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Forcings which preserve large cardinals

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The aim of this article is to summarise the three lectures that I gave at the Hejnice Winter School during February 2–4 2010. In those lectures I covered the following topics:

1. What are large cardinals?

2. Forcings which preserve large cardinals:

a. Failures of GCH.
b. Cardinal characteristics at large cardinals.
c. *L*-like universes and large cardinals.

Much of what is said here under Topics 1 and 2(a) is a repeat of what can be found in my article appearing in the Proceedings of the 2009 Hejnice Winter School; topics 2(b) and 2(c) were not treated there.

Mixing forcing with large cardinals is an old idea. Two landmark results in the area were Silver's (see [20]), showing that the GCH can fail at a measurable cardinal, and Prikry's ([26]), showing that a measurable cardinal can be forced to be singular without collapsing cardinals. Subsequently there has been a vast amount of work in this area, typically motivated by questions concerning the possible behaviours of the

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generalised continuum function $\kappa \mapsto 2^\kappa$. As Easton showed ([10]) that this function can behave almost arbitrarily when restricted to regular cardinals, the emphasis has been on its behaviour at singulars ("the singular cardinal problem"). Nevertheless, interesting results have also been proved through the use of forcings that preserve large cardinal properties, and that is the emphasis of this article.

Large cardinals

κ is strongly inaccessible iff it is uncountable, regular and closed under the generalised continuum function, i.e., if α is less than κ then so is 2^α. Strongly inaccessible cardinals are large in the sense that there existence implies the consistency of ZFC and is therefore unprovable in ZFC. Whereas many interesting consistency results can be shown starting with just an inaccessible, stronger large cardinal properties are often required. A natural and important strengthening is that of a *measurable* cardinal:

κ is *measurable* iff:
$\kappa > \aleph_0$,
$\exists$ nonprincipal, κ-complete ultrafilter on κ.

This is a perfectly good definition, but proving things about measurable cardinals often demands an alternative, equivalent definition, phrased in terms of (elementary) embeddings. Let V denote the universe of sets and M an inner model (i.e., a transitive proper class that satisfies the axioms of ZFC). A definable $j : V \to M$ is an *embedding* iff j is not the identity and j is elementary, i.e. preserves the truth of formulas with parameters.

Fix such a $j : V \to M$. The *critical point* of j is the least ordinal κ such that $j(\kappa) \neq \kappa$. It is easy to show that such a κ must exist and is an uncountable cardinal. Many large cardinal notions are defined in terms of critical points in the following way: κ is "large" iff κ is the critical point of an embedding $j : V \to M$ where M is "large", i.e., where M contains a "large" amount of V. The two most basic notions are:

κ is $H(\lambda)$-strong iff $H(\lambda) \subseteq M$,
κ is λ-supercompact iff $M^\lambda \subseteq M$.

Fact: Measurable = $H(\kappa^+)$-strong = κ-supercompact.

Other notions of large involve the embeddings j itself:

κ is superstrong iff $H(j(\kappa)) \subseteq M$,
κ is hyperstrong iff $H(j(\kappa)^+) \subseteq M$,
κ is n-superstrong iff $H(j^n(\kappa)) \subseteq M$, ($n$ finite)
κ is ω-superstrong iff $H(j^\omega(\kappa)) \subseteq M$,

where $j^\omega(\kappa)$ is the supremum of the $j^n(\kappa)$'s for finite n. An important result of Kunen says that this as far as one can go:

Kunen's Theorem. ([21]) More than ω-superstrong is inconsistent, i.e. one cannot have $H(j^\omega(\kappa)^+) \subseteq M$.

I pause for a moment and consider the

Question. Why study large cardinals?

Here is one reason:

Even with large cardinals, set theory is *incomplete*: For many φ, both ZFC + φ and ZFC+ $\sim \varphi$ are consistent. But set theory with large cardinals seems to be *consistency complete*: For almost all φ, either φ is inconsistent or we have

Con(ZFC + LC) $\rightarrow$ Con(ZFC + φ)

for some large cardinal axiom LC; moreover we often get

Con(ZFC + φ) $\rightarrow$ Con(ZFC + lc)

where lc is another large cardinal axiom, almost as strong as LC. Thus the conclusion we come to is: We *need* large cardinals to show consistency and to measure consistency strength.

There is another reason to study large cardinals. Forcing is even more interesting when they exist! Examples:

a. The failure of GCH at a measurable.

Increasing 2^κ for a measurable κ with κ-Cohen is painful, with κ-Laver regrettable, but with κ-Sacks perfect! Thus large cardinals make interesting distinctions between forcings.

b. Cardinal characteristics at a measurable (new area). The characteristics

$$\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{e}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{m}, \mathfrak{p}, \mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{u}$$

can be considered at a measurable κ. This demands the use of iterated forcing with uncountable supports, a challenging topic in the current theory of forcing.

c. Forcing combinatorial principles at a measurable.

There are some nice surprises here with Jensen's $\square$ Principle. The general question of "large cardinal tolerance", i.e., the study of the large cardinal properties which are consistent with given combinatorial principles, is a very interesting topic of study.

Beyond the scope of this article are

d. Singular cardinal problems (Prikry-type forcings, see [19]).

This area of set theory continues to be very active, thanks largely to the recent dramatic discoveries emanating from Shelah's PCF theory ([28]).

Forcings that preserve large cardinals: Failures of GCH

We pose now the following general

Question. Suppose κ is a large cardinal and G is P-generic over V. Is κ still a large cardinal in $V[G]$?

The most common approach to this question is to use the *lifting method* of Silver. Suppose we are given $j : V \to M$ and G is P-generic over V. Let P^* denote $j(P)$.

Goal: Find a G^* which is P^*-generic over M such that $j[G] \subseteq G^*$.

