\int_{t-\tau}^{t} \mathrm{e}^{W(s)}\,\mathrm{d}s(\mathrm{e}^{W(t)} - 1)^2 \\ &\quad + \frac{\alpha^2}{2}\int_{t-\tau}^{t} \mathrm{e}^{W(s)}(\mathrm{e}^{W(s-\tau)} - 1)^2\,\mathrm{d}s. \end{aligned} \tag{2.4}$$

Define

$$V_2(t) = \alpha^2 \int_{t-\tau}^{t} \Big[\int_{\nu}^{t} \mathrm{e}^{W(s)}(1 - \mathrm{e}^{W(s-r)})^2\,\mathrm{d}s\Big]\,\mathrm{d}\nu,$$

then the derivative of V_2 along the solution of (2.2) is

$$\dot{V}_2(t)|_{(2.2)} = \alpha^2\tau e^{W(t)}(1-e^{W(t-\tau)})^2 - \alpha^2\int_{t-\tau}^{t} e^{W(s)}(1-e^{W(s-\tau)})^2\,ds. \tag{2.5}$$

Choose ε so that $1-\frac{3}{2}\alpha\tau(e^{\alpha\tau}+\varepsilon)>0$, define $V_3(t)=\alpha^2\tau L\int_{t-\tau}^{t}(e^{W(s)}-1)^2\,ds$, where $L=e^{\alpha\tau}+\varepsilon$. Then

$$\dot{V}_3(t)|_{(2.2)} = \alpha^2\tau L(e^{W(t)}-1)^2 - \alpha^2\tau L(e^{W(t-\tau)}-1)^2. \tag{2.6}$$

Let $V(t)=V_1(t)+V_2(t)+V_3(t)$. It is easy to see that $V(t)$ is bounded below and the derivative of V along the solution of (2.2) is

$$\begin{aligned}\dot{V}(t)|_{(2.2)} = {} & -\alpha(e^{W(t)}-1)^2 + \frac{\alpha^2}{2}\int_{t-\tau}^{t} e^{W(s)}\,ds\,(e^{W(t)}-1)^2 \\ & + \frac{\alpha^2}{2}\int_{t-\tau}^{t} e^{W(s)}(e^{W(s-\tau)}-1)^2\,ds + \alpha^2\tau e^{W(t)}(1-e^{W(t-\tau)})^2 \\ & - \alpha^2\int_{t-\tau}^{t} e^{W(s)}(1-e^{W(s-\tau)})^2\,ds + \alpha^2\tau L(e^{W(t)}-1)^2 \\ & - \alpha^2\tau L(e^{W(t-\tau)}-1)^2.\end{aligned}$$

Since $\limsup\limits_{t\to+\infty} W(t)\leqslant\alpha\tau$, we have $e^{W(t)}\leqslant L$ for t sufficiently large. We have from the assumption that

$$\dot{V}(t)|_{(2.2)} \leqslant -\alpha\Big(1-\frac{3\alpha\tau}{2}L\Big)(e^{W(t)}-1)^2 \leqslant 0.$$

It is easy to see that $\dot{V}=0$ if and only if $W=0$, i.e. $z=d$, hence the Invariant Principle implies $z(t)\to d$ as $t\to+\infty$. The proof is complete. □

3. Boundedness and persistence

In this section, we discuss the boundedness of the solutions and the persistence of the system (1.6). A lemma will be given for convenience.

Lemma 3.1. *Consider the equation*

$$\begin{aligned}\frac{\partial v}{\partial t} &= \overline{D}\Delta v + vg(v(t-\tau,x)), \quad t>0,\ x\in\Omega,\\ \frac{\partial v}{\partial \mathbf{n}} &= 0, \quad t>0,\ x\in\partial\Omega,\\ v(s,x) &\geqslant 0, \quad s\in[-\tau,0],\ x\in\Omega.\end{aligned} \tag{3.1}$$

If $v(0,x)\not\equiv 0$, the following statements are valid.

(i) *If the function g satisfies $g(v)\leqslant \alpha(1-(1/d)v)$, then the solution of the equation* (3.1) *has the property*

$$\limsup_{t\to+\infty}\max_{\overline{\Omega}} v(t,\cdot)\leqslant d\mathrm{e}^{\alpha\tau}. \tag{3.2}$$

(ii) *If the function g satisfies $g(v)\geqslant \alpha(1-(1/d)v)$, then the solution of the equation* (3.1) *has the property*

$$\liminf_{t\to+\infty}\min_{\overline{\Omega}} v(t,\cdot)\geqslant d\mathrm{e}^{A\alpha\tau}, \quad \textit{where } A=1-\mathrm{e}^{\alpha\tau}<0. \tag{3.3}$$

Proof. Since $g(v)\leqslant\alpha(1-v/d)$, we are led to consider the FDE in the form

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= z\alpha\Big(1-\frac{1}{d}z(t-\tau)\Big), \quad t>0,\\ z(s) &= \max_{\overline{\Omega}} v(s,\cdot)\geqslant 0, \quad s\in[-\tau,0].\end{aligned}$$

Then Lemma 2.1 and the assumption $v(0,x)\not\equiv 0$ imply (3.2) by the comparison method. Inequality (3.3) is true by similar arguments. □

The above lemma leads to

Theorem 3.1. *Suppose that (u,w) is a solution of* (1.6). *The following statements are valid:*

$$\limsup_{t\to+\infty}\max_{\overline{\Omega}} u(t,\cdot)\leqslant 1; \tag{3.4}$$

$$\limsup_{t\to+\infty}\max_{\overline{\Omega}} w(t,\cdot)\leqslant \frac{1+k_2}{\beta_2}\mathrm{e}^{\alpha\tau}; \tag{3.5}$$

$$\liminf_{t\to+\infty}\min_{\overline{\Omega}} w(t,\cdot)\geqslant \frac{k_2}{\beta_2}\mathrm{e}^{A\alpha\tau}, \quad A=1-\mathrm{e}^{\alpha\tau}<0, \tag{3.6}$$

provided $w_0(s,x)$ is nonnegative with $w_0(s,x)\not\equiv 0$.

P r o o f. In view of $1-u-\beta_1 w/(u+k_1)\leqslant 1-u$, the first inequality is true and there is $T_1>0$ such that $u(t,x)\leqslant 1+\varepsilon$ for $t>T_1$, $x\in\Omega$. It follows that

$$1-\frac{\beta_2 w(t-\tau,x)}{k_2}\leqslant 1-\frac{\beta_2 w(t-\tau)}{u(t-\tau)+k_2}\leqslant 1-\frac{\beta_2 w(t-\tau)}{1+\varepsilon+k_2}$$

for $t>T_1+\tau$, $x\in\Omega$. Using Lemma 3.1 again, we have inequality (3.6) and

$$\limsup_{t\to\infty}\max_{\overline{\Omega}} w(t,\cdot)\leqslant\frac{1+\varepsilon+k_2}{\beta_2}\mathrm{e}^{\alpha\tau}.$$

Let $\varepsilon\to 0$, then (3.5) is valid, too. All the inequalities are valid and the theorem is proved. $\square$

