Communications in Mathematics

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Communications in Mathematics, Vol. 22 (2014), No. 1, 57--69

Persistent URL: `http://dml.cz/dmlcz/143906`

VOLUME
22/2014
No. 1

ISSN 1804-1388
(Print)
ISSN 2336-1298
(Online)

COMMUNICATIONS IN MATHEMATICS

Available online at
http://cm.osu.cz

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Communications in Mathematics 22 (2014) 57–69

Existence of entropy solutions for degenerate quasilinear elliptic equations in L^1

Albo Carlos Cavalheiro

Abstract. In this article, we prove the existence of entropy solutions for the Dirichlet problem

$$(P)\begin{cases} -\operatorname{div}[\omega(x)\mathcal{A}(x,u,\nabla u)] = f(x) - \operatorname{div}(G), & \text{in } \Omega \\ u(x) = 0, & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 2$, $f \in L^1(\Omega)$ and $G/\omega \in [L^{p'}(\Omega,\omega)]^N$.

1 Introduction

The main purpose of this article (see Theorem 2) is to establish the existence of entropy solutions for the Dirichlet problem

$$(P)\begin{cases} -\operatorname{div}[\omega(x)\mathcal{A}(x,u,\nabla u)] = f(x) - \operatorname{div}(G) & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, ω is a weight function (i.e., a locally integrable function on $\mathbb{R}^N$ such that $0 < \omega(x) < \infty$ a.e. $x \in \mathbb{R}^N$), $f \in L^1(\Omega)$, $G = (g_1, \dots, g_N)$ with $G/\omega \in [L^{p'}(\Omega,\omega)]^N$, and the function

$$\mathcal{A}\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

satisfies the following conditions:

(H1) $x \mapsto \mathcal{A}(x,s,\xi)$ is measurable on Ω for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and $(s,\xi) \mapsto \mathcal{A}(x,s,\xi)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for almost all $x \in \Omega$.

(H2) $\langle \mathcal{A}(x,s,\xi_1) - \mathcal{A}(x,s,\xi_2), \xi_1 - \xi_2 \rangle > 0$, whenever $\xi_1, \xi_2 \in \mathbb{R}^N$, $\xi_1 \neq \xi_2$ (where $\langle \cdot,\cdot \rangle$ denotes the usual inner product in $\mathbb{R}^N$).

2010 MSC: 35J70, 35J60
Key words: Degenerate elliptic equations, entropy solutions, weighted Sobolev spaces

(H3) $\langle \mathcal{A}(x,s,\xi),\xi\rangle \geq \lambda\,|\xi|^p$, with $1<p<\infty$, and $\lambda>0$.

(H4) $|\mathcal{A}(x,s,\xi)| \leq K(x) + h_1(x)\,|s|^{p/p'} + h_2(x)\,|\xi|^{p/p'}$, where K, h_1 and h_2 are positive functions, with $h_1 \in L^\infty(\Omega)$, $h_2 \in L^\infty(\Omega)$ and $K \in L^{p'}(\Omega,\omega)$ (where $1/p+1/p'=1$).

If $f/\omega \in L^{p'}(\Omega,\omega)$ (with $1<p<\infty$), the problem (P) has been studied in [4], and in this case the problem (P) has a solution $u \in W^{1,p}(\Omega,\omega)$. However, since $L^1(\Omega)$ is not a subspace of $W^{-1,p'}(\Omega,\omega)$ so when we want to consider $f \in L^1(\Omega)$ a different theory is needed.

In [1], a new concept of solution has been introduced for the elliptic equation

$$\begin{cases} -\operatorname{div}[a(x,\nabla u)] = f(x) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(when $f \in L^1(\Omega)$) namely entropy solution. In [3] the author studied the degenerate elliptic equation $Lu=f$, where L is a degenerate elliptic operator in divergence form, i.e.,

$$Lu = -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u),$$

and $f \in L^1(\Omega)$. Note that, in the proof of our main result, many ideas have been adapted from [1] and [3].

For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [5], [6], [8], [9] and [13]).

A class of weights, which is particularly well understood, is the class of A_p weights that was introduced by B. Muckenhoupt in the early 1970s (see [11]).

We propose to solve the problem (P) by approximation with variational solutions: we take $f_n \in C_0^\infty(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$, $G_n/\omega \in [L^{p'}(\Omega,\omega)]^N$ such that $G_n/\omega \to G/\omega$ in $[L^{p'}(\Omega,\omega)]^N$, we find a solution $u_n \in W_0^{1,p}(\Omega,\omega)$ for the problem with right-hand side f_n and G_n, and we will try to pass to the limit as $n\to\infty$.

2 Definitions and basic results

By a weight we shall mean a locally integrable function ω on $\mathbb{R}^N$ such that $0<\omega(x)<\infty$ for a.e. $x\in\mathbb{R}^N$. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^N$ through integration. This measure will be denoted by μ. Thus, $\mu(E)=\displaystyle\int_E \omega(x)\,\mathrm{d}x$ for measurable sets $E\subset\mathbb{R}^N$.

