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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 54 (2015), No. 1, 19–31

Persistent URL: `http://dml.cz/dmlcz/144365`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **54**, 1 (2015) 19–31

On a Construction of Modular *GMS*-algebras

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(Received April 19, 2013)

Abstract

In this paper we investigate the class of all modular *GMS*-algebras which contains the class of *MS*-algebras. We construct modular *GMS*-algebras from the variety $\underline{\mathbf{K}}_2$ by means of $\underline{K}_2$-quadruples. We also characterize isomorphisms of these algebras by means of $\underline{K}_2$-quadruples.

Key words: *MS*-algebras, *GMS*-algebras, K_2-algebras, Kleene algebras, isomorphisms.

2010 Mathematics Subject Classification: 06D05, 06D30

1 Introduction

T. S. Blyth and J. C. Varlet [2] have studied the variety of *MS*-algebras as a common abstraction of de Morgan algebras and Stone algebras. D. Ševčovič [12] investigated a larger variety of algebras containing *MS*-algebras, the so-called generalized *MS*-algebras (*GMS*-algebras). In such algebras the distributive identity need not be necessarily satisfied. In [4] T. S. Blyth and J. C. Varlet presented a construction of some *MS*-algebras from the subvariety $\mathbf{K}_2$ (the so-called K_2-algebras) from Kleene algebras and distributive lattices. This was a construction by means of triples which were successfully used in construction of Stone algebras (see [6], [7]), distributive p-algebras (see [9]), modular p-algebras (see [10]), etc. T. S. Blyth and J. V. Varlet [5] improved their construction from [4] by means of quadruples and they showed that each member of $\mathbf{K}_2$ can be constructed in this way. In [8] M. Haviar presented a simple quadruple construction of K_2-algebras which works with pairs of elements only. He also proved that there exists a one-to-one correspondence between locally bounded K_2-algebras and decomposable K_2-quadruples. Recently, A. Badawy, D. Guffová and M. Haviar [1] introduced the class of decomposable *MS*-algebras. They presented a triple construction of decomposable *MS*-algebras. Moreover, they proved that there exists a one-to-one correspondence between the decomposable *MS*-algebras and the decomposable *MS*-triples.

The aim of this paper is to investigate a subvariety of *GMS*-algebras containing the variety of *MS*-algebras, the so-called modular *GMS*-algebras. We construct modular *GMS*-algebras from the variety $\underline{\mathbf{K}}_2$ ($\underline{K}_2$-algebras) from Kleene algebras and modular lattices by means of $\underline{K}_2$-quadruples. Also we define an isomorphism between two $\underline{K}_2$-quadruples and we show that two $\underline{K}_2$-algebras are isomorphic if and only if their associated $\underline{K}_2$-quadruples are isomorphic.

2 Preliminaries

An *MS*-algebra is an algebra $(L; \vee, \wedge, ^\circ, 0, 1)$ of type $(2, 2, 1, 0, 0)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation $^\circ$ satisfies

$$x \le x^{\circ\circ}, \quad (x \wedge y)^\circ = x^\circ \vee y^\circ, \quad 1^\circ = 0.$$

The class **MS** of all *MS*-algebras forms a variety. The members of the subvariety **M** of **MS** defined by the identity $x = x^{\circ\circ}$ are called de Morgan algebras and the members of the subvariety **K** of **M** defined by the identity $x \wedge x^\circ \le y \vee y^\circ$ are called Kleene algebras. The subvariety $\mathbf{K}_2$ of **MS** is defined by the additional two identities

$$x \wedge x^\circ = x^{\circ\circ} \wedge x^\circ, \quad x \wedge x^\circ \le y \vee y^\circ.$$

The class **S** of all Stone algebras is a subvariety of **MS** and is characterized by the identity $x \wedge x^\circ = 0$. The subvariety **B** of **MS** characterized by the identity $x \vee x^\circ = 1$ is the class of Boolean algebras.