If there is such a G^* then $j : V \to M$ lifts to $j^* : V[G] \to M[G^*]$, defined by $j^*(\sigma^G) = j(\sigma)^{G^*}$. To see that this is well-defined, argue as follows:

$\sigma_0^G = \sigma_1^G \to p \Vdash \sigma_0 = \sigma_1$ for some $p \in G \to j(p) \Vdash j(\sigma_0) = j(\sigma_1)$ for some $p \in G \to j(\sigma_0)^{G^*} = j(\sigma_1)^{G^*}$ as $j[G] \subseteq G^*$.

A similar argument shows that j^* is elementary. Now if G^* can moreover be chosen in $V[G]$ then κ is still measurable in $V[G]$ (and usually as "large" in $V[G]$ as it was in V).

Ultrapowers

To successfully apply the lifting method one often needs a special $j : V \to M$ to start with, as described in the next result.

Theorem 1 *(Ultrapower Theorem) Suppose that κ is $H(\lambda)$-strong, i.e., there is $j : V \to M$ with critical point κ such that $H(\lambda) \subseteq M$.*
(a) (Extender ultrapower) If $\lambda \leq j(\kappa)$ then j can be modified so that: $M = \{j(f)(a) \mid f : H(\kappa) \to V,\ a \in H(\lambda)\}$.
(b) (Hyperextender ultrapower) If $\lambda = j(\kappa)^+$ then j can be modified so that: $M = \{j(f)(a) \mid f : H(\kappa^+) \to V,\ a \in H(j(\kappa)^+)\}$.
(c) (2-Hyperextender ultrapower) If $\lambda \leq j^2(\kappa)$ then j can be modified so that: $M = \{j(f)(a) \mid f : H(j(\kappa)) \to V,\ a \in H(\lambda)\}$.
(d) The $n+1$-Hyperextender ultrapower uses $f : H(j^n(\kappa)) \to V$; the ω-Hyperextender ultrapower uses $f : H(j^\omega(\kappa)) \to V$.

Proof of (a): Define $H = \{j(f)(a) \mid f : H(\kappa) \to V, a \in H(\lambda)\} \prec M$, $k : H \simeq M'$ the transitive collapse, $j' : V \to M'$ by $j' = k \circ j$.

The proofs of (b), (c) and (d) are similar. □

Easy lifting

Sometimes it is easy to lift $j : V \to M$ to $j^* : V[G] \to M[G^*]$.

Recall: $j : V \to M$ has critical point κ, G is P-generic over V, $P^* = j(P)$ and we want a G^* which is P^*-generic over M satisfying $j[G] \subseteq G^*$. We say that *j lifts for P*.

Small forcing. Suppose that P belongs to $H(\kappa)$ (P is small). Then j lifts for P.

Proof. $P^* = j(P) = P$. Take $G^* = G$. Then G^* is P^*-generic over $M \subseteq V$ and $j[G] = G \subseteq G^*$, trivially! □

P is κ^+ *distributive* iff the intersection of κ-many open dense sets is dense.

Theorem 2 *Suppose that $j : V \to M$ is given by an extender ultrapower, i.e., $M = \{j(f)(a) \mid f : H(\kappa) \to V, a \in H(\lambda)\}$ for some $\lambda \leq j(\kappa)$, $H(\lambda) \subseteq M$. Suppose that P is κ^+ distributive in V. Then j lifts for P.*

Proof. Suppose that $D \in M$ is open dense on $P^* = j(P)$. Write $D = j(f)(a)$ where $f : H(\kappa) \to V$, $a \in H(\lambda)$. We can assume that $f(x)$ is open dense on P for each $x \in H(\kappa)$. By the κ^+ distributivity of P there is $p \in G$ which belongs to each $f(x)$. It follows that $j(p)$ belongs to each $j(f)(y)$, $y \in H(j(\kappa))^M$ and therefore to $j(f)(a)$. So $j[G]$ "generates" the P^*-generic $G^* = \{p^* \in P^* \mid j(p) \leq p^*$ for some p in $G\}$. □

So P-lifting is nontrivial only when P has size at least κ and adds κ-sequences. A good example is κ-Cohen forcing. Does j lift for κ-Cohen forcing? Bad news!

Theorem 3 *Let P be κ-Cohen forcing. Then no $j : V \to M$ lifts for P.*

Here is the problem:

Suppose that $C \subseteq \kappa$ is generic for κ-Cohen. We want to lift $j : V \to M$ to j^* : : $V[C] \to M[C^*]$. I.e., we want to find C^* which is $j(\kappa)$-Cohen generic over M and which "extends" C, i.e., such that $C = C^* \cap \kappa$. But this is impossible! Proper initial segments of C^* must belong to M, but C does not even belong to V, which contains M!

So we need the forcing to add C^* to be defined not in M but in a model that *already has C*. The solution is to force not just at κ, but at all inaccessible $\alpha \leq \kappa$, via an iteration

$$P = P(\alpha_0) * P(\alpha_1) * \cdots * P(\kappa)$$

where $P(\alpha)$ denotes α-Cohen forcing. We should lift not just $P(\kappa) = \kappa$-Cohen forcing, but the entire iteration P. (We "prepare below κ".)

But what is the support of the iteration

$$P = P(\alpha_0) * P(\alpha_1) * \cdots * P(\kappa) \ ?$$

Use *Easton support*, i.e., require that for p in $P = P(\alpha_0) * P(\alpha_1) * \cdots * P(\kappa)$, $\mathrm{Support}(p) = \{i \mid p \restriction i \nVdash p(\alpha_i) \text{ is trivial}\}$ has bounded intersection with each inaccessible. Then for regular λ, P factors as:

$$P(\leq \lambda) * P(> \lambda)$$

where $P(\leq \lambda)$ has "size" λ and $P(> \lambda)$ is λ^+-closed (descending sequences of length λ have lower bounds). As in the proof of Easton's theorem, this gives cofinality preservation.