Definition 1. Let $1\leq p<\infty$. A weight ω is said to be an A_p-weight, if there is a positive constant $C=C(p,\omega)$ such that for every ball $B\subset\mathbb{R}^N$

$$\left(\frac{1}{|B|}\int_B \omega(x)\,\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)\,\mathrm{d}x\right)^{p-1} \leq C \quad \text{if } p>1,$$

$$\left(\frac{1}{|B|}\int_B \omega(x)\,\mathrm{d}x\right)\left(\operatorname*{ess\,sup}_{x\in B}\frac{1}{\omega(x)}\right) \leq C \quad \text{if } p=1,$$

where $|\cdot|$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$.

If $1 < q \leq p$, then $A_q \subset A_p$ (see [8], [9] or [14] for more information about A_p-weights). As an example of an A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^N$, is in A_p if and only if, $-N < \alpha < N(p-1)$ (see [12], Chapter IX, Corollary 4.4). If $\varphi \in BMO(\mathbb{R}^N)$ then $\omega(x) = \mathrm{e}^{\alpha\,\varphi(x)} \in A_2$ for some $\alpha > 0$ (see [12]).

Remark 1. If $\omega \in A_p$, $1 < p < \infty$, then

$$\left(\frac{|E|}{|B|}\right)^p \leq C\,\frac{\mu(E)}{\mu(B)}$$

for all measurable subsets E of B (see *15.5 strong doubling property* in [9]). Therefore if $\mu(E) = 0$ then $|E| = 0$. Thus, if $\{u_n\}$ is a sequence of functions defined in B and $u_n \to u$ μ-a.e. then $u_n \to u$ a.e..

Definition 2. Let ω be a weight. We shall denote by $L^p(\Omega,\omega)$ $(1 \leq p < \infty)$ the Banach space of all measurable functions f defined in Ω for which

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)\,\mathrm{d}x\right)^{1/p} < \infty\,.$$

We denote $[L^{p'}(\Omega,\omega)]^N = L^{p'}(\Omega,\omega) \times \cdots \times L^{p'}(\Omega,\omega)$.

Remark 2. If $\omega \in A_p$, $1 < p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(\Omega,\omega) \subset L^1_{\mathrm{loc}}(\Omega)$ (see [14], Remark 1.2.4). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 3. Let $\Omega \subset \mathbb{R}^N$ a bounded open set, $1 < p < \infty$, k a nonnegative integer and $\omega \in A_p$. We shall denote by $W^{k,p}(\Omega,\omega)$, the weighted Sobolev spaces, the set of all functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$, $1 \leq |\alpha| \leq k$. The norm in the space $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x)\,\mathrm{d}x + \sum_{1\leq|\alpha|\leq k} \int_\Omega |D^\alpha u(x)|^p \omega(x)\,\mathrm{d}x\right)^{1/p}. \tag{1}$$

We also define the space $W^{k,p}_0(\Omega,\omega)$ as the closure of $C^\infty_0(\Omega)$ with respect to the norm

$$\|u\|_{W^{k,p}_0(\Omega,\omega)} = \left(\sum_{1\leq|\alpha|\leq k} \int_\Omega |D^\alpha u(x)|^p \omega(x)\,\mathrm{d}x\right)^{1/p}.$$

The dual space of $W^{1,p}_0(\Omega,\omega)$ is the space $[W^{1,p}_0(\Omega,\omega)]^* = W^{-1,p'}(\Omega,\omega)$,

$$W^{-1,p'}(\Omega,\omega) = \left\{T = f - \mathrm{div}(G) : G = (g_1,\ldots,g_N), \frac{f}{\omega}, \frac{g_j}{\omega} \in L^{p'}(\Omega,\omega)\right\}.$$

It is evident that a weight function ω which satisfies $0 < C_1 \leq \omega(x) \leq C_2$, for a.e. $x \in \Omega$, gives nothing new (the space $\mathrm{W}^{k,p}(\Omega,\omega)$ is then identical with the classical Sobolev space $\mathrm{W}^{k,p}(\Omega)$). Consequently, we study all such weight function ω that either vanish in $\Omega \cup \partial\Omega$ or increase to infinity (or both).

We need the following basic result.

Theorem 1. *(The weighted Sobolev inequality) Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let ω be an A_p-weight, $1 < p < \infty$. Then there exist positive constants C_Ω and δ such that for all $f \in C_0^\infty(\Omega)$ and $1 \le \eta \le N/(N-1) + \delta$*

$$\|f\|_{L^{\eta p}(\Omega,\omega)} \le C_\Omega \|\nabla f\|_{L^p(\Omega,\omega)}. \tag{2}$$

Proof. See [6], Theorem 1.3. □

Definition 4. We say that $u \in \mathcal{T}_0^{1,p}(\Omega,\omega)$ if $T_k(u) \in W_0^{1,p}(\Omega,\omega)$, for all $k > 0$, where the function $T_k\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \le k \\ k \operatorname{sign}(s), & \text{if } |s| > k. \end{cases}$$

Remark 3. (i) Note that for given $h > 0$ and $k > 0$ we have

$$T_h(u - T_k(u)) = \begin{cases} 0, & \text{if } |u| \le k \\ (|u| - k) \operatorname{sign}(u), & \text{if } k < |u| \le k + h \\ h \operatorname{sign}(u), & \text{if } |u| > k + h. \end{cases}$$

And if $\alpha \in \mathbb{R}$, $\alpha \ne 0$, we have $T_k(\alpha u) = \alpha\, T_{k/|\alpha|}(u)$.