A generalized de Morgan algebra (or GM-algebra) is a universal algebra $(L; \vee, \wedge, ^-, 0, 1)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and the unary operation of involution $^-$ satisfies the identities

$$GM_1\colon x = x^{--}, \quad GM_2\colon (x \wedge y)^- = x^- \vee y^-, \quad GM_3\colon 1^- = 0.$$

A modular GM-algebra L is a GM-algebra where $(L; \vee, \wedge, 0, 1)$ is a modular lattice. A modular generalized Kleene algebra (modular GK-algebra) L is a modular GM-algebra satisfying the identity $x \wedge x^\circ \le x \vee y^\circ$.

A generalized *MS*-algebra (or *GMS*-algebra) is a universal algebra $(L; \vee, \wedge, ^\circ, 0, 1)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and the unary operation $^\circ$ satisfies the identities

$$GMS_1\colon x \le x^{\circ\circ}, \quad GMS_2\colon (x \wedge y)^\circ = x^\circ \vee y^\circ, \quad GMS_3\colon 1^\circ = 0.$$

The class of all GM-algebras is a subvariety of the variety of all *GMS*-algebras.

A modular *GMS*-algebra is a *GMS*-algebra $(L; \vee, \wedge, ^\circ, 0, 1)$ where $(L; \vee, \wedge, 0, 1)$ is a modular lattice.

The class of all modular *GMS*-algebras forms a variety. The class **MS** is a subvariety of the variety of all modular *GMS*-algebras. Then the varieties **B**, **M**, **S** and $\mathbf{K}_2$ are subvarieties of the variety of all modular *GMS*-algebras.

The class $\underline{\mathbf{S}}$ of all modular *S*-algebras is a subvariety of the variety of all modular *GMS*-algebras and is characterized by the identity $x \wedge x^\circ = 0$. It is known that the class **S** is a subvariety of $\underline{\mathbf{S}}$.

The main immediate consequences of these axioms are summarized in the following result.

Lemma 2.1 *Let L be a GMS-algebra. Then we have*

(1) $0^\circ = 1$,

(2) $x \leq y$ *implies* $x^\circ \geq y^\circ$,

(3) $x^\circ = x^{\circ\circ\circ}$,

(4) $(x \vee y)^\circ = x^\circ \wedge y^\circ$,

(5) $(x \wedge y)^{\circ\circ} = x^{\circ\circ} \wedge y^{\circ\circ}$,

(6) $(x \vee y)^{\circ\circ} = x^{\circ\circ} \vee y^{\circ\circ}$.

Consequently, if L is a modular *GMS*-algebra, then the set $L^{\circ\circ} = \{x \in L \colon x^{\circ\circ} = x\}$ is a modular *GM*-algebra and a subalgebra of L such that the mapping $x \mapsto x^{\circ\circ}$ is a homomorphism of L onto $L^{\circ\circ}$, and $D(L) = \{x \in L \colon x^\circ = 0\}$ is a filter of L, the elements of which are called dense.

For an arbitrary lattice L, the set $F(L)$ of all filters of L ordered under set inclusion is a lattice. It is known that $F(L)$ is a modular lattice if and only if L is modular. Let $a \in L$; $[a)$ denotes the filter of L generated by a.

For any modular *GMS*-algebra L, the relation Φ defined by

$$x \equiv y \ (\Phi) \quad \Leftrightarrow \quad x^{\circ\circ} = y^{\circ\circ}$$

is a congruence relation on L and $L/\Phi \cong L^{\circ\circ}$ holds. Each congruence class contains exactly one element of $L^{\circ\circ}$ which is the largest element in the congruence class, the largest element of $[x]\Phi$ is $x^{\circ\circ}$ which is denoted by $\max[x]\Phi$. Hence Φ partition L into $\{F_c \colon c \in L^{\circ\circ}\}$, where $F_c = \{x \in L \colon x^{\circ\circ} = c\}$. Obviously, $F_0 = \{0\}$ and $F_1 = \{x \in L \colon x^{\circ\circ} = 1\} = D(L)$.