Theorem 4 *Assume GCH. Let $P = P(\leq \kappa) = P(\alpha_0) * P(\alpha_1) * \cdots * P(\kappa)$ be the iteration of α-Cohen for inaccessible $\alpha \leq \kappa$ described above. Suppose that $j : V \to M$ is an extender ultrapower witnessing the $H(\lambda)$-strength of κ for some regular λ less than the least inaccessible above κ. Then j lifts for P.*

Let's prove this. Let $C(\leq \kappa) = C(\alpha_0) * C(\alpha_1) * \cdots * C(\kappa)$ denote the P-generic and $V^* = V[C(\leq \kappa)]$.

We want to lift $j : V \to M$ to

$j^* : V[C(\leq \kappa)] \to M[C^*(\leq \kappa) * C^*(\beta_0) * C^*(\beta_1) * \cdots * C^*(j(\kappa))]$,

where the β_i's are the inaccessibles of M between κ and $j(\kappa)$ and the C^*'s are chosen in $V^* = V[C(\leq \kappa)]$.

Set $C^*(\leq \kappa) = C(\leq \kappa)$.

The middle part: Take $\langle C^*(\beta) \mid \kappa < \beta < j(\kappa)\rangle = C^*(\kappa, j(\kappa))$ to be any generic in V^* (why are there any ???).

The last lift: Take $C^*(j(\kappa))$ to be any generic in V^* for $j(\kappa)$-Cohen forcing of $M[C^*(\leq \kappa) * C^*(\kappa, j(\kappa))]$ containing the condition $C(\kappa) = C^*(\kappa)$ (again, why are there any ???).

So we have to explain the two ???'s in the following diagram:

$j^* : V[C(\leq \kappa)] \to M[C(\leq \kappa) * C^*(\kappa, j(\kappa))??? * C^*(j(\kappa))???]$.

The middle part: We want a generic $C^*(\kappa, j(\kappa))$ in $V^* = V[C(\leq \kappa)]$ for $P^*(\kappa, j(\kappa)) = P^*(\beta_0) * P^*(\beta_1) * \cdots$, a forcing which is β_0-closed and has $j(\kappa)$-many maximal antichains in $M[C(\leq \kappa)]$.

Recall that the original $j : V \to M$ was an extender ultrapower witnessing $H(\lambda)$-strength for some regular $\lambda < \beta_0$. Using this we have:

Claim.
(a) $M^\kappa \cap V \subseteq M$.
(b) $j(\kappa)$ can be written in V as the union of κ^+-many subsets, each an element of M of size λ in M.

Given (a) and (b): The κ^+-cc of $P(\leq \kappa)$ implies that (a) also holds for the models $M[C(\leq \kappa)]$, $V[C(\leq \kappa)]$:

$$M[C(\leq \kappa)]^\kappa \cap V[C(\leq \kappa)] \subseteq M[C(\leq \kappa)].$$

Therefore $P^*(\kappa, j(\kappa))$ is κ^+-closed in $V[C(\leq \kappa)]$. But then (b) and the λ^+ closure of $P^*(\kappa, j(\kappa))$ in $M[C(\leq \kappa)]$ implies that we can build a $P^*(\kappa, j(\kappa))$-generic in κ^+ steps. So we are done with the first ???; but we must prove the Claim.

Proof that $M^\kappa \cap V \subseteq M$:

Given $j(f_0)(a_0), j(f_1)(a_1), \cdots$ of length κ define $f : H(\kappa) \to V$ by $f(\langle x_0, x_1, \cdots \rangle) = = \langle f_0(x_0), f_1(x_1), \cdots \rangle$; then $j(f)(\langle a_0, a_1, \cdots \rangle)$ is the κ-sequence of the $j(f_i)(a_i)$'s and $\langle a_0, a_1, \cdots \rangle$ is an element of $H(\lambda)$.

Proof that $j(\kappa)$ can be written in V as the union of κ^+-many subsets, each an element of M of size λ in M:

Every ordinal less than $j(\kappa)$ is of the form $j(f)(a)$ where $f : H(\kappa) \to V$ and $a \in H(\lambda)$; but we may assume $f : H(\kappa) \to \kappa$ (simply redefine $f(x)$ to be 0 if $f(x)$ is not an ordinal $< \kappa$; this won't affect $j(f)(a)$). So $j(\kappa)$ is the union of the sets $A(f) = \{j(f)(a) \mid a \in \in H(\lambda)\}$, $f : H(\kappa) \to \kappa$, each of which has size λ in M by GCH, and again by GCH there are only κ^+-many such sets.

Now we turn to the second ???: Here is the diagram again:

$j^* : V[C(< \kappa) * C(\kappa)] \to M[C(\leq \kappa) * C^*(\kappa, j(\kappa)) * C^*(j(\kappa))???]$

We need a generic in V^* for $P^*(j(\kappa))$ = the $j(\kappa)$-Cohen forcing of $M[C(\leq \kappa) * C^*(\kappa, j(\kappa))]$ containing the condition $C(\kappa)$.

This is similar to the previous case. We have:

Claim.
(a) $M[C^*(< j(\kappa))]^\kappa \cap V^* \subseteq M[C^*(< j(\kappa))]$.
(b) $P^*(j(\kappa))$ has $(j(\kappa)^+)^{M[C^*(<j(\kappa))]} = j(\kappa^+)$ many maximal antichains in $M[C^*(< j(\kappa))]$ and $j(\kappa^+)$ can be written in V^* as the union of κ^+ many subsets, each an element of M of size λ in M.

For (a) we need only show $\mathrm{Ord}^\kappa \cap V^* \subseteq M[C^*(< j(\kappa))]$, which follows from $\mathrm{Ord}^\kappa \cap$ $\cap V^* \subseteq M[C^*(\leq \kappa)]$.