(ii) If $u \in W^{1,1}_{\text{loc}}(\Omega,\omega)$ then we have

$$\nabla T_k(u) = \chi_{\{|u|<k\}} \nabla u$$

where χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{R}^N$.

Definition 5. Let $f \in L^1(\Omega)$, $G/\omega \in [L^{p'}(\Omega,\omega)]^N$ and $u \in \mathcal{T}_0^{1,p}(\Omega,\omega)$. We say that u is an entropy solution to problem (P) if

$$\int_\Omega \big\langle \mathcal{A}(x,u,\nabla u), \nabla T_k(u-\varphi)\big\rangle\, \omega \,\mathrm{d}x = \int_\Omega f T_k(u-\varphi)\,\mathrm{d}x + \int_\Omega \big\langle G, \nabla T_k(u-\varphi)\big\rangle\,\mathrm{d}x \tag{3}$$

for all $k > 0$ and all $\varphi \in W_0^{1,p}(\Omega,\omega) \cap L^\infty(\Omega)$.

We recall that the gradient of u which appears in (3) is defined as in Remark 2.8 of [3], that is to say that $\nabla u = \nabla T_k(u)$ on the set where $|u| < k$.

Remark 4. Note that if $u_1, u_2 \in W_0^{1,p}(\Omega,\omega)$ then

$$\varphi = T_k(u_1 + u_2) \in W_0^{1,p}(\Omega,\omega) \cap L^\infty(\Omega)$$

and we have

$$\nabla\varphi = \nabla T_k(u_1 + u_2) = \nabla(u_1 + u_2)\, \chi_{\{|u_1+u_2| \le k\}}.$$

Definition 6. Let $0 < p < \infty$ and let ω be a weight function. We define the weighted Marcinkiewicz space $\mathcal{M}^p(\Omega,\omega)$ as the set of all measurable functions $f\colon \Omega \to \mathbb{R}$ such that the function

$$\Gamma_f(k) = \mu(\{x \in \Omega : |f(x)| > k\}),\ k > 0,$$

satisfies an estimate of the form $\Gamma_f(k) \le Ck^{-p}$, $0 < C < \infty$.

Remark 5. If $1 \le q < p$ and $\Omega \subset \mathbb{R}^N$ is a bounded set, we have that

$$L^p(\Omega,\omega) \subset \mathcal{M}^p(\Omega,\omega) \text{ and } \mathcal{M}^p(\Omega,\omega) \subset L^q(\Omega,\omega)$$

(the proof follows the lines of Theorem 2.18.8 in [10]).

Lemma 1. *Let $u \in \mathcal{T}_0^{1,p}(\Omega,\omega)$ and $\omega \in A_p$, $1 < p < \infty$, be such that*

$$\frac{1}{k}\int_{\{|u|<k\}} |\nabla u|^p \omega \,\mathrm{d}x \le M\,, \tag{4}$$

for every $k > 0$. Then $u \in \mathcal{M}^{p_1}(\Omega,\omega)$, where $p_1 = \eta\,(p-1)$ (where η is the constant in Theorem 1). More precisely, there exists $C > 0$ such that $\Gamma_u(k) \le CM^{\eta}k^{-p_1}$.

Proof. See Lemma 3.3 in [3]. □

Lemma 2. *Let $u \in \mathcal{T}_0^{1,p}(\Omega,\omega)$, where $\omega \in A_p$, $1 < p < \infty$, be such that*

$$\frac{1}{k}\int_{\{|u|<k\}} |\nabla u|^p \omega \,\mathrm{d}x \le M\,,$$

for every $k > 0$. Then $|\nabla u| \in \mathcal{M}^{p_2}(\Omega,\omega)$, where $p_2 = p\,p_1/(p_1+1)$ (with η as in Lemma 2 and $p_1 = \eta(p-1)$). More precisely, there exists $C > 0$ such that

$$\Gamma_k(|\nabla u|) \le CM^{(p_1+\eta)/(p_1+1)}k^{-p_2}.$$

Proof. See Lemma 3.4 in [3]. □

3 Main Result

In this section, we prove the main result of this paper. We need the following result.

Lemma 3. *Let $\omega \in A_p$, $1 < p < \infty$ and a sequence $\{u_n\}$, $u_n \in W_0^{1,p}(\Omega,\omega)$ satisfies*

(i) $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega,\omega)$ and μ-a.e. in Ω;

(ii) $\displaystyle\int_\Omega \langle \mathcal{A}(x,u_n,\nabla u_n) - \mathcal{A}(x,u_n,\nabla u), \nabla(u_n - u)\rangle\,\omega\,\mathrm{d}x \to 0$ with $n \to \infty$.