Definition 3.1. The problem (1.6) is said to have the persistence property if for any nonnegative initial data (u_0,w_0) with $u_0(s,x)\not\equiv 0, w_0(s,x)\not\equiv 0$ for $(s,x)\in[-\tau,0]\times\Omega$ there exists a positive constant $\eta=\eta(u_0,w_0)$ such that the corresponding solution (u,w) of (1.6) satisfies

$$\liminf_{t\to\infty}\min_{\overline{\Omega}} u(t,\cdot)\geqslant\eta,\quad \liminf_{t\to\infty}\min_{\overline{\Omega}} w(t,\cdot)\geqslant\eta.$$

Theorem 3.2. *Suppose that $k_1>\beta_1(k_2/\beta_2)\mathrm{e}^{\alpha\tau}$, let $K=((k_1/\beta_1)\mathrm{e}^{-\alpha\tau}-k_2/\beta_2)\beta_2$. If $\beta_1(k_2/\beta_2)\mathrm{e}^{A\alpha\tau}>(1-K)(K+k_1)$, then (1.6) is persistent.*

P r o o f. Suppose (u,w) is a solution of (1.6) with $u_0(x)\geqslant 0$, $w_0(x)\geqslant 0$ and $u_0(x)\not\equiv 0$, $w_0(x)\not\equiv 0$. By the assumption, we can choose $\varepsilon_1>0$ small enough so that

$$\beta_1\Big(\frac{k_2}{\beta_2}\mathrm{e}^{A\alpha\tau}-\varepsilon_1\Big)>(1-K)(K+k_1). \tag{3.7}$$

Then we know from (3.6) that there is $T_2>0$ such that $w(t,\cdot)\geqslant(k_2/\beta_2)\mathrm{e}^{A\alpha\tau}-\varepsilon_1$ for all $t\geqslant T_2$. It follows that

$$1-u-\frac{\beta_1 w}{u+k_1}\leqslant\frac{(1-u)(u+k_1)-\beta_1\big((k_2/\beta_2)\mathrm{e}^{A\alpha\tau}-\varepsilon_1\big)}{u+k_1}\quad\text{for } t\geqslant T_2,\ x\in\Omega.$$

In view of $k_1>\beta_1(k_2/\beta_2)\mathrm{e}^{\alpha\tau}$, there is $\eta_1>0$ with $(1-\eta_1)(\eta_1+k_1)-\beta_1((k_2/\beta_2)\mathrm{e}^{A\alpha\tau}-\varepsilon_1)=0$ such that $\limsup\limits_{t\to+\infty}\max\limits_{\overline{\Omega}} u(t,\cdot)\leqslant\eta_1$. Since (3.7) yields $\eta_1<K$, we can choose ε_2 such that $K-\eta_1>\varepsilon_2>0$ and $T_3>T_2$ such that $u(t,\cdot)\leqslant\eta_1+\varepsilon_2$ for all $t\geqslant T_3$. It follows that

$$1-\frac{\beta_2 w(t-\tau)}{(u+k_2)}\leqslant 1-\frac{\beta_2 w(t-\tau)}{(\eta_1+\varepsilon_2+k_2)}\quad\text{for all } t\geqslant T_3,\ x\in\Omega.$$

From Lemma 3.1 we have

$$\limsup_{t\to+\infty} \max_{\overline{\Omega}} w(t,\cdot) \leqslant \eta_2 = \frac{\eta_1 + \varepsilon_2 + k_2}{\beta_2} \mathrm{e}^{\alpha\tau}.$$

In view of $\eta_2 < ((K + k_2)/\beta_2)\mathrm{e}^{\alpha\tau} = k_1/\beta_1$, we can choose $\varepsilon_3 > 0$ with $\eta_2 + \varepsilon_3 < k_1/\beta_1$ and $T_4 > 0$ such that $w(t,\cdot) \leqslant \eta_2 + \varepsilon_3 < k_1/\beta_1$ for all $t \geqslant T_4$. It follows that

$$1 - u - \frac{\beta_1 w}{(u + k_1)} \geqslant \frac{(1-u)(u+k_1) - \beta_1(\eta_2 + \varepsilon_3)}{u + k_1}, \quad t \geqslant T_4,\ x \in \Omega.$$

Thus there is $\eta_3 > 0$ with $(1 - \eta_3)(\eta_3 + k_1) - \beta_1(\eta_2 + \varepsilon_3) = 0$ such that

$$\liminf_{t\to+\infty} \min_{\overline{\Omega}} u(t,\cdot) \geqslant \eta_3. \tag{3.8}$$

Inequality (3.6) together with (3.8) yields the conclusion of this theorem. □

4. Stability of constant steady states

We discuss the stability of the constant steady states in this section.

4.1. Constant steady states. Obviously, $(0,0)$, $(1,0)$, $(0, k_2/\beta_2)$ are the three pairs of boundary constant steady states of (1.6). A proposition from [7] is to guarantee existence and uniqueness of a positive constant steady state.

Proposition 4.1. *System* (1.6) *has a unique interior equilibrium* $E_1 = (u^*, w^*)$ *(i.e.,* $u^* > 0, w^* > 0$*) if the following condition holds:*

$$\frac{k_2}{\beta_2} < \frac{k_1}{\beta_1}. \tag{4.1}$$

Moreover, we have another proposition to guarantee existence of two positive constant steady states from the graphs of functions $f_1(x) = (1-x)(x+k_1)/\beta_1$, $f_2(x) = (x + k_2)/\beta_2$.

Proposition 4.2. *System* (1.6) *has two interior equilibria* $E_2 = (u_1^*, w_1^*)$ *and* $E_3 = (u_2^*, w_2^*)$ *(i.e.,* $u_i^* > 0$*,* $w_i^* > 0$*) if the following condition holds:*

$$\frac{k_2}{\beta_2} > \frac{k_1}{\beta_1},\ \ 0 < k_1 < 1,\ \ \Delta = (\beta_1 - \beta_2 + \beta_2 k_1)^2 - 4\beta_2(\beta_1 k_2 - \beta_2 k_1) > 0. \tag{4.2}$$

If $u_1^* > u_2^*$, then $u_1^* > (1-k_1)/2 > u_2^*$ and $w_1^* > w_2^*$. So $E_2 = (u_1^*, w_1^*)$ is the larger interior equilibrium and $E_3 = (u_2^*, w_2^*)$ is the smaller interior equilibrium.

4.2. Local stability. Motivated by [9], we plan to discuss the local stability of the constant steady states by analyzing the eigenvalues. So first we give two lemmas which are helpful in judging the signs of the real parts of the eigenvalues. Lemma 4.1 is from [14].

Lemma 4.1. *All roots of the equation* $(z+a)\mathrm{e}^z + b = 0$*, where* a *and* b *are real numbers, have negative real parts if and only if*

$$a > -1, \quad a + b > 0, \quad b < \varrho \sin\varrho - a\cos\varrho,$$

where $\varrho = \pi/2$ *if* $a = 0$*, or* ϱ *is the root of* $\varrho = -a\tan\varrho$ *in* $(0,\pi)$ *if* $a \neq 0$.

Lemma 4.2 is a consequence of Lemma 4.1.

Lemma 4.2. *Supposing* $\mu \geqslant 0$*,* $\tau \geqslant 0$*, all the roots of the equation*

$$\mu + \alpha \mathrm{e}^{\lambda\tau} = \lambda \tag{4.3}$$

have positive real parts if $\alpha\tau < \pi/2$.

P r o o f. Suppose $z = -\lambda\tau$, then the equation (4.3) becomes

$$(z + \mu\tau)\mathrm{e}^z + \alpha\tau = 0. \tag{4.4}$$

We note that if all roots of (4.4) have negative real parts, then all roots of (4.3) have positive real parts. Obviously, $\mu\tau > -1$ and $\mu\tau + \alpha\tau > 0$, hence we only need to prove the third condition of Lemma 4.1.