Now we introduce certain modular *GMS*-algebras, which are called $\underline{K}_2$-algebras.

Definition 2.2 A modular *GMS*-algebra L is called a $\underline{K}_2$-algebra if $L^{\circ\circ}$ is a distributive lattice and L satisfies the identities $x \wedge x^\circ = x^{\circ\circ} \wedge x^\circ$ and $x \wedge x^\circ \leq y \vee y^\circ$.

The class $\underline{\mathbf{K}}_2$ of all $\underline{K}_2$-algebras contains the class $\mathbf{K}_2$. Clearly, the classes $\underline{\mathbf{S}}$, **S**, **M**, **K** and **B** are subclasses of the class $\underline{\mathbf{K}}_2$.

Theorem 2.3 *Let* $L \in \underline{\mathbf{K}}_2$. *Then*

(1) $x = x^{\circ\circ} \wedge (x \vee x^\circ)$ *for every* $x \in L$,

(2) $L^{\circ\circ} = \{x \in L\colon x = x^{\circ\circ}\}$ *is a Kleene algebra,*

(3) $L^{\wedge} = \{x \wedge x^{\circ}\colon x \in L\} = \{x \in L\colon x \leq x^{\circ}\}$ *is an ideal of* L*,*

(4) $L^{\vee} = \{x \vee x^{\circ}\colon x \in L\} = \{x \in L\colon x \geq x^{\circ}\}$ *is a filter of* L*,*

(5) $D(L) = \{x \in L\colon x^{\circ} = 0\}$ *is a filter of* L *and* $D(L) \subseteq L^{\vee}$*.*

Proof (1) Since $x \leq x^{\circ\circ}$, then by modularity of L we get

$$\begin{aligned} x^{\circ\circ} \wedge (x \vee x^{\circ}) &= (x^{\circ\circ} \wedge x^{\circ}) \vee x \\ &= (x \wedge x^{\circ}) \vee x \text{ by Definition 2.2} \\ &= x. \end{aligned}$$

(2) It is obvious.

(3) Clearly $0 \in L^{\wedge}$. Let $x, y \in L^{\wedge}$. Then $x \leq x^{\circ}$ and $y \leq y^{\circ}$. By Definition 2.2, we get $x = x \wedge x^{\circ} \leq y \vee y^{\circ} = y^{\circ}$. It follows that $x^{\circ} \geq y^{\circ\circ} \geq y$. Then $x^{\circ} \wedge y^{\circ} \geq x, y$ implies $x^{\circ} \wedge y^{\circ} \geq x \vee y$. Now

$$(x \vee y) \wedge (x \vee y)^{\circ} = (x \vee y) \wedge (x^{\circ} \wedge y^{\circ}) = x \vee y.$$

Consequently $x \vee y \leq (x \vee y)^{\circ}$ and $x \vee y \in L^{\wedge}$. Let $x \in L^{\wedge}$ be such that $z \leq x$ for some $z \in L$. Then $z \leq x \leq x^{\circ} \leq z^{\circ}$. Hence $z \in L^{\wedge}$. Then $L^{\wedge}$ is an ideal of L.

(4) By duality of (3).

(5) It is obvious. □

Corollary 2.4 *Let* L *be a modular GMS-algebra. Then for all* $x \in L$ *the following conditions are equivalent:*

(1) $x = x^{\circ\circ} \wedge (x \vee x^{\circ})$*,*

(2) $x \wedge x^{\circ} = x^{\circ\circ} \wedge x^{\circ}$*.*

Now we reformulate the definition of polarization given in [Definition 1(iii), 11] as follows.

Definition 2.5 Let K be a Kleene algebra and D be a modular lattice with 1. A mapping $\varphi\colon K \to F(D)$ is called a polarization if φ is a (0,1)-homomorphism such that $a\varphi = D$ for every $a \in K^{\vee}$ and $a\varphi$ is a principal filter of D for every $a \in K^{\wedge}$.