Then $u_n \to u$ in $W_0^{1,p}(\Omega,\omega)$.

Proof. The proof of this lemma follows the lines of Lemma 5 in [2]. □

Theorem 2. *Let $\omega \in A_p$, $1 < p < \infty$, and $\mathcal{A}(x,s,\xi)$ satisfies the conditions (H1), (H2), (H3) and (H4). Then, there exists an entropy solution u of problem (P). Moreover, $u \in \mathcal{M}^{p_1}(\Omega,\omega)$ and $|\nabla u| \in \mathcal{M}^{p_2}(\Omega,\omega)$, with $p_1 = \eta\,(p-1)$ and $p_2 = p_1\,p/(p_1+1)$ (where η is the constant in Theorem 1).*

Proof. Considering a sequence $\{f_n\}$, $f_n \in C_0^\infty(\Omega)$, which

$$f_n \to f \text{ in } L^1(\Omega) \text{ and } \|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}\,,$$

and a sequence $\{G_n\}$, with $G_n/\omega \in [L^{p'}(\Omega,\omega)]^N$ such that

$$\frac{G_n}{\omega} \to \frac{G}{\omega} \text{ in } [L^{p'}(\Omega,\omega)]^N \text{ and } \left\|\frac{|G_n|}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \leq \left\|\frac{|G|}{\omega}\right\|_{L^{p'}(\Omega,\omega)}.$$

For each n, there exists a solution $u_n \in W_0^{1,p}(\Omega,\omega)$ of the Dirichlet problem

$$(P_n)\begin{cases} -\operatorname{div}[\omega(x)\mathcal{A}(x,u_n,\nabla u_n)] = f_n(x) - \operatorname{div}(G_n), & \text{in } \Omega \\ u_n(x) = 0, & \text{on } \partial\Omega \end{cases}$$

(by Theorem 1.1 in [4]) that is,

$$\int_\Omega \omega\,\langle \mathcal{A}(x,u_n,\nabla u_n), \nabla\varphi\rangle\,\mathrm{d}x = \int_\Omega f_n\varphi\,\mathrm{d}x + \int_\Omega \langle G_n, \nabla\varphi\rangle\,\mathrm{d}x\,, \tag{5}$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega)$. For $\varphi = T_k(u_n)$ we obtain in (5) that

$$\int_\Omega \omega\,\langle \mathcal{A}(x,u_n,\nabla u_n), \nabla T_k(u_n)\rangle\,\mathrm{d}x = \int_\Omega f_n T_k(u_n)\,\mathrm{d}x + \int_\Omega \langle G_n, \nabla T_k(u_n)\rangle\,\mathrm{d}x. \tag{6}$$

Using (H3) and Remark 3 (ii) we have,

$$\begin{aligned}\int_\Omega \omega\,\langle \mathcal{A}(x,u_n,\nabla u_n), \nabla T_k(u_n)\rangle\,\mathrm{d}x &= \int_\Omega \omega\,\langle \mathcal{A}(x,u_n,\nabla T_k(u_n)), \nabla T_k(u_n)\rangle\,\mathrm{d}x \\ &\geq \lambda \int_\Omega |\nabla T_k(u_n)|^p \omega\,\mathrm{d}x.\end{aligned}$$

We also have

$$\left|\int_\Omega f_n T_k(u_n)\,\mathrm{d}x\right| \leq \int_\Omega |f_n|\,|T_k(u_n)|\,\mathrm{d}x \leq k\|f_n\|_{L^1(\Omega)} \leq k\|f\|_{L^1(\Omega)}\,,$$

and using Young's inequality there exists a constant $C_1 > 0$ such that

$$\begin{aligned}\left|\int_\Omega \langle G_n, \nabla T_k(u_n)\rangle\,\mathrm{d}x\right| &\leq \int_\Omega \left|\frac{G_n}{\omega}\right| |\nabla T_k(u_n)|\,\omega\,\mathrm{d}x \\ &\leq \left(\int_\Omega \left|\frac{G_n}{\omega}\right|^{p'} \omega\,\mathrm{d}x\right)^{1/p'} \left(\int_\Omega |\nabla T_k(u_n)|^p \omega\,\mathrm{d}x\right)^{1/p} \\ &\leq \frac{\lambda}{2}\int_\Omega |\nabla T_k(u_n)|^p\omega\,\mathrm{d}x + C_1\int_\Omega \left|\frac{G_n}{\omega}\right|^{p'} \omega\,\mathrm{d}x \\ &\leq \frac{\lambda}{2}\int_\Omega |\nabla T_k(u_n)|^p\omega\,\mathrm{d}x + C_1\int_\Omega \left|\frac{G}{\omega}\right|^{p'} \omega\,\mathrm{d}x\,.\end{aligned}$$