If $\mu = 0$, then $\alpha\tau < \pi/2$ implies the third condition of Lemma 4.1 is valid.

If $\mu > 0$, then $\alpha\tau < \varrho\sin\varrho - \mu\tau\cos\varrho$ implies the third condition of Lemma 4.1 is valid, where ϱ is the root of $\varrho = -\mu\tau\tan\varrho$ in $(0,\pi)$ from Lemma 4.1. From the graph of $\varrho = -\mu\tau\tan\varrho$ in $(0,\pi)$ we know that $\pi/2 < \varrho < \pi$. And calculation yields

$$\varrho\sin\varrho - \mu\tau\cos\varrho = \varrho\sin\varrho + \frac{\varrho}{\tan\varrho}\cos\varrho = \varrho\sin\varrho + \frac{\varrho\cos^2\varrho}{\sin\varrho} = \frac{\varrho}{\sin\varrho}$$

and

$$\left(\frac{\varrho}{\sin\varrho}\right)' = \frac{1+\mu\tau}{\sin^3\varrho} > 0.$$

Thus it follows that $\varrho\sin\varrho - \mu\tau\cos\varrho > \lim\limits_{\varrho\to\pi/2} \varrho/\sin\varrho = \pi/2$. Hence, $\alpha\tau < \pi/2$ implies $\alpha\tau < \varrho\sin\varrho - \mu\tau\cos\varrho$, so the third condition of Lemma 4.1 is valid, too. Thus all roots of (4.4) have negative real parts. That means that all roots of (4.3) have positive real parts. The proof is complete. □

The linearized system of (1.6) is

$$\begin{aligned} \frac{\partial u}{\partial t} &= D\Delta u + \theta_1 u + \theta_2 w, \\ \frac{\partial w}{\partial t} &= \Delta w + \theta_3 w + \theta_4 w(t-\tau) + \theta_5 w(t-\tau). \end{aligned} \tag{4.5}$$

The corresponding linearized eigenvalue problem is

$$\begin{aligned} -D\Delta\xi - \theta_1\xi - \theta_2\eta &= \lambda\xi \quad \text{in } \Omega, \\ -\Delta\eta - \theta_3\eta - \theta_4 e^{\lambda\tau}\eta - \theta_5 e^{\lambda\tau}\xi &= \lambda\eta \quad \text{in } \Omega, \\ \frac{\partial\xi}{\partial\mathbf{n}} = \frac{\partial\eta}{\partial\mathbf{n}} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.6}$$

Of course θ_i $(i = 1,2,3,4)$ take different values for each constant steady state. We list them below.

For $(0, k_2/\beta_2)$, θ_i $(i = 1,2,3,4)$ are taken as

$$\theta_1 = 1 - \frac{\beta_1 k_2}{\beta_2 k_1}, \quad \theta_2 = 0, \quad \theta_3 = 0, \quad \theta_4 = -\alpha, \quad \theta_5 = \frac{\alpha}{\beta_2} \tag{4.7}$$

for (u^*, w^*),

$$\theta_1 = 1 - 2u^* - \frac{\beta_1 k_1 w^*}{(u^*+k_1)^2}, \quad \theta_2 = -\frac{\beta_1 u^*}{u^*+k_1}, \quad \theta_3 = 0, \quad \theta_4 = -\alpha, \quad \theta_5 = \frac{\alpha}{\beta_2} \tag{4.8}$$

for (u_1^*, w_1^*),

$$\begin{aligned} \theta_1 &= 1 - 2u_1^* - \frac{\beta_1 k_1 w_1^*}{(u_1^*+k_1)^2} = \frac{u_1^*}{u_1^*+k_1}(1 - 2u_1^* - k_1) < 0, \\ \theta_2 &= -\frac{\beta_1 u_1^*}{u_1^*+k_1}, \quad \theta_3 = 0, \quad \theta_4 = -\alpha, \quad \theta_5 = \frac{\alpha}{\beta_2} \end{aligned} \tag{4.9}$$

for (u_2^*, w_2^*),

$$\begin{aligned} \theta_1 &= 1 - 2u_2^* - \frac{\beta_1 k_1 w_2^*}{(u_2^*+k_1)^2} = \frac{u_2^*}{u_2^*+k_1}(1 - 2u_2^* - k_1) > 0, \\ \theta_2 &= -\frac{\beta_1 u_2^*}{u_2^*+k_1}, \quad \theta_3 = 0, \quad \theta_4 = -\alpha, \quad \theta_5 = \frac{\alpha}{\beta_2}. \end{aligned} \tag{4.10}$$

As is well known, the constant steady states are linearly stable if (4.6) has no eigenvalue λ with $\operatorname{Re}\lambda \leqslant 0$. Therefore, we will discuss the linear stability for each constant steady state by analyzing the linearized eigenvalue problem. It is of interest to note that the linear stabilities depend on the parameters.

Theorem 4.1. *The positive constant steady state (u^*, w^*) is linearly stable if β_1 is sufficiently small and $\alpha\tau < \pi/2$.*

Proof. A contradiction argument will be used by assuming that (1.6) has a positive constant steady state $(u^*_{(i)}, w^*_{(i)})$ which is linearly unstable for a sequence $\{\beta_1^{(i)}\}$ with $\beta_1^{(i)} \to 0$, where $i \geqslant 1$. Thus there exists λ_i with $\operatorname{Re}(\lambda_i) \leqslant 0$ and $(\xi_i, \eta_i) \neq (0,0)$ such that

$$\begin{aligned} -D\Delta\xi_i - \theta_1^{(i)}\xi_i - \theta_2^{(i)}\eta_i &= \lambda_i\xi_i \quad \text{in } \Omega,\\ -\Delta\eta_i - \theta_4 e^{\lambda_i\tau}\eta_i - \theta_5 e^{\lambda_i\tau}\xi_i &= \lambda_i\eta_i \quad \text{in } \Omega,\\ \frac{\partial\xi_i}{\partial\mathbf{n}} = \frac{\partial\eta_i}{\partial\mathbf{n}} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.11}$$

where $\theta_1^{(i)} = 1 - 2u^*_{(i)} - \beta_1^{(i)} k_1 w^*_{(i)}/(u^*_{(i)} + k_1)^2$, $\theta_2^{(i)} = -(\beta_1^{(i)} u^*_{(i)})/(u^*_{(i)} + k_1)$.