For (b), note that every $\alpha < j(\kappa^+)$ can be written as $j(f)(a)$ with $f : H(\kappa) \to \kappa^+$, $a \in H(\lambda)$, and there are still only κ^+-many such f's. So we can build a $P^*(j(\kappa))$-generic in V^* containing $C(\kappa)$.

So we have succeeded in lifting $j : V \to M$ to $j : V^* = V[C(\leq \kappa)] \to M[C^*(\leq j(\kappa))]$ in V^*, where $C(\leq \kappa)$ results by iterating α-Cohen forcing for inaccessible $\alpha \leq \kappa$. □

Now we would like to make this work with α-Cohen forcing replaced by $\mathrm{Cohen}(\alpha, \alpha^{++})$, a forcing that adds α^{++}-many α-Cohen sets and therefore kills the GCH at α.

It doesn't work! Here is the problem:

Assuming that the original $j : V \to M$ witnessed $H(\kappa^{++})$-strength (to allow $C^*(\kappa) =$ $= C(\kappa)$), all goes well until the last lift: we *can* choose $C^*(\gamma)$ for M-inaccessible $\gamma < j(\kappa)$ and lift $j : V \to M$ to

$j' : V[C(< \kappa)] \to M[C^*(< j(\kappa))]$.

We then need to find a generic $C^*(j(\kappa))$ for $P^*(j(\kappa))$ = the $\mathrm{Cohen}(j(\kappa), j(\kappa^{++}))$-forcing of $M[C^*(< j(\kappa))]$ which contains $j'[C(\kappa)]$ to get:

$j^* : V[C(\leq \kappa)] \to M[C^*(< j(\kappa)) * C^*(j(\kappa))???]$.

But $P^*(j(\kappa)) = \mathrm{Cohen}(j(\kappa), j(\kappa^{++}))$ is a big forcing: it has size κ^{++} and won't have a generic in $V[C(\leq \kappa)]$! Even worse, whereas before $j'[C(\kappa)]$ was equal to $C(\kappa)$, now $j'[C(\kappa)]$ is a complicated set of conditions.

Here is the solution: Use $\mathrm{Sacks}(\kappa, \kappa^{++})$ instead of $\mathrm{Cohen}(\kappa, \kappa^{++})$.

Now we want to lift $j : V \to M$ to

$j^* : V[S(\leq \kappa)] \to M[S(\leq \kappa) * S^*(\kappa, j(\kappa)) * S^*(j(\kappa))]$.

The nice thing now is that we don't have to build a generic $S^*(j(\kappa))$ for $P^*(j(\kappa)) =$ $= \mathrm{Sacks}(j(\kappa), j(\kappa^{++}))$ containing $j'[S(\kappa)]$, because in fact $j'[S(\kappa)]$ (almost) generates one for us!

I'll illustrate this with just $\mathrm{Sacks}(\kappa, 1) = \kappa$-Sacks: A condition is a κ-tree, i.e. a subtree T of $2^{<\kappa}$ such that:

i. T has no terminal nodes and is $< \kappa$-closed, i.e., the union of a $(< \kappa)$ increasing sequence of nodes in T is a node in T.

ii. T has "CUB splitting": For some CUB $C(T) \subseteq \kappa$, $\sigma \in T$ "splits" in T iff the length of σ belongs to $C(T)$.

If G is generic then the intersection of the κ-trees in G gives us a function $g : \kappa \to 2$, which uniquely determines G.

Now prepare as before, iterating for $\kappa + 1$ steps, but with α-Sacks instead of α-Cohen. Then as before we obtain an embedding

$$j' : V[S(<\kappa)] \to M[S^*(<j(\kappa))].$$

To extend j' further we want to find a generic $S^*(j(\kappa))$ for the $j(\kappa)$-Sacks of $M[S^*(<$ $< j(\kappa))]$ which contains $j'[S(\kappa)]$.

But in fact there are only two possible choices for $S^*(j(\kappa))$. This uses the following

Claim. The intersection of the $j(C)$, C CUB in κ, is $\{\kappa\}$.

Assume this Claim. For any CUB C in κ there are κ-trees T in the generic $S(\kappa)$ which only split on C. Thus by the Claim the intersection of the $j(T)$, $T \in S(\kappa)$ splits only at κ and is therefore the union of exactly two $b_0, b_1 : j(\kappa) \to 2$ which first disagree at κ (a "Tuning Fork"). As $S^*(j(\kappa))$ must contain each $j(T)$, $T \in S(\kappa)$, b_0, b_1 are the only candidates for the desired $j(\kappa)$-Sacks generic! It can be shown that both b_0, b_1 are indeed $j(\kappa)$-Sacks generic.

Now we prove the Claim. We assume that $j : V \to M$ is an extender ultrapower witnessing the $H(\kappa^{++})$-strength of κ, so $M = \{j(f)(a) \mid f : H(\kappa) \to V, a \in H(\kappa^{++})\}$. We must show that if α does not equal κ then α fails to belong to $j(C)$ for some CUB C in κ. We may assume that α lies between κ and $j(\kappa)$; write $\alpha = j(f)(a)$ for some $f : H(\kappa) \to \kappa$, $a \in H(\kappa^{++})$. We take C to be $\{\beta < \kappa \mid \beta$ is a limit cardinal and $H(\beta)$ is closed under $f\}$, a CUB subset of κ. Then $j(C) = \{\beta < j(\kappa) \mid \beta$ is a limit cardinal of M and $H(\beta)^M$ is closed under $j(f)\}$. If $\beta > \kappa$ belongs to $j(C)$ then $j(f)(b) < \beta$ for all $b \in H(\kappa^{++})^M = H(\kappa^{++})$, so in particular $\kappa < \alpha = j(f)(a) < \beta$. Thus α does not belong to $j(C)$. This proves the Claim.

A similar result holds for Sacks(κ, κ^{++}) (joint work with Katie Thompson [17]). A condition is a function $p : \kappa^{++} \to \kappa$-Sacks which is trivial on all but κ many $i < \kappa^{++}$.