Hence in (6) we obtain

$$\lambda \int_\Omega |\nabla T_k(u_n)|^p \omega \,\mathrm{d}x \le k\,\|f\|_{L^1(\Omega)} + \frac{\lambda}{2}\int_\Omega |\nabla T_k(u_n)|^p \omega \,\mathrm{d}x + C_1 \int_\Omega \left|\frac{G}{\omega}\right|^{p'} \omega \,\mathrm{d}x\,,$$

and

$$\begin{aligned}\int_\Omega |\nabla T_k(u_n)|^p \omega \,\mathrm{d}x &\le \frac{k}{\lambda}\Big(\|f\|_{L^1(\Omega)} + C_1 \Big\|\frac{G}{\omega}\Big\|^{p'}_{L^{p'}(\Omega,\omega)}\Big)\\ &= C_2\, k\,, \quad \text{for all } k>0\,. \end{aligned} \tag{7}$$

By Lemma 1 and Lemma 2, we have that the sequence $\{u_n\}$ is bounded in $\mathcal{M}^{p_1}(\Omega,\omega)$ (with $p_1 = \eta\,(p-1)$ and $\{|\nabla u_n|\}$ is bounded in $\mathcal{M}^{p_2}(\Omega,\omega)$ (with $p_2 = p_1\,p/(p_1+1)$). Moreover, $\{u_n\}$ is a Cauchy sequence in μ-measure. Consequently, there exists a function u and a subsequence, that we will still denote by $\{u_n\}$, such that

$$u_n \to u \quad \mu\text{-a.e. in } \Omega, \tag{8}$$

and $u_n \to u$ a.e. in Ω (by Remark 1).

Using (7) and (8), we have

$$\begin{aligned} &T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W^{1,p}_0(\Omega,\omega),\\ &T_k(u_n) \to T_k(u) \quad \text{strongly in } L^p(\Omega,\omega) \text{ and } \mu\text{-a.e. in } \Omega, \end{aligned} \tag{9}$$

for all $k>0$. Hence $T_k(u) \in W^{1,p}_0(\Omega,\omega)$.

Furthermore, by the weak lower semicontinuity of the norm $W^{1,p}_0(\Omega,\omega)$, we have that (7) still holds for u, that is,

$$\int_\Omega |\nabla T_k(u)|^p \omega \,\mathrm{d}x \le C_2\, k\,.$$

Applying Lemma 1 and Lemma 2, we have that $u \in \mathcal{M}^{p_1}(\Omega,\omega)$ (with $p_1 = \eta(p-1)$) and $|\nabla u| \in \mathcal{M}^{p_2}(\Omega,\omega)$ (with $p_2 = p_1\,p/(p_1+1)$).

- We need to show that $T_k(u_n) \to T_k(u)$ strongly in $W^{1,p}_0(\Omega,\omega)$, for all $k>0$.

Let $h>k$ and applying (5) with function

$$\varphi_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\,,$$

we get

$$\int_\Omega \omega\,\langle \mathcal{A}(x,u_n,\nabla u_n), \nabla\varphi_n\rangle \,\mathrm{d}x = \int_\Omega f_n \varphi_n \,\mathrm{d}x + \int_\Omega \langle G_n, \nabla\varphi_n\rangle \,\mathrm{d}x\,. \tag{10}$$

If we set $M = 4k+h$, we have $\nabla\varphi_n = 0$ for $|u_n| > M$. Hence, since condition (H3) implies that $\mathcal{A}(x,s,0)=0$, we can write

$$\int_\Omega \omega\,\langle \mathcal{A}(x,T_M(u_n),\nabla T_M(u_n)), \nabla\varphi_n\rangle \,\mathrm{d}x = \int_\Omega f_n \varphi_n \,\mathrm{d}x + \int_\Omega \langle G_n, \nabla\varphi_n\rangle \,\mathrm{d}x\,. \tag{11}$$

In the left-hand side of (11), we have

$$\begin{aligned}
&\int_\Omega \omega\big\langle \mathcal{A}(x,T_M(u_n),\nabla T_M(u_n)),\nabla T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\big\rangle\,\mathrm{d}x\\
&=\int_{\{|u_n|\le k\}} \omega\big\langle \mathcal{A}(x,T_M(u_n),\nabla T_M(u_n)),\nabla T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\big\rangle\,\mathrm{d}x\\
&+\int_{\{|u_n|> k\}} \omega\big\langle \mathcal{A}(x,T_M(u_n),\nabla T_M(u_n)),\nabla T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\big\rangle\,\mathrm{d}x
\end{aligned} \tag{12}$$