Assuming that $\|\xi_i\|^2_{L^2} + \|\eta_i\|^2_{L^2} = 1$, from (4.11) we have

$$\lambda_i = \int_\Omega |\nabla\xi_i|^2 - \int_\Omega \theta_1^{(i)}|\xi_i|^2 - \int_\Omega \theta_2^{(i)}\eta_i\bar{\xi_i} + \int_\Omega |\nabla\eta_i|^2 - \int_\Omega \theta_4 e^{\lambda_i\tau}|\eta_i|^2 - \int_\Omega \theta_5 e^{\lambda_i\tau}\xi_i\overline{\eta}_i.$$

Note that $\theta_1^{(i)} \to -1$, $\theta_2^{(i)} \to 0$, $u^*_{(i)} \to 1$, $w^*_{(i)} \to (1+k_2)/\beta_2$ as $i \to \infty$, $\beta_1^{(i)} \to 0$, hence $\{\operatorname{Im}\lambda_i\}$ and $\{\operatorname{Re}(\lambda_i)\}$ are bounded and so $\{\lambda_i\}$ is bounded. Without loss of generality, assume $\lambda_i \to \lambda$, then $\operatorname{Re}\lambda \leqslant 0$. We can also assume that $\xi_i \to \xi$ and $\eta_i \to \eta$ since $\{\xi_i\}$ and $\{\eta_i\}$ are bounded. Taking the limit in (4.11), we have

$$\begin{aligned} -D\Delta\xi + \xi &= \lambda\xi \quad \text{in } \Omega,\\ -\Delta\eta - \theta_4 e^{\lambda\tau}\eta - \theta_5 e^{\lambda\tau}\xi &= \lambda\eta \quad \text{in } \Omega,\\ \frac{\partial\xi}{\partial\mathbf{n}} = \frac{\partial\eta}{\partial\mathbf{n}} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.12}$$

If $\xi \neq 0$, then λ is an eigenvalue of the problem

$$\begin{aligned} -D\Delta\varphi + \varphi &= \lambda\varphi \quad \text{in } \Omega,\\ \frac{\partial\varphi}{\partial\mathbf{n}} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

It is clear that $\lambda \geqslant 1$, which is impossible. Hence, $\xi = 0$ and thus $\eta \neq 0$. Substituting $\xi = 0$ into the second equation of (4.12), we have

$$-\Delta\eta + \alpha e^{\lambda\tau}\eta = \lambda\eta \quad \text{in } \Omega. \tag{4.13}$$

Since $\eta \neq 0$, so λ is an eigenvalue of the problem

$$\begin{aligned} -\Delta\varphi + \alpha \mathrm{e}^{\lambda\tau}\varphi &= \lambda\varphi \quad \text{in } \Omega, \\ \frac{\partial\varphi}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.14}$$

Let $0 = \mu_0 < \mu_1 < \mu_2 < \ldots < \ldots$ denote the eigenvalues of the operator $-\Delta$ on $\mathbb{X}$ with the homogeneous Neumann boundary condition, then there is μ_i such that the eigenvalue λ satisfies

$$\mu_i + \alpha \mathrm{e}^{\lambda\tau} = \lambda.$$

It follows from Lemma 4.2 that $\mathrm{Re}\,\lambda > 0$, which is also a contradiction. Hence the theorem is valid. $\square$

Theorem 4.2. *The boundary constant steady state* $(0, k_2/\beta_2)$ *is linearly stable if* $k_1/\beta_1 < k_2/\beta_2$.

P r o o f. Note that due to (4.6) and (4.7) it is sufficient to prove that all eigenvalues of the linearized eigenvalue problem

$$\begin{aligned} -D\Delta\xi - \Big(1 - \frac{\beta_1 k_2}{\beta_2 k_1}\Big)\xi &= \lambda\xi \quad \text{in } \Omega, \\ -\Delta\eta + \alpha \mathrm{e}^{\lambda\tau}\eta - \frac{\alpha}{\beta_2}\mathrm{e}^{\lambda\tau}\xi &= \lambda\eta \quad \text{in } \Omega, \\ \frac{\partial\varphi}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.15}$$

have positive real parts, i.e., $\mathrm{Re}\,\lambda > 0$. This is valid because the eigenvalues satisfy $\lambda > \beta_1 k_2/\beta_2 k_1 - 1 > 0$ by virtue of the first equation of (4.15). Hence, the theorem is proved. $\square$

Theorem 4.1 is the result for system (1.6) that has a unique interior equilibrium. But how about that system (1.6) if it has two interior equilibria? The following two theorems answer the question.

Theorem 4.3. *Suppose that* $\alpha\tau < \pi/2$, *then the positive constant steady state* (u_1^*, w_1^*) *is linearly stable if* $\beta_2 \to \infty$.

P r o o f. If the conclusion is not true, we assume that (1.6) has a positive steady state (u_{1i}^*, w_{1i}^*) which is linearly unstable for a sequence $\{\beta_2^{(i)}\}$ with $\beta_2^{(i)} \to \infty$, where $i \geqslant 1$. Thus there exists λ_i with $\mathrm{Re}\,(\lambda_i) \leqslant 0$ and $(\xi_i, \eta_i) \neq (0, 0)$ such that

$$\begin{aligned} -D\Delta\xi_i - \theta_1\xi_i - \theta_2\eta_i &= \lambda_i\xi_i \quad \text{in } \Omega, \\ -\Delta\eta_i - \theta_4 \mathrm{e}^{\lambda_i\tau}\eta_i - \theta_5^{(i)}\mathrm{e}^{\lambda_i\tau}\xi_i &= \lambda_i\eta_i \quad \text{in } \Omega, \\ \frac{\partial\xi_i}{\partial \mathbf{n}} = \frac{\partial\eta_i}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.16}$$

where $\theta_5^{(i)} = \alpha/\beta_2^{(i)}$. Similarly to the proof of Theorem 4.1, we can assume $\lambda_i \to \lambda$, then $\operatorname{Re}(\lambda) \leqslant 0$. Taking the limit in (4.16), we have

$$
\begin{aligned}
-D\Delta\xi - \theta_1\xi - \theta_2\eta &= \lambda\xi \quad \text{in } \Omega,\\
-\Delta\eta - \theta_4 \mathrm{e}^{\lambda\tau}\eta &= \lambda\eta \quad \text{in } \Omega,\\
\frac{\partial\xi}{\partial\mathbf{n}} = \frac{\partial\eta}{\partial\mathbf{n}} &= 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{4.17}
$$

From the second equation of (4.17), we obtain $\operatorname{Re}\lambda > 0$ if $\eta \neq 0$ from Lemma 4.2, which is a contradiction. Then $\eta = 0$. Substituting it into the first equation of (4.17), we get

$$
-D\Delta\xi - \Big(1 - 2u_1^* - \frac{\beta_1 k_1 w_1^*}{u_1^* + k_1}\Big)\xi = \lambda\xi.
$$

Then $\lambda > 0$ if $\xi \neq 0$, because $1 - 2u_1^* - \beta_1 k_1 w_1^*/(u_1^* + k_1) < 0$, which is also a contradiction. So, (u_1^*, w_1^*) is linearly stable if β_2 is sufficiently large. □

Theorem 4.4. *The positive constant steady state (u_2^*, w_2^*) is linearly unstable if $\beta_2 \to \infty$.*

Proof. For this constant steady state, $\theta_1 > 0$, $\beta_2 \to \infty$ implies that $\theta_1\theta_4 < \theta_2\theta_5 < 0$. If we can find that (4.6) has at least one eigenvalue whose real part $\operatorname{Re}(\lambda) < 0$, the conclusion will be true. In view of the fact that the operator $-\Delta$ subject to the homogeneous Neumann boundary condition has many eigenvalues $0 = \mu_0 < \mu_1 < \ldots < \mu_n < \ldots$, if λ is the eigenvalue of (4.6) corresponding to $\mu = 0$, then

$$
\begin{vmatrix} -\lambda - \theta_1 & -\theta_2 \\ -\theta_5 \mathrm{e}^{\lambda\tau} & -\lambda - \theta_4 \mathrm{e}^{\lambda\tau} \end{vmatrix} = 0,
$$

hence

$$
(\lambda + \theta_1)(\lambda + \theta_4 \mathrm{e}^{\lambda\tau}) = \theta_2\theta_5 \mathrm{e}^{\lambda\tau}.
$$