Prepare as before, iterating for $\kappa + 1$ steps, but with Sacks(α, α^{++}) at inaccessible stages $\alpha \leq \kappa$. As before we obtain an embedding

$$j' : V[S(<\kappa)] \to M[S^*(<j(\kappa))].$$

To extend j' further we want to find a generic $S^*(j(\kappa))$ for the Sacks$(j(\kappa), j(\kappa^{++}))$ of $M[S^*(<j(\kappa))]$ which contains $j'[S(\kappa)]$, where $S(\kappa)$ is the Sacks(κ, κ^{++})-generic, yielding

$$j^* : V[S(\leq \kappa)] \to M[S^*(< j(\kappa))][S^*(j(\kappa)].$$

Now what happens is this:

For $i < j(\kappa^{++})$ *in the range of* j, the intersection of the $j(p)(i)$ is a tuning fork $b^i_0, b^i_1 : : j(\kappa) \to 2$.

For $i < j(\kappa^{++})$ *not in the range of* j, the intersection of the $j(p)(i)$ is a single $b^i : : j(\kappa) \to 2$.

And if for $i < j(\kappa^{++})$ we take the b^i_0 for i in the range of j and the b^i for i not in the range of j then we obtain a Sacks($j(\kappa)$, $j(\kappa^{++})$)-generic. This generic contains $j'[S(\kappa)]$ by its definition (and is almost generated by it).

In conclusion: The fusion property for κ-Sacks is a good substitute for κ^+-distributivity, and therefore works better than κ-Cohen.

Some other applications of "fusion lifting" are:

(with Magidor [15]) Assume GCH, let κ be measurable and let α be any cardinal at most κ^{++}. Then there is a cofinality-preserving forcing extension in which there are exactly α normal measures on κ. If κ is $H(\kappa^{++})$-strong, then there is a cofinality-preserving forcing extension in which GCH fails at κ and there is a unique normal measure on κ.

This uses variants of κ-Sacks, tuning forks and nonstationary support iterations.

(with Dobrinen [9]) Assume GCH and let κ be $H(\lambda)$-strong where $\lambda > \kappa$ is weakly compact. Then there is a forcing extension in which κ is still measurable and the tree property holds at κ^{++}.

This extends the tuning fork method from a κ-Sacks product to a κ-Sacks iteration (of length λ).

(with Honzík [14]) (Special Case) Assume GCH and F is an Easton function of the form $F(\kappa)$ = the least λ such that $H(\lambda^+) \vDash \varphi(\kappa)$ for some formula φ. Then there is a cofinality-preserving forcing extension in which $2^\gamma = F(\gamma)$ for all regular γ and every κ which is $H(F(\kappa))$-strong in the ground model remains measurable.

This uses the tuning fork method and matrices of conditions to lift an embedding.

Forcings that preserve large cardinals: Cardinal characteristics

This is a new area; we consider three examples: $\mathfrak{d}(\kappa)$, CofSym(κ) and $\mathfrak{s}(\kappa)$.

Cummings and Shelah proved an Easton-type theorem for the function $\kappa \mapsto \mathfrak{d}(\kappa)$. In particular:

Theorem 5 *(Cummings-Shelah) Assume GCH and κ regular. Then in a cofinality-preserving extension, $\kappa^+ = \mathfrak{d}(\kappa) < 2^\kappa$.*

Their proof goes as follows: First apply Cohen(κ, κ^{++}) to make $2^\kappa = \kappa^{++}$. Then iterate κ-Hechler forcing for κ^+ steps, adding at each step a function $f : \kappa \to \kappa$ which eventually dominates all ground model functions. A condition in κ-Hechler is a pair (s, f) where

$s : |s| \to \kappa$, $|s| < \kappa$
$f : \kappa \to \kappa$

$(t, g) \leq (s, f)$ iff $t \supseteq s$, g dominates f, t dominates f on $|t| \setminus |s|$. This is κ-closed and κ^+-cc.

In the resulting model $\mathfrak{d}(\kappa) = \kappa^+$. Now we pose the basic

Question: Can one have $\mathfrak{d}(\kappa) < 2^\kappa$ for a measurable κ?

Assume GCH, κ is $H(\kappa^{++})$-strong and $j : V \to M$ witnesses this via an extender ultrapower. Our strategy is to prepare up to κ using Cohen(α, α^{++}) followed by an α^+ iteration of α-Hechler, and lift the embedding:

$$V[CH(\leq \kappa)] \to M[CH(< j(\kappa)) * CH(j(\kappa))].$$

But this doesn't work!

We already saw the problems with lifting for Cohen(κ, κ^{++}); but κ-Hechler presents even more serious difficulties: Consider

$$j^* : V[H(\leq \kappa)] \to M[H^*(< j(\kappa)) * H^*(j(\kappa))],$$

where the $H(\alpha), H^*(\alpha)$ are generic for α-Hechler forcing. Now we want the $j(\kappa)$-Hechler generic $H^*(j(\kappa))$ to extend the κ-Hechler generic $H(\kappa)$. Let $h^* : j(\kappa) \to j(\kappa)$ be the $j(\kappa)$-Hechler generic function associated with $H^*(j(\kappa))$ and $h : \kappa \to \kappa$ the κ-Hechler generic function associated with $H(\kappa)$. Then:

For any $f : \kappa \to \kappa$ in V, h dominates f beyond some $\alpha < \kappa$; so

For any $f : \kappa \to \kappa$ in V, h^* dominates $j(f)$ beyond (the same) ordinal $\alpha < \kappa$, and in particular $j(f)(\kappa) < h^*(\kappa)$.

But we have seen that the intersection of the $j(C)$, C club in κ is $\{\kappa\}$ and from this it follows that the $j(f)(\kappa)$ for $f : \kappa \to \kappa$ are cofinal in $j(\kappa)$. So $h^*(\kappa)$ cannot be defined!