(a) If $|u_n|\le k$. Since $h>k$, if $|u_n|\le k<h$, then $T_h(u_n)=T_k(u_n)=u_n$. Hence,

$$u_n-T_h(u_n)+T_k(u_n)-T_k(u)=u_n-T_k(u)\,.$$

We also have that $|u_n-u|\le 2k$. Then, since $\nabla T_M(u_n)=\nabla T_k(u_n)$ (because $|u_n|\le k<M$),

$$\begin{aligned}
&\int_{\{|u_n|\le k\}} \omega\big\langle \mathcal{A}(x,T_M(u_n),\nabla T_M(u_n)),\nabla T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\big\rangle\,\mathrm{d}x\\
&\qquad=\int_{\{|u_n|\le k\}} \omega\big\langle \mathcal{A}(x,T_k(u_n),\nabla T_k(u_n)),\nabla (T_k(u_n)-T_k(u))\big\rangle\,\mathrm{d}x\\
&\qquad=\int_{\Omega} \omega\big\langle \mathcal{A}(x,T_k(u_n),\nabla T_k(u_n)),\nabla (T_k(u_n)-T_k(u))\big\rangle\,\mathrm{d}x\,.
\end{aligned}$$

(b) If $|u_n|>k$. Since u_n, $T_k(u_n)$ and $T_k(u)$ are in $W_0^{1,p}(\Omega,\omega)$, if

$$|u_n-T_h(u_n)+T_k(u_n)-T_k(u)|\le 2k\,,$$

we obtain

$$\begin{aligned}
\nabla T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u)) &= \nabla(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\\
&=\nabla u_n-\nabla T_h(u_n)+\nabla T_k(u_n)-\nabla T_k(u)\\
&=\nabla u_n-\nabla T_h(u_n)-\nabla T_k(u)
\end{aligned}$$

(because $\nabla T_k(u_n)=0$ if $|u_n|>k$). There are two possible cases as follows:

(i) If $k<|u_n|<h$, we have $\nabla T_h(u_n)=\nabla u_n$. Then

$$\nabla T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))=-\nabla T_k(u).$$

(ii) If $h<|u_n|\le M$, we have that $\nabla T_h(u_n)=0$. Then

$$\nabla T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))=\nabla u_n-\nabla T_k(u)=\nabla T_M(u_n)-\nabla T_k(u).$$

Since $\langle \mathcal{A}(x,s,\xi),\xi\rangle\ge\lambda|\xi|^p\ge 0$, in both cases we obtain

$$\begin{aligned}
\big\langle \mathcal{A}(x,T_M(u_n),\nabla T_M(u_n)),\nabla T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\big\rangle&\\
\ge -\big\langle \mathcal{A}(x,T_M(u_n),\nabla T_M(u_n),\nabla T_k(u)\big\rangle&\\
\ge -\big|\mathcal{A}(x,T_M(x),\nabla T_M(x))\big|\big|\nabla T_k(u)\big|.&
\end{aligned}$$

Therefore we obtain in (12)

$$\begin{aligned}
&\int_\Omega \omega\big\langle \mathcal{A}(x,T_M(u_n),\nabla T_M(u_n)),\nabla T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\big\rangle\,\mathrm{d}x\\
&=\int_{\{|u_n|\le k\}} \omega\big\langle \mathcal{A}(x,T_M(u_n),\nabla T_M(u_n)),\nabla T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\big\rangle\,\mathrm{d}x\\
&\quad+\int_{\{|u_n|>k\}} \omega\big\langle \mathcal{A}(x,T_M(u_n),\nabla T_M(u_n)),\nabla T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\big\rangle\,\mathrm{d}x\\
&\ge\int_\Omega \omega\big\langle \mathcal{A}(x,T_k(u_n),\nabla T_k(u_n)),\nabla(T_k(u_n)-T_k(u))\big\rangle\,\mathrm{d}x\\
&\quad-\int_{\{|u_n|>k\}} \omega\big|\mathcal{A}\big(x,T_M(u_n),\nabla T_M(u_n)\big)\big|\big|\nabla T_k(u)\big|\,\mathrm{d}x\,.
\end{aligned}$$

Hence, in (11) we obtain

$$\begin{aligned}
&\int_\Omega \omega\big\langle \mathcal{A}(x,T_k(u_n),\nabla T_k(u_n))-\mathcal{A}(x,T_k(u_n),\nabla T_k(u)),\nabla(T_k(u_n)-T_k(u))\big\rangle\,\mathrm{d}x\\
&\le\int_{\{|u_n|>k\}} \omega|\mathcal{A}(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|\,\mathrm{d}x\\
&\quad+\int_\Omega f_nT_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\,\mathrm{d}x\\
&\quad+\int_\Omega \big\langle G_n,\nabla T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\big\rangle\,\mathrm{d}x\\
&\quad-\int_\Omega \omega\big\langle \mathcal{A}(x,T_k(u_n),\nabla T_k(u)),\nabla(T_k(u_n)-T_k(u))\big\rangle\,\mathrm{d}x\,. \qquad (13)
\end{aligned}$$

Considering the test function $\psi_n=T_{2k}(u_n-T_h(u_n))$ in (5), we have

$$\int_\Omega \omega\,\langle \mathcal{A}(x,u_n,\nabla u_n),\nabla\psi_n\rangle\,\mathrm{d}x=\int_\Omega f_n\psi_n\,\mathrm{d}x+\int_\Omega\langle G_n,\nabla\psi_n\rangle\,\mathrm{d}x\,,$$

and by (7) we obtain

$$\int_\Omega |\nabla T_{2k}(u_n-T_h(u_n))|^p\omega\,\mathrm{d}x\le C_2(2k+1),\ \text{ for all } k\ge 1\,.$$