Define $f_1(\lambda) = (\lambda + \theta_1)(\lambda + \theta_4 \mathrm{e}^{\lambda\tau})$, then $f_1(0) = \theta_1\theta_4$, $f_1(-\theta_1) = 0$. Define $f_2(\lambda) = \theta_2\theta_5 \mathrm{e}^{\lambda\tau}$, then $f_2(0) = \theta_2\theta_5$, $f_2(-\theta_1) = \theta_2\theta_5 \mathrm{e}^{-\theta_1\tau} < 0$. Thus $\theta_1\theta_4 < \theta_2\theta_5$ and the intermediate value theorem implies that there exists a λ between $-\theta_1$ and 0 being the eigenvalue of (4.6), thus (u_2^*, w_2^*) is linearly unstable. □

4.3. Global stability. As is well known, the results in [8] have been applied in studying the global stability for the delayed diffusive systems (see [12]). But they cannot be applied to study (1.6) because it is impossible to find a pair of suitable upper-lower solutions. Thus we plan to investigate (1.6) by virtue of the stability for the FDE (2.1) as well as some monotonous iterative sequences.

Theorem 4.5. *Suppose that one of the following conditions is satisfied:*

(1) $k_1 \geqslant 1$ *and* $k_1/\beta_1 < (k_2/\beta_2)\mathrm{e}^{A\alpha\tau}$;

(2) $k_1 < 1$ *and* $((1+k_1)/2)^2 < \beta_1(k_2/\beta_2)\mathrm{e}^{A\alpha\tau}$.

Let $1 - \frac{3}{2}\alpha\tau\mathrm{e}^{\alpha\tau} > 0$. *Then*

$$\lim_{t\to\infty}(u(t,\cdot), w(t,\cdot)) = (0, k_2/\beta_2) \quad \textit{uniformly on } \Omega$$

provided $w_0(s,x) \not\equiv 0$, $s \in [-\tau, 0]$, $x \in \Omega$.

Proof. Choose ε small enough to have $k_1/\beta_1 < (k_2/\beta_2)\mathrm{e}^{A\alpha\tau} - \varepsilon$ and $((1+k_1)/2)^2 < \beta_1((k_2/\beta_2)\mathrm{e}^{A\alpha\tau} - \varepsilon)$. From (3.6) we have T_1 such that $w(t,x) \geqslant (k_2/\beta_2)\mathrm{e}^{A\alpha\tau} - \varepsilon$ for $t > T_1$ and $x \in \overline{\Omega}$, hence both (1) and (2) guarantee $(1-u)(u+k_1) < \beta_1 w$ for $t > T_1$, $x \in \overline{\Omega}$. It follows from the first equation of (1.6) that

$$\lim_{t\to+\infty} u(t,\cdot) = 0 \quad \text{uniformly on } \Omega. \tag{4.18}$$

Together with the condition $w_0(s,x) \not\equiv 0$, $s \in [-\tau, 0]$, $x \in \Omega$, we have $T_2 > T_1$ such that $0 < u(t,x) < \varepsilon$ for $t > T_2$, $x \in \overline{\Omega}$ and $0 < w(t,x)$ for $t > T_2$, $x \in \overline{\Omega}$. It leads to

$$1 - \frac{\beta_2 w(t-\tau)}{k_2} < 1 - \frac{\beta_2 w(t-\tau)}{u(t-\tau)+k_2} < 1 - \frac{\beta_2 w(t-\tau)}{\varepsilon + k_2}, \quad t > T_2 + \tau,\ x \in \Omega.$$

Consider the following two FDE, one being

$$\begin{aligned} \dot{w}_1(t) &= \alpha w_1\Big(1 - \frac{\beta_2 w_1(t-\tau)}{k_2}\Big), \quad t > T_2 + \tau, \\ w_1(s) &= \min_{\overline{\Omega}} w(s,x) > 0, \quad s \in [T_2, T_2+\tau], \end{aligned} \tag{4.19}$$

the other

$$\begin{aligned} \dot{w}_2(t) &= \alpha w_2\Big(1 - \frac{\beta_2 w_2(t-\tau)}{\varepsilon + k_2}\Big), \quad t > T_2 + \tau, \\ w_2(s) &= \max_{\overline{\Omega}} w(s,x) > 0, \quad s \in [T_2, T_2+\tau]. \end{aligned} \tag{4.20}$$

By virtue of Lemma 2.2, $w_1(t)$ and $w_2(t)$ possess the property that

$$\lim_{t\to+\infty} w_1(t) = \frac{k_2}{\beta_2}, \quad \lim_{t\to+\infty} w_2(t) = \frac{\varepsilon + k_2}{\beta_2}. \tag{4.21}$$

By using the comparison results, (4.21) implies

$$\frac{k_2}{\beta_2} \leqslant \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} w(t,\cdot) \leqslant \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} w(t,\cdot) \leqslant \frac{\varepsilon + k_2}{\beta_2}.$$

Letting $\varepsilon \to 0$, we have

$$\lim_{t\to+\infty} w(t,\cdot) = \frac{k_2}{\beta_2} \quad \text{uniformly on } \Omega. \tag{4.22}$$

From (4.18) and (4.22), we conclude that the theorem is valid. $\square$

The global stability of $(0, k_2/\beta_2)$ is obtained by virtue of the stability of (2.1). Next, we discuss the stability of the state (u^*, w^*) by using the stability of (2.1) as well as some monotonous iterative sequences.

Theorem 4.6. *Suppose that* $k_1/\beta_1 > (1+k_2)/\beta_2$ *and* $1 - \frac{3}{2}\alpha\tau e^{\alpha\tau} > 0$. *If* $\beta_1 - \beta_2 k_1 \leqslant 0$, *then* (u^*, w^*) *is globally stable provided* $w_0(s,x) \not\equiv 0$, $s \in [-\tau, 0]$, $x \in \Omega$.

Proof. Choose $\varepsilon_1 > 0$ with $(1+k_2)/\beta_2 + \varepsilon_1 < k_1/\beta_1$. Define $P_1 = k_2/\beta_2$, then we can choose $0 < M_1 < 1$ and $\varepsilon_1 > 0$ such that

$$(1 - (M_1 - \varepsilon_1))((M_1 - \varepsilon_1) + k_1) - \beta_1(P_1 - \varepsilon_1) = 0. \tag{4.23}$$

Define $Q_1 = (M_1 + k_2)/\beta_2$. Since $Q_1 > P_1$, there is N_1 with $0 < N_1 \leqslant M_1 < 1$ such that

$$(1 - (N_1 + \varepsilon_1))((N_1 + \varepsilon_1) + k_1) - \beta_1(Q_1 + \varepsilon_1) = 0. \tag{4.24}$$

Since $u \geqslant 0$ and $w_0(s,x) \not\equiv 0$, $s \in [-\tau, 0]$, $x \in \Omega$, there is $T_0 > 0$ such that for $t > T_0$, we have $1 - \beta_2 w/(u + k_2) \geqslant 1 - \beta_2 w/k_2$ and $0 < w(t,x)$, $x \in \overline{\Omega}$. Hence we consider the FDE in the form

$$\begin{aligned} \dot{w}_3(t) &= \alpha w_3\Big(1 - \frac{\beta_2 w_3(t-\tau)}{k_2}\Big), \quad t > T_0 + \tau,\\ w_3(s) &= \min_{\overline{\Omega}} w(s,\cdot) > 0, \quad s \in [T_0, T_0 + \tau]. \end{aligned}$$