However we have already solved this problem: We showed that κ remains measurable after iterating Sacks(α, α^{++}) for inaccessible $\alpha \leq \kappa$. This factors as

$$\text{(Iteration of Sacks}(\alpha, \alpha^{++})\text{ below }\kappa) * \text{Sacks}(\kappa, \kappa^{++}).$$

A forcing is κ^κ bounding iff every function $f : \kappa \to \kappa$ that it adds is dominated by such a function from the ground model. Any κ-cc forcing is κ^κ bounding, and fusion shows that Sacks(κ, κ^{++}) is also κ^κ bounding. It follows that the above iteration is κ^κ bounding and therefore over a model of GCH forces $\mathfrak{d}(\kappa) = \kappa^+ < 2^\kappa = \kappa^{++}$.

Remark. With enough supercompactness, it can be shown that the κ-Cohen with κ-Hechler strategy does work, and indeed one can get κ measurable with any reasonable values for $\mathfrak{d}(\kappa)$, $\mathfrak{b}(\kappa)$ and 2^κ, where $\mathfrak{b}(\kappa)$ is the bounding number at κ, i.e., the smallest size of a subset of ${}^\kappa\kappa$ which is not bounded in ${}^\kappa\kappa$ under the order of eventual domination.

The Cardinal Characteristic CofSym(κ)

Let κ be regular. Sym(κ) denotes the group of permutations of κ under composition. CofSym(κ) denotes the least λ such that Sym(κ) is the union of a strictly increasing λ-chain of subgroups. Macpherson and Neumann [22] showed that CofSym(κ) is greater than κ. Sharp and Thomas [27] showed that for any regular κ, one can force CofSym(κ) to be greater than κ^+.

Theorem 6 *(F-Zdomskyy* [18]*) Suppose that κ is $H(\kappa^{++})$-strong. Then in a forcing extension, κ is measurable and CofSym$(\kappa) = \kappa^{++}$.*

The Sharp-Thomas proof (based on a forcing of Mekler-Shelah [23]) does not appear to work; instead one uses an iteration of Miller(κ) (a version of Miller forcing at κ with continuous club-splitting) mixed with a variant of κ-Sacks forcing. It is another lifting argument using fusion.

Question. Is it consistent that CofSym$(\kappa) = \kappa^{+++}$ for a measurable κ?

The Cardinal Characteristic $\mathfrak{s}(\kappa)$

Fix κ regular. For x, y subsets of κ of size κ, *x splits y* iff both $y \setminus x$ and $y \cap x$ have size κ. $\mathfrak{s}(\kappa)$ is the least size of a splitting family of subsets of κ, i.e., a family sufficient to split every size κ subset of κ.

Facts (see [30]*).* For κ regular and uncountable: κ is inaccessible iff $\mathfrak{s}(\kappa) \geq \kappa$ and κ is weakly compact iff $\mathfrak{s}(\kappa) > \kappa$. Relative to a supercompact, it is consistent to have a measurable κ with $\mathfrak{s}(\kappa) = \kappa^{++}$.
(Zapletal) $\mathfrak{s}(\kappa) > \kappa^+$ for an uncountable regular κ requires an α of Mitchell order α^{++} (this is slightly weaker than $H(\alpha^{++})$-strong).

Question. Can one obtain a measurable κ with $\mathfrak{s}(\kappa) = \kappa^{++}$ from an α which is $H(\alpha^{++})$-strong?

Forcings that preserve large cardinals: L-like Universes

The axiom $V = L$ is very powerful in the sense that it resolves many problems in set theory. On the other hand, it is powerless for establishing consistency; the latter role is played by large cardinal axioms. It is therefore natural to ask: Can we have the advantages of both $V = L$ and large cardinals simultaneously?

There are two approaches to this question.

The Inner model programme: Show that a universe with large cardinals has an *inner model* which is L-like and has large cardinals.

The outer model programme: Show that a universe with large cardinals has an *outer model* which is L-like and has large cardinals.

The first approach uses fine structure theory and iterated ultrapowers; however the second approach is much easier, and uses only forcing.

We should first say what we mean by "L-like". Some examples of L-like properties are the following.

GCH
Definable Wellorders of the Universe
Jensen's $\diamond$, $\square$ and Morass Principles
Condensation Principles

We will examine to what extent these principles can be forced without damaging large cardinal properties. (Some references for this work are [1, 2, 3, 4, 5, 6, 7, 12, 13].)

Forcing GCH

We simply iterate α^+-Cohen for regular α with Easton support. ω_1-Cohen forces CH by collapsing $2^{\aleph_0}$ to ω_1. Then ω_2-Cohen forces GCH at ω_1 by collapsing 2^{ω_1} to ω_2, etc. Now let's show that all of the large cardinal properties we have introduced will be preserved by this forcing. Throughout our discussion below, $j : V \to M$ will denote an embedding witnessing a large cardinal property and κ will denote its critical point.

Preserving a superstrong: Want a lifting of $j : V \to M$ to

$j^* : V[G(< \kappa) * G[\kappa, \infty)] \to$
$M[G^*(< \kappa) * G^*[\kappa, j(\kappa)) * G^*[j(\kappa), \infty)]$.

The forcings P (to add G) and $P^* = j(P)$ (to add G^*) agree strictly below $j(\kappa)$ since $j : V \to M$ is superstrong; but they may take different limits at $j(\kappa)$:

$P^*(< j(\kappa)) = $ DirLim of $P^*(< \alpha)$, $\alpha < j(\kappa)$
$P(< j(\kappa)) = $ InvLim of $P(< \alpha)$, $\alpha < j(\kappa)$, if $j(\kappa)$ singular (can this happen?)