Now using that $T_{2k}(u_n-T_h(u_n))\rightharpoonup T_{2k}(u-T_h(u))$ weakly in $W_0^{1,p}(\Omega,\omega)$ (by (9) and Remark 3 (i)), we have

$$\int_\Omega |\nabla T_{2k}(u-T_h(u))|^p\omega\,\mathrm{d}x\le C_2(2k+1)\,. \qquad (14)$$

We have (by Remark 3 (i) and (ii) and (14))

$$\begin{aligned}
\int_\Omega |G|\big|\nabla T_{2k}(u-T_h(u))\big|\,\mathrm{d}x &= \int_{\{h<|u|<2k+h\}} |G||\nabla u|\,\mathrm{d}x \\
&\leq \bigg(\int_{\{|u|\geq h\}} |G/\omega|^{p'}\omega\,\mathrm{d}x\bigg)^{1/p'} \bigg(\int_{\{h<|u|<2k+h\}} |\nabla u|^p\omega\,\mathrm{d}x\bigg)^{1/p} \\
&= \bigg(\int_{\{|u|\geq h\}} |G/\omega|^{p'}\omega\,\mathrm{d}x\bigg)^{1/p'} \bigg(\int_\Omega |\nabla T_{2k}(u-T_h(u))|^p\omega\,\mathrm{d}x \\
&= C_3\bigg(\int_{\{|u|\geq h\}} |G/\omega|^{p'}\omega\,\mathrm{d}x\bigg)^{1/p'},
\end{aligned}$$

where C_3 depends on k but not on h. Therefore we have

$$\lim_{h\to\infty}\int_\Omega \big\langle G, \nabla T_{2k}(u-T_h(u))\big\rangle\,\mathrm{d}x = 0\,.$$

We also have (by Theorem 1)

$$\begin{aligned}
\int_\Omega |T_{2k}(u-T_h(u))|^p\omega\,\mathrm{d}x &\leq C_\Omega\int_\Omega |\nabla T_{2k}(u-T_h(u))|^p\omega\,\mathrm{d}x \\
&\leq C_\Omega\, C_2(2k+1).
\end{aligned}$$

Moreover, by Lebesgue's theorem, we obtain

$$\lim_{h\to\infty}\int_\Omega f\,T_{2k}(u-T_h(u))\,\mathrm{d}x = 0\,.$$

We can fix a positive real number h_ε sufficiently large to have

$$\int_\Omega fT_{2k}(u-T_{h_\varepsilon}(u))\,\mathrm{d}x + \int_\Omega \big\langle G, \nabla T_{2k}(u-T_{h_\varepsilon}(u))\big\rangle\,\mathrm{d}x \leq \varepsilon. \tag{15}$$

Considering $h = h_\varepsilon$ in (13) (and $M = M_\varepsilon = 4k + h_\varepsilon$), by (H4) and (7), we have

$$\begin{aligned}
&\int_\Omega \big|\mathcal{A}(x, T_M(u_n), \nabla T_M(u_n))\big|^{p'}\omega\,\mathrm{d}x \\
&\quad\leq \int_\Omega \Big(K(x) + h_1(x)|T_M(u_n)|^{p/p'} + h_2(x)|\nabla T_M(u_n)|^{p/p'}\Big)^{p'}\omega\,\mathrm{d}x \\
&\quad\leq C\bigg[\int_\Omega K^{p'}(x)\,\omega\,\mathrm{d}x + \int_\Omega h_1^{p'}(x)|T_M(u_n)|^p\omega\,\mathrm{d}x \\
&\quad\quad + \int_\Omega h_2^{p'}(x)|\nabla T_M(u_n)|^p\omega\,\mathrm{d}x\bigg] \\
&\quad\leq C\Big(\|K\|^{p'}_{L^{p'}(\Omega,\omega)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega |T_M(u_n)|^p\omega\,\mathrm{d}x \\
&\quad\quad + \|h_2\|^{p'}_{L^\infty(\Omega)}\int_\Omega |\nabla T_M(u_n)|^p\omega\,\mathrm{d}x\Big) \\
&\quad\leq C\Big(\|K\|^{p'}_{L^{p'}(\Omega,\omega)} + \|h_1\|^{p'}_{L^\infty(\Omega)}M^p\mu(\Omega) + \|h_2\|^{p'}_{L^\infty(\Omega)}M\,C_2\Big),
\end{aligned}$$

that is, $|\mathcal{A}(x, T_M(u_n), \nabla T_M(u_n))|$ is bounded in $L^{p'}(\Omega, \omega)$.

Moreover, $\chi_{\{|u_n|>k\}}|\nabla T_k(u)| \to 0$ in $L^p(\Omega, \omega)$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \int_{\{|u_n|>k\}} \big|\mathcal{A}(x, T_M(u_n), \nabla T_M(u_n))\big| \, |\nabla T_k(u)| \, \omega \, \mathrm{d}x = 0. \tag{16}$$

Furthermore, we have that

$$T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rightharpoonup T_{2k}(u - T_h(u)),$$

weakly in $W_0^{1,p}(\Omega, \omega)$, as $n \to \infty$.