The assumption $1 - \frac{3}{2}\alpha\tau e^{\alpha\tau} > 0$ implies $\lim_{t\to+\infty} w_3(t) = P_1$ by Lemma 2.2, which yields

$$\liminf_{t\to+\infty} \min_{\overline{\Omega}} w(t,\cdot) \geqslant P_1, \tag{4.25}$$

by comparison results. Then there is $T_1 > 0$ such that $w(t,x) \geqslant P_1 - \varepsilon_1$ for $t > T_1$, $x \in \Omega$. It follows that

$$1 - u - \frac{\beta_1 w}{u + k_1} \leqslant \frac{(1-u)(u+k_1) - \beta_1(P_1 - \varepsilon_1)}{u + k_1},$$

and we obtain

$$\limsup_{t\to+\infty} \max_{\overline{\Omega}} u(t,\cdot) \leqslant M_1 - \varepsilon_1. \tag{4.26}$$

Hence, there is $T_2 > T_1$ such that $u(t-\tau,\cdot) \leqslant M_1$ for $t > T_2 + \tau$, and thus

$$1 - \frac{\beta_2 w(t-\tau)}{u(t-\tau)+k_2} \leqslant 1 - \frac{\beta_2 w(t-\tau)}{M_1+k_2} \quad \text{for } t > T_2+\tau,\ x \in \Omega.$$

Consider the FDE in the form

$$\begin{aligned} \dot w_4(t) &= \alpha w_4\Big(1 - \frac{\beta_2 w_4(t-\tau)}{M_1+k_2}\Big), \quad t > T_2+\tau,\\ w_4(s) &= \max_{\overline{\Omega}} w(s,\cdot), \quad s \in [T_2, T_2+\tau]. \end{aligned} \tag{4.27}$$

It is easy to see from Lemma 2.2 that $\lim\limits_{t\to+\infty} w_4(t) = (M_1+k_2)/\beta_2 = Q_1$, and so

$$\limsup_{t\to+\infty} \max_{\overline{\Omega}} w(t,\cdot) \leqslant Q_1. \tag{4.28}$$

It follows that there is $T_3 > T_2$ such that

$$1 - u - \frac{\beta_1 w}{u+k_1} \geqslant \frac{(1-u)(u+k_1) - \beta_1(Q_1+\varepsilon_1)}{u+k_1} \quad \text{for } t > T_3,\ x \in \Omega.$$

Consequently, we have from (4.24) that

$$\liminf_{t\to+\infty} \min_{\overline{\Omega}} u(t,\cdot) \geqslant N_1 + \varepsilon_1. \tag{4.29}$$

Summarize the above discussion, we have from (4.25), (4.26), (4.28), (4.29) that

$$\begin{aligned} P_1 \leqslant \liminf_{t\to+\infty} \min_{\overline{\Omega}} w(t,\cdot) &\leqslant \limsup_{t\to+\infty} \max_{\overline{\Omega}} w(t,\cdot) \leqslant Q_1,\\ N_1+\varepsilon_1 \leqslant \liminf_{t\to+\infty} \min_{\overline{\Omega}} u(t,\cdot) &\leqslant \limsup_{t\to+\infty} \max_{\overline{\Omega}} u(t,\cdot) \leqslant M_1 - \varepsilon_1. \end{aligned}$$

Similarly we can claim for $n \geqslant 2$ that

$$\begin{aligned} P_n \leqslant \liminf_{t\to+\infty} \min_{\overline{\Omega}} w(t,\cdot) &\leqslant \limsup_{t\to+\infty} \max_{\overline{\Omega}} w(t,\cdot) \leqslant Q_n,\\ N_n + \frac{\varepsilon_1}{n} \leqslant \liminf_{t\to+\infty} \min_{\overline{\Omega}} u(t,\cdot) &\leqslant \limsup_{t\to+\infty} \max_{\overline{\Omega}} u(t,\cdot) \leqslant M_n - \frac{\varepsilon_1}{n}. \end{aligned} \tag{4.30}$$

The sequences are defined as

$$P_n = \frac{N_{n-1}+k_2}{\beta_2}, \quad Q_n = \frac{M_n+k_2}{\beta_2},$$
$$\Big(1-\Big(M_n-\frac{\varepsilon_1}{n}\Big)\Big)\Big(\Big(M_n-\frac{\varepsilon_1}{n}\Big)+k_1\Big)-\beta_1\Big(P_n-\frac{\varepsilon_1}{n}\Big)=0,$$
$$\Big(1-\Big(N_n+\frac{\varepsilon_1}{n}\Big)\Big)\Big(\Big(N_n+\frac{\varepsilon_1}{n}\Big)+k_1\Big)-\beta_1\Big(Q_n+\frac{\varepsilon_1}{n}\Big)=0.$$

They satisfy

$$\begin{aligned} &(4.31) \quad 0 < N_1 \leqslant N_2 \leqslant \ldots \leqslant N_n \leqslant M_n \leqslant M_{n-1} \leqslant \ldots \leqslant M_2 \leqslant M_1 < 1,\\ &\frac{k_2}{\beta_2} = P_1 \leqslant P_2 \leqslant \ldots \leqslant P_n \leqslant Q_n \leqslant Q_{n-1} \leqslant \ldots \leqslant Q_2 \leqslant Q_1 = \frac{M_1+k_2}{\beta_2}. \end{aligned}$$

Since all the sequences are monotonous and bounded, there are $\overline{M}$, $\overline{N}$, $\overline{P}$, $\overline{Q}$ being the limits of the sequences, respectively. They satisfy

$$\overline{P} = \frac{\overline{N}+k_2}{\beta_2}, \quad \overline{Q} = \frac{\overline{M}+k_2}{\beta_2}.$$
$$(1-\overline{M})(\overline{M}+k_1)=\beta_1\overline{P}, \quad (1-\overline{N})(\overline{N}+k_1)=\beta_1\overline{Q}.$$

Then we have

$$\begin{aligned} \beta_2(1-\overline{M})(\overline{M}+k_1)-\beta_1(\overline{N}+k_2)&=0,\\ \beta_2(1-\overline{N})(\overline{N}+k_1)-\beta_1(\overline{M}+k_2)&=0. \end{aligned} \tag{4.32}$$

Thus

$$(\beta_2+\beta_1-\beta_2k_1)(\overline{M}-\overline{N})-\beta_2(\overline{M}^2-\overline{N}^2)=0.$$

If $\overline{M} > \overline{N}$, we have

$$\overline{M}+\overline{N} = 1+\frac{\beta_1}{\beta_2}-k_1, \quad \text{i.e.,} \quad \overline{N} = 1+\frac{\beta_1}{\beta_2}-k_1-\overline{M}.$$

Substituting it into (4.32), we have

$$\frac{\beta_2}{\beta_1}(1-\overline{M})(\overline{M}+k_1)-\Big(1+\frac{\beta_1}{\beta_2}-k_1-\overline{M}\Big)=0.$$

We have $\overline{M} > 1$ or $\overline{M} < 0$ from the assumption $\beta_1-\beta_2k_1 \leqslant 0$ and the graphs of functions $f(x) = (\beta_2/\beta_1)(1-x)(x+k_1)$ and $g(x) = 1+\beta_1/\beta_2-k_1-x$, which contradicts (4.31). Thus $\overline{M} = \overline{N}$. Together with the facts that $0 < \overline{M} = \overline{N} < 1$, $0 < \overline{P} = \overline{Q}$, we have $\overline{M} = \overline{N} = u^*$, $\overline{P} = \overline{Q} = w^*$ because (u^*, w^*) is the unique positive steady state of the equation (1.6). Thus (4.30) implies

$$\lim_{t\to+\infty}(u(t,\cdot), w(t,\cdot)) = (u^*, w^*) \quad \text{uniformly on } \Omega,$$

which means (u^*, w^*) is globally stable. □

As far as we know, there are few papers considering the case that two equilibria exist, but it is interesting to consider which equilibrium will attract the solution. We believe the smaller equilibrium is unstable due to Theorem 4.4 and the larger equilibrium will not be globally stable. So we find an attractive area for the larger equilibrium in the following theorem.