Fact. Suppose that κ is superstrong and let δ be the least ordinal of the form $j(\kappa)$ for some superstrong $j : V \to M$ with critical point κ. Then δ has cofinality κ^+.

So we do have to deal with a singular $j(\kappa)$. But it is easy to show:

$G(< j(\kappa)) \cap P^*(< j(\kappa))$ is generic over M for $P^*(< j(\kappa))$,

so we can simply take this to be $G^*(< j(\kappa))$.

Now we are done, as $P[\kappa, \infty)$ is κ^+-distributive and this implies that the image of $G[\kappa, \infty)$ generates a $P^*[j(\kappa), \infty)$-generic.

Preserving a Hyperstrong: We want a lifting of $j : V \to M$ to

$j^* : V[G(< \kappa) * G(\kappa) * G[\kappa^+, \infty)] \to$
$M[G^*(< \kappa) * G^*[\kappa, j(\kappa)) * G^*(j(\kappa)) * G^*[j(\kappa)^+, \infty)]$.

Now P and P^* agree up to $j(\kappa)$, so we would like to take $G^*(\leq j(\kappa))$ to be $G(\leq j(\kappa))$; we must however ensure that this contains $j[G(\leq \kappa)]$. We first lift j to $j' : V[G(< \kappa)] \to M[G^*(< j(\kappa))]$ and then observe that $j'[G(\kappa)]$ has a greatest lower bound in the forcing $P^*(j(\kappa))$. So we simply assume that $G(j(\kappa))$ was chosen below this greatest lower bound.

Finally in analogy to the superstrong case, the κ^{++}-distributivity of $P[\kappa^+, \infty)$ implies that the image of $G[\kappa^+, \infty)$ generates a $P^*[j(\kappa)^+, \infty)$-generic.

Preserving a 2*-superstrong:* We want a lifting of $j : V \to M$ to

$j^* : V[G(< \kappa) * G[\kappa, j(\kappa)) * G[j(\kappa), \infty)] \to$
$M[G^*(< \kappa) * G^*[\kappa, j^2(\kappa)) * G^*[j^2(\kappa), \infty)]$.

This time P^* and P agree strictly below $j^2(\kappa)$, P^* takes a direct limit at $j^2(\kappa)$ and P possibly takes an inverse limit there, as $j^2(\kappa)$ may be singular. This singularity can occur:

Fact. Suppose that κ is 2-superstrong and let δ be the least ordinal of the form $j^2(\kappa)$ for some 2-superstrong $j : V \to M$ with critical point κ. If $j : V \to M$ is a 2-superstrong embedding with $j^2(\kappa) = \delta$ then j is continuous at $j(\kappa)$ and therefore δ has cofinality $j(\kappa)$.

So as before we take $G^*(< j^2(\kappa))$ to be $G(< j^2(\kappa)) \cap P^*(< j^2(\kappa))$. We can ensure that $j[G(< j(\kappa))]$ is contained in $G(< j^2(\kappa))$, as the former has a greatest lower bound in the forcing $P(< j^2(\kappa))$.

And the $j(\kappa)^+$-distributivity of $P[j(\kappa), \infty)$ implies that the image of $G[j(\kappa), \infty)$ generates a $P^*[j^2(\kappa), \infty)$-generic.

Finally, for the ω-superstrong case we choose $G(< j^\omega(\kappa))$ to contain a condition forcing $j[G(< j^n(\kappa))] \subseteq G(< j^{n+1}(\kappa))$ for each n, and show:

Claim. $G(< j^\omega(\kappa)) \cap P^*(< j^\omega(\kappa))$ is $P^*(< j^\omega(\kappa))$-generic over M.

The proof of the Claim uses an argument regarding the "reduction" of dense sets. (For further details see [12].)

Forcing Definable Wellorders

We have:

Lemma 7 *(Asperó-F* [1, 2]*) Preserving a proper class of ω-superstrongs it is possible to force GCH together with a wellorder of V whose restriction to $H(\kappa^+)$ is definable over $H(\kappa^+)$ for uncountable regular κ, uniformly.*

Thus one gets a wellorder of $H(\aleph_{\omega+1})$ which is only definable over $H(\aleph_{\omega+2})$, not over $H(\aleph_{\omega+1})$, as one might hope. This leaves a nice open problem:

Question. With set-forcing, can one always add a definable wellorder of $H(\aleph_{\omega+1})$?

Note: One cannot expect to force a definable wellorder of $H(\omega_1)$; this is not possible if there is a proper class of Woodin cardinals, for example, as then Projective Determinacy holds in all set-generic extensions.

Another note: It is definitely not always possible to force a definable wellorder of $H(\lambda^+)$ for singular λ: This is contradicted by an elementary embedding from $L[H(\lambda^+)]$ to itself with critical point less than λ, using Kunen's proof that there is no nontrivial elementary embedding of V to itself.

Forcing $\diamondsuit$

In this case we iterate α-Cohen forcing for all regular α. It is easy to see that this forces $\diamondsuit_\alpha$ for all regular α and preserves cofinalities, assuming GCH.

Preserving a superstrong: We want to lift $j : V \to M$ to:

$j^* : V[G(< \kappa) * G(\kappa) * G(\kappa, j(\kappa)) * G[j(\kappa), \infty)] \to$
$M[G^*(< j(\kappa)) * G^*(j(\kappa)) * G^*[j(\kappa)^+, \infty)]$.

As before we can take $G^*(< j(\kappa))$ to be $G(< j(\kappa))$. The new concern is:

How do we choose $G^*(j(\kappa))$?

Note that we can't set $G^*(j(\kappa)) = G(j(\kappa))$ as $j(\kappa)$ is in general singular in V, so $G(j(\kappa))$ is not even defined!