Hence, by (9), (15) and (16), passing to the limit in (13), we have

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega \big\langle \mathcal{A}(x, T_k(u_n), \nabla T_k(u_n)) - \mathcal{A}(x, T_k(u_n), \nabla T_k(u)), \nabla (T_k(u_n) - T_k(u))\big\rangle \, \omega \, \mathrm{d}x \\ \leq \int_\Omega f T_{2k}(u - T_{h_\varepsilon}(u)) \, \mathrm{d}x + \int_\Omega \langle G, \nabla T_{2k}(u - T_{h_\varepsilon}(u))\rangle \, \mathrm{d}x \leq \varepsilon,\end{aligned}$$

for all $\varepsilon > 0$, that is,

$$\int_\Omega \big\langle \mathcal{A}(x, T_k(u_n), \nabla T_k(u_n)) - \mathcal{A}(x, T_k(u_n), \nabla T_k(u)), \nabla (T_k(u_n) - T_k(u))\big\rangle \, \omega \to 0,$$

as $n \to \infty$. Applying Lemma 3 we get

$$T_k(u_n) \to T_k(u) \tag{17}$$

strongly in $W_0^{1,p}(\Omega, \omega)$ for every $k > 0$. This convergence implies that, for every fixed $k > 0$

$$\mathcal{A}\big(x, T_k(u_n), \nabla T_k(u_n)\big) \to \mathcal{A}\big(x, T_k(u), \nabla T_k(u)\big) \tag{18}$$

in $(L^{p'}(\Omega, \omega))^N = L^{p'}(\Omega, \omega) \times \cdots \times L^{p'}(\Omega, \omega)$.

• Finally, we need to show that u is an entropy solution to Dirichlet problem (P). Let us take $\psi_n = T_k(u_n - \varphi)$ as test function in (5), with $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$. We obtain,

$$\int_\Omega \omega \, \big\langle \mathcal{A}(x, u_n, \nabla u_n), \nabla \psi_n \big\rangle \, \mathrm{d}x = \int_\Omega f_n \psi_n \, \mathrm{d}x + \int_\Omega \big\langle G_n, \nabla \psi_n \big\rangle \, \mathrm{d}x. \tag{19}$$

If $M = k + \|\varphi\|_{L^\infty(\Omega)}$ and $n > M$, we have

$$\begin{aligned}&\int_\Omega \omega \, \big\langle \mathcal{A}(x, u_n, \nabla u_n), \nabla T_k(u_n - \varphi)\big\rangle \, \mathrm{d}x \\ &\qquad = \int_\Omega \omega \, \big\langle \mathcal{A}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_k(u_n - \varphi)\big\rangle \, \mathrm{d}x.\end{aligned}$$

Hence, in (19) we obtain

$$\int_\Omega \omega \, \langle \mathcal{A}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_k(u_n - \varphi) \rangle \, \mathrm{d}x = \int_\Omega f_n T_k(u_n - \varphi) \, \mathrm{d}x + \int_\Omega \langle G, \nabla T_k(u_n - \varphi) \rangle \, \mathrm{d}x \,. \quad (20)$$

Therefore, by (9) and (18), passing to the limit as $n \to \infty$ in (20), we obtain

$$\int_\Omega \omega \, \big\langle \mathcal{A}(x, u, \nabla u), \nabla T_k(u - \varphi) \big\rangle \, \mathrm{d}x = \int_\Omega f \, T_k(u - \varphi) \, \mathrm{d}x + \int_\Omega \big\langle G, \nabla T_k(u - \varphi) \big\rangle \, \mathrm{d}x$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$ and for each $k > 0$.

Therefore u is an entropy solution of problem (P). □

Example 1. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, the weight function

$$\begin{aligned} \omega(x, y) &= (x^2 + y^2)^{-1/2} \quad (\omega \in A_3), \\ f(x, y) &= \frac{\cos(xy)}{(x^2 + y^2)^{1/3}}, \\ G(x, y) &= \big((x^2 + y^2) \sin(xy), (x^2 + y^2)^{-1/3} \cos(xy)\big) \end{aligned}$$

and $\mathcal{A} \colon \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, $\mathcal{A}((x, y), s, \xi) = |\xi| \, \xi$. By Theorem 2, the problem

$$(P) \begin{cases} -\operatorname{div}[(x^2 + y^2)^{-1/2} \mathcal{A}(x, u, \nabla u)] = \dfrac{\cos(xy)}{(x^2 + y^2)^{1/3}} - \operatorname{div}(G(x, y)), & \text{in } \Omega \\ u(x, y) = 0, & \text{on } \partial\Omega \end{cases}$$

has an entropy solution.

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Received: 24 February, 2014
Accepted for publication: 4 April, 2014
Communicated by: Geoff Prince