Theorem 4.7. *Suppose that the equation* $(1-x)(x+k_1)-(\beta_1(1+k_2)/\beta_2)\mathrm{e}^{\alpha\tau}=0$ *has two roots. Denote the larger root by* R. *Suppose that* $k_2/\beta_2 > k_1/\beta_1$ *and* $1-\frac{3}{2}\alpha\tau\mathrm{e}^{\alpha\tau}>0$. *If* $\beta_1-\beta_2 k_1 \leqslant 0$, *then* (u_1^*, w_1^*) *is stable provided* $0 \leqslant w_0(s,x) \leqslant ((1+k_2)/\beta_2)\mathrm{e}^{\alpha\tau}$, $R \leqslant u_0(s,x) \leqslant 1$, $s \in [-\tau, 0]$, $x \in \overline{\Omega}$.

P r o o f. We claim that the subset $[R,1]\times[0,((1+k_2)/\beta_2)\mathrm{e}^{\alpha\tau}]$ is invariant. We only need to prove that $w(t,x) \leqslant ((1+k_2)/\beta_2)\mathrm{e}^{\alpha\tau}$ and $u(t,x) \geqslant R$ for $t>0$, $x\in\Omega$. Consider the equation

$$\dot{z}_1 = \alpha z_1\Big(1-\frac{\beta_2 z_1(t-\tau)}{1+k_2}\Big) \leqslant \alpha z_1\Big(1-\frac{\beta_2 z_1(t)}{1+k_2}\mathrm{e}^{-\alpha\tau}\Big);$$

it follows that $w(t,x) \leqslant ((1+k_2)/\beta_2)\mathrm{e}^{\alpha\tau}$ for $t>0$, $x\in\Omega$. Then $u(t,x)\geqslant R$ for $t>0$, $x\in\Omega$, because

$$1-u-\frac{\beta_1 w}{u+k_1} \geqslant 1-u-\frac{\beta_1(1+k_2)}{(u+k_1)\beta_2}\mathrm{e}^{\alpha\tau}.$$

Next, like in the proof of Theorem 4.6, we define some monotonous sequences $\{\widehat{P}_n\}$, $\{\widehat{Q}_n\}$, $\{\widehat{M}_n\}$, $\{\widehat{N}_n\}$ to press the solution from both sides and guarantee that the solution converges to (u_1^*, w_1^*). The initial terms of the sequences are defined as $\widehat{P}_1=(R+k_2)/\beta_2$, $\widehat{M}_1-\varepsilon_1$ is the larger root of the equation $(1-u)(u+k_1)-\beta_1(\widehat{P}_1-\varepsilon_1)=0$, $\widehat{Q}_1=(\widehat{M}_1+k_2)/\beta_2$, $\widehat{N}_1-\varepsilon_1$ is the larger root of the equation $(1-u)(u+k_1)-\beta_1(\widehat{Q}_1+\varepsilon_1)=0$, ε_1 has the same meaning as in the proof of Theorem 4.6. The sequences are defined as

$$\widehat{P}_n=\frac{\widehat{N}_{n-1}+k_2}{\beta_2},$$

$\widehat{M}_n-\dfrac{\varepsilon_1}{n}$ is the larger root of the equation $(1-u)(u+k_1)-\beta_1\Big(\widehat{P}_n-\dfrac{\varepsilon_1}{n}\Big)=0,$

$$\widehat{Q}_n=\frac{\widehat{M}_n+k_2}{\beta_2},$$

$\widehat{N}_n-\dfrac{\varepsilon_1}{n}$ is the larger root of the equation $(1-u)(u+k_1)-\beta_1\Big(\widehat{Q}_n+\dfrac{\varepsilon_1}{n}\Big)=0.$

As in the proof of Theorem 4.6, we also can prove that

$$\widehat{P}_n \leqslant \liminf_{t\to+\infty} \min_{\widehat{\Omega}} w(t,\cdot) \leqslant \limsup_{t\to+\infty} \max_{\overline{\Omega}} w(t,\cdot) \leqslant \widehat{Q}_n,$$
$$\widehat{N}_n + \frac{\varepsilon_1}{n} \leqslant \liminf_{t\to+\infty} \min_{\overline{\Omega}} u(t,\cdot) \leqslant \limsup_{t\to+\infty} \max_{\overline{\Omega}} u(t,\cdot) \leqslant \widehat{M}_n - \frac{\varepsilon_1}{n},$$

and $\lim_{n\to+\infty} \widehat{M}_n = \lim_{n\to+\infty} \widehat{N}_n = u_1^*$, $\lim_{n\to+\infty} \widehat{P}_n = \lim_{n\to+\infty} \widehat{Q}_n = w_1^*$ because (u_1^*, w_1^*) is the unique positive steady state of the equation (1.6) in the invariant subset $[R,1] \times [0, ((1+k_2)/\beta_2)\mathrm{e}^{\alpha\tau}]$. Thus the theorem is proved. □

5. Numerical simulation

In this section we give some examples to illustrate our main results on the convergence of problem (1.6).

Example 5.1. In system (1.6), let $\alpha = 1$, $\tau = 0.1$, $\beta_1 = 1$, $\beta_2 = 0.5$, $k_1 = 2$, $k_2 = 2$.

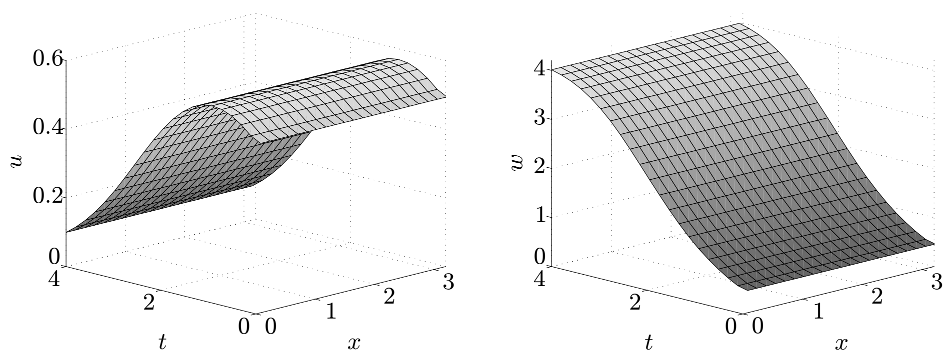

Figure 5.1

In this example, we can verify that the conditions in Theorem 4.5 are valid by simple calculation, hence the solutions converge to $(0, k_2/\beta_2) = (0, 4)$.