3 The triple associated with a $\underline{K}_2$-algebra

Let $L \in \underline{\mathbf{K}}_2$. $L^{\vee}$ is a filter of L, and $L^{\vee}$ is a modular lattice with the largest element 1. So $F(L^{\vee})$ is also a modular lattice. Consider the map $\varphi(L)\colon L^{\circ\circ} \to F(L^{\vee})$ defined by the following way

$$a\varphi(L) = \{x \in L^{\vee} : x \geq a^{\circ}\} = [a^{\circ}) \cap L^{\vee}, \quad a \in L^{\circ\circ}.$$

Lemma 3.1 *Let* $L \in \underline{\mathbf{K}}_2$*. Then* $\varphi(L)$ *is a polarization of* $L^{\circ\circ}$ *into* $F(L^{\vee})$*.*

Proof It is easy to check that $0\varphi(L) = [1)$, $1\varphi(L) = L^\vee$ and $(a \wedge b)\varphi(L) = a\varphi(L) \cap b\varphi(L)$. Now we show that $(a \vee b)\varphi(L) = a\varphi(L) \vee b\varphi(L)$. Since $a, b \le a \vee b$, then $a\varphi(L) \vee b\varphi(L) \subseteq (a \vee b)\varphi(L)$. For the converse, let $t \in (a \vee b)\varphi(L) = [a^\circ \wedge b^\circ) \cap L^\vee$. Put $x = a \vee (a^\circ \wedge t)$. Then $x^\circ = a^\circ \wedge (a \vee t^\circ) = (a^\circ \wedge a) \vee (a^\circ \wedge t^\circ) \le a \vee (a^\circ \wedge t) = x$ since $L^{\circ\circ}$ is distributive and $t^\circ \le t$. Thus $x \in L^\vee$. Moreover,

$$a^\circ \wedge (b^\circ \vee x) = a^\circ \wedge (b^\circ \vee (a \vee (a^\circ \wedge t))) = (a^\circ \wedge (a \vee b^\circ)) \vee (a^\circ \wedge t) \le t,$$

since $a^\circ \wedge (a \vee b^\circ) = (a^\circ \wedge a) \vee (a^\circ \wedge b^\circ) \le t$. Now, $t \in [a^\circ) \vee [b^\circ \vee x) \subseteq [a^\circ) \vee ([b^\circ) \cap L^\vee)$. But $t \in L^\vee$ and $F(L)$ is a modular lattice, hence

$$t \in ([a^\circ) \vee ([b^\circ) \cap L^\vee)) \cap L^\vee = ([a^\circ) \cap L^\vee) \vee ([b^\circ) \cap L^\vee) = a\varphi(L) \vee b\varphi(L).$$

Thus $\varphi(L)$ is (0,1)-lattice homomorphism. If $a \in L^{\circ\circ}$, then $(a \vee a^\circ)\varphi(L) = [a^\circ \wedge a) \cap L^\vee = L^\vee$ and $(a \wedge a^\circ)\varphi(L) = [a^\circ \vee a)$. Then φ is a polarization. $\square$

Definition 3.2 A triple (K, D, φ) is said to be a $\underline{K}_2$-triple if

(1) $(K; \vee, \wedge, 0, 1)$ is a Kleene algebra,

(2) D is a modular lattice with 1,

(3) $\varphi\colon K \to F(D)$ is a polarization.

Let L be a $\underline{K}_2$-algebra. Then $(L^{\circ\circ}, L^\vee, \varphi(L))$ is the triple associated with L and this triple is a $\underline{K}_2$-triple.