The solution is to use a superstrong $j : V \to M$ with $j(\kappa)$ as small as possible (and therefore of cofinality κ^+):

Each relevant dense D in M is of the form $j(f)(a)$ for some $f : H(\kappa) \to H(\kappa^+)$, some $a \in H(j(\kappa))$.
Choose $\alpha_0 < \alpha_1 < \cdots$ cofinal in $j(\kappa)$ of length κ^+ and a list $f_0, f_1, \ldots$ of all relevant f's. Then for each $i < \kappa^+$ consider the collection

$S_i = \{D \mid D$ is dense and of the form $j(f_i)(a)$ for some $a \in H(\alpha_i^+)\}$.

Each S_i has size $< j(\kappa)$ and $P^*(j(\kappa))$ is $j(\kappa)$-distributive. Also M is κ-closed in V. So we can build a $P^*(j(\kappa))$-generic in κ^+ steps, hitting the dense sets in S_i at step i.

Preserving Hyperstrength: This is easier, as $P^*(j(\kappa))$ now equals $P(j(\kappa))$. One only needs to guarantee that the image of $G(\kappa) * G(\kappa^+)$ is contained in $G^*(j(\kappa)) * G^*(j(\kappa)^+)$, which is possible as this image has a greatest lower bound in the forcing $P^*(j(\kappa)) * P^*(j(\kappa)^+)$.

Preserving 2*-superstrength:* The new task here is to build $G^*(j^2(\kappa))$. As observed before, for a minimal $j^2(\kappa)$, j is continuous at $j(\kappa)$; from this it follows using the $j(\kappa)$-distributivity of $P(j(\kappa))$ that the image of $G(j(\kappa))$ will in fact generate the desired generic $G^*(j^2(\kappa))$.

Forcing $\square$

(Global) $\square$ asserts that one can assign CUB subsets C_α of ordertype $< \alpha$ to singular limit ordinals α which cohere: If $\bar{\alpha}$ is a limit point of C_α then $C_{\bar{\alpha}}$ is just an initial segment of C_α. Global $\square$ is the conjunction of two weaker properties:

$\square$ on the Singular Cardinals: This is $\square$ where C_α is only defined for singular *cardinals* α.

$\square_\kappa$ for all (uncountable cardinals) κ, where $\square_\kappa$ is $\square$ restricted to ordinals between κ and κ^+.

Forcing $\square$, preserving superstrength: This is very similar to forcing $\diamond$. At regular stage α force $\square$ below α in the natural way. The main problem is to build $C(j(\kappa))$, as $j(\kappa)$ can be singular. Again the trick is to choose $j : V \to M$ witnessing the superstrength of κ with smallest possible $j(\kappa)$; then $j(\kappa)$ will have cofinality κ^+, enabling a construction of $C(j(\kappa))$ in κ^+ steps.

But now something unexpected happens: Solovay (later improved by Jensen) showed that $\square$ contradicts large cardinals! A weakening of Jensen's result can be stated as follows:

Lemma 8 *(Jensen) If κ is hyperstrong then $\square_\kappa$ fails.*

Jensen's argument is essentially that if $\vec{C}$ witnesses $\square_\kappa$ and $j : V \to M$ witnesses hyperstrength, then there is a problem with the $\square_{j(\kappa)}$-sequence $j(\vec{C})$ in M at the ordinal $\alpha = \sup i[\kappa^+]$. In fact Jensen shows that $\square_\kappa$ fails for all κ which are *subcompact*, a property weaker than hyperstrength. κ is *subcompact* iff for any $A \subseteq H(\kappa^+)$ there are $\bar{\kappa} < \kappa$, $\bar{A} \subseteq H(\bar{\kappa})$ and an elementary embedding $i : (H(\bar{\kappa}^+), \bar{A}) \to (H(\kappa^+), A)$ with critical point $\bar{\kappa}$.

$\square$ on the Singular Cardinals is also contradicted by large cardinals, but now the large cardinal strength is greater. $j : V \to M$ is *inaccessibly hyperstrong* iff $H(\lambda) \subseteq M$ for some inaccessible λ greater than κ; we say *almost inaccessibly hyperstrong* if λ is only required to be inaccessible in M.

Theorem 9 *(Cummings-F* [6]*) (a) If κ is inaccessibly hyperstrong then $\square$ fails on the singular cardinals below κ.*
(b) One can force $\square$ on the singular cardinals preserving almost inaccessible hyperstrength.

Forcing Morasses

The only work so far on forcing morasses in the presence of large cardinals is for the Gap 1 case. I showed that one can do this for a single ω-superstrong ([12]) and with A. Brooke-Taylor ([4]) for all ω-superstrongs simultaneously. We also force *universal* morasses, which by an observation of Donder implies the consistency of "tree-like continuous scales" at very large cardinals.

Forcing Condensation

There are different formulations of Condensation (see [13]). Club-Condensation, which holds in L, is very strong and contradicts the existence of an ω_1-Erdős cardinal. Stationary Condensation can be forced preserving ω-superstrongs. Better is Strong Condensation, which holds in the known core models and can also be forced preserving ω-superstrength. But the best of all is Strong Condensation with Acceptability, which better captures the condensation properties of core models. Peter Holy and I show that one can force this preserving ω-superstrongs; this is especially important when combined with some work of Neeman-Schimmerling:

(Neeman-Schimmerling [25]) Given a Σ^2_1 indescribable 1-Gap the Proper Forcing Axiom for c^+ linked forcings holds in a proper forcing extension.

The above hypothesis is a bit above a subcompact in strength.

(Neeman [24]) The previous result is optimal if there is a "sufficiently L-like" model with a Σ^2_1 indescribable 1-Gap.

(F-Holy [13]) One can force a "sufficiently L-like" model with a Σ^2_1 indescribable 1-Gap. Therefore:

(F-Holy [13]) It is consistent with the existence of a proper class of subcompacts that the Proper Forcing Axiom for c^+ linked forcings fails in all proper set-forcing extensions.

This gives a "quasi lower bound" on the consistency strength of PFA(c^+ linked).

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