Example 5.2. In system (1.6), let $\alpha = 1$, $\tau = 0.1$, $\beta_1 = 0.1$, $\beta_2 = 0.5$, $k_1 = 1$, $k_2 = 1$.

In this example, we can verify that the conditions in Theorem 4.6 are valid by simple calculation, hence the solutions converge to $(u^*, w^*) = (0.8, 3.6)$.

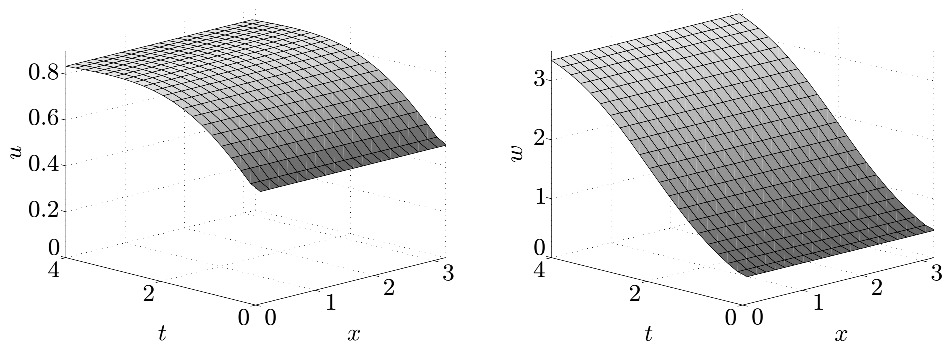

Figure 5.2

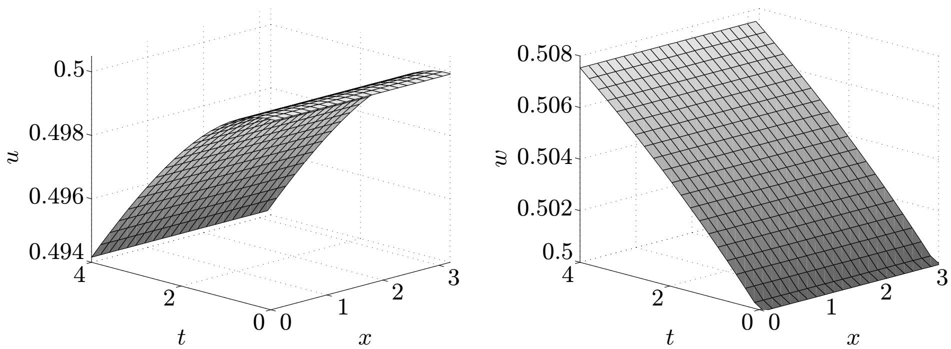

Figure 5.3

Example 5.3. In system (1.6), let $\alpha = 0.1$, $\tau = 0.1$, $\beta_1 = 1$, $\beta_2 = 64$, $k_1 = 1/2$, $k_2 = 33$.

In this example, we can verify that the conditions in Theorem 4.7 are valid by simple calculation, hence the solutions converge to $(u_1^*, w_1^*) = (0.449264, 0.522650)$.

6. Conclusion and discussion

First we state the biological meaning of the parameters of the model (1.6). Our model (1.6) is derived from the model (1.1), which is discussed by Nindjin in [7]. Model (1.1) describes a prey population x which serves as food for a predator with population y. The parameters a_1, a_2, b, c_1, c_2, k_1, k_2 are assumed to be only of positive values: a_1 and a_2 are the growth rates of prey x and predator y, respectively, b measures the strength of competition among individuals of species x, c_1 is the maximum value of the per capita reduction rate of x due to y, k_1 and k_2 measure the extent to which environment provides protection to prey x and to predator y respectively, and c_2 has a similar meaning as c_1. So the parameters of model (1.6) have similar biological meaning. β_1 describes the predators capability of capturing food, while k_1 indicates the preys reaction to the predator. When β_1 decreases and k_1 increases, i.e., k_1/β_1 is increasing, the speed of the preys growth will increase, thus its survival ability. On the other hand, $1/\beta_2$ can be used to describe the transition from prey to predator after the prey is captured by the predator; $1/k_2$ indicates the reliability of the predator to the prey; $k_2 = 0$ means the predator is completely dependent on the prey. The larger k_2, the lower reliability the predator has to the prey, thus a better survival rate in an environment where the prey is scarce. Due to the above reasons, we can use k_1/β_1 and k_2/β_2 to describe the survivability of the prey u and the predator w respectively.

Next we express the biological meaning of the equilibria of the system (1.6). The interior equilibria imply co-existence of the prey and the predator. The boundary equilibrium $(0, k_2/\beta_2)$ implies there is only predator left in the environment, and the prey is eliminated. Although there exists no prey in this case, due to the food diversity of the predator, it can still survive, with its number limited. Note that the un-stabilities of $(1, 0)$ and $(0, 0)$ imply that an invasion of a minor quantity of predator or an invasion of a minor quantity of prey will cause a change of the long time behavior of the system moving to either $(0, k_2/\beta_2)$ or the interior equilibria.

Now we try to give biological explanation of several different cases according to our previous theorems.

(1) When k_1/β_1 is small and k_2/β_2 is large, the prey has weaker survivability. The system shifts towards equilibrium $(0, k_2/\beta_2)$, thus the prey in the environment becomes extinct (refer to Theorems 4.2 and Theorem 4.5).

(2) $k_1/\beta_1 < k_2/\beta_2$, but $k_2/\beta_2 - k_1/\beta_1$ is small. That means there is a little difference between the survivability of the prey and the survivability of the predator. So there are two interior equilibria and the smaller equilibrium (u_2^*, w_2^*) is unstable (refer to Theorem 4.4). And it is reasonable to suppose that either a rather small transition from prey to predator or a rather large initial density of the prey can result in the stability of the larger interior equilibrium (u_1^*, w_1^*) (refer to Theorem 4.3 and Theorem 4.7).

(3) When k_1/β_1 is large and k_2/β_2 is small, the prey has stronger survivability. There exists a positive equilibrium in this case (see Proposition 4.1). Furthermore, if β_1/β_2 becomes small, i.e., we have lower foraging ability of the predator or lower transition rate from prey to predator, the prey has relatively stronger survivability, and it will not be extinct. Thus the system is persistent (see Theorem 3.2). In addition, when the foraging ability of the predator which is denoted by β_1 is small enough,

(u^*, w^*) is locally stable (refer to Theorem 4.1) and when $k_1/\beta_1 > (1+k_2)/\beta_2$, (u^*, w^*) is globally stable (see Theorem 4.6).

As in [7], our results also reveal the fact that the delay τ plays an important role in the dynamic system (1.6). The local and global stability of the equilibria involve restrictions on the length of time delay. These restrictions are due to the assumption $\alpha\tau < \pi/2$ in Theorem 4.1 and Theorem 4.3 and the assumption $1 - \frac{3}{2}\alpha\tau e^{\alpha\tau} > 0$ in Theorem 4.5, Theorem 4.6 and Theorem 4.7. Therefore, it is obvious that the delay has a destabilized effect on the equilibria $(0, k_2/\beta_2)$, (u^*, w^*) and (u_1^*, w_1^*).

To conclude, our system model exhibits specific biological significance, and the theorems provided in this paper agree with the natural rules of ecology. In addition, the effect of the delay is indicated in our paper.

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