Lemma 3.3 *Let (K, D, φ) be a $\underline{K}_2$-triple. Then we have*

$$a\varphi \cap (b\varphi \vee c\varphi) = (a\varphi \cap b\varphi) \vee (a\varphi \cap c\varphi) \textit{ for every } a, b, c \in K.$$

Lemma 3.4 *Let (K, D, φ) be a $\underline{K}_2$-triple. Then we have*

(i) for every $a \in K$ and for every $y \in D$ there exists an element $t \in D$ such that

$$a\varphi \cap [y) = [t),$$

(ii) for every $a \in K$ and for every $y \in D$ there exists an element $t \in a^\circ\varphi$ such that

$$a\varphi \vee [y) = a\varphi \vee [t),$$

(iii) for every $a, b \in K$ and for every $y \in D$ there exists an element $t \in D$ such that

$$((a\varphi \cap b^\circ\varphi) \vee [y)) \cap (a^\circ\varphi \vee b\varphi \vee [y)) = [t).$$

Proof For any $a \in K$, there is $d_a \in D$ such that $(a \wedge a^\circ)\varphi = a\varphi \cap a^\circ\varphi = [d_a)$ as $a \wedge a^\circ \in K^\wedge$ and φ is a polarization. Recall that $F(D)$ is a modular lattice.

(i). For all $a \in K, a \wedge a^\circ \in K^\wedge, a \vee a^\circ \in K^\vee$. Then there exists $d_a \in D$ such that $a\varphi \cap a^\circ\varphi = [d_a)$ and $a\varphi \vee a^\circ\varphi = (a \vee a^\circ)\varphi = D$. Therefore, there exist elements $x_1 \in a\varphi$ and $z_1 \in a^\circ\varphi$ such that $x_1, z_1 \le d_a$ and $x_1 \wedge z_1 \le y$.

We notice that $x_1 \vee z_1 \in a\varphi \cap a^\circ\varphi$. Hence $x_1 \vee z_1 = d_a$. We claim $t = x_1 \vee y$. Clearly $t \in a\varphi \cap [y)$. Conversely, let $v \in a\varphi \cap [y)$. Then

$$\begin{aligned} v &\geq (v \wedge x_1) \vee y \\ &= ((v \wedge x_1) \vee (x_1 \wedge z_1)) \vee y \\ &= (((v \wedge x_1) \vee z_1) \wedge x_1) \vee y \text{ by modularity of } D \\ &= (d_a \wedge x_1) \vee y \\ &= x_1 \vee y \text{ as } (v \wedge x_1) \vee z_1 = d_a \geq x_1. \end{aligned}$$

Hence $v \geq x_1 \vee y = t$, and therefore $a\varphi \cap [y) = [t)$.

(ii). It is enough to show that $a^\circ\varphi \cap (a\varphi \vee [y)) = [t)$, for some $t \in D$ since then $t \in a^\circ\varphi$ and $[t) \vee a\varphi = (a^\circ\varphi \cap (a\varphi \vee [y))) \vee a\varphi = (a\varphi \vee [y)) \cap (a^\circ\varphi \vee a\varphi) = a\varphi \vee [y)$, from modularity of $F(D)$. Let $x_1 \in a\varphi$, $z_1 \in a^\circ\varphi$, $x_1 \wedge z_1 \leq y$ and $x_1, z_1 \leq d_a$. We claim that $t = z_1 \vee (x_1 \wedge y)$. Evidently, $t \in a^\circ\varphi \cap (a\varphi \vee [y))$. Conversely, let $v \in a^\circ\varphi \cap (a\varphi \vee [y))$. Then $v \geq v \wedge z_1 \in a^\circ\varphi$ and there is $x \in a\varphi$ with $v \geq x \wedge y \geq (x \wedge x_1) \wedge y$. Denote $z_0 = v \wedge z_1$ and $x_0 = x \wedge x_1$. Hence

$$v \geq (x_0 \wedge y) \vee z_0 \geq (x_0 \wedge x_1 \wedge z_1) \vee z_0 = (x_0 \wedge z_1) \vee z_0 = (x_0 \vee z_0) \wedge z_1 = z_1,$$

because $x_0 \vee z_0 = d_a \geq z_1$. This implies

$$\begin{aligned} v &\geq (x_0 \wedge y) \vee z_1 \\ &= (x_0 \wedge y) \vee (x_1 \wedge z_1) \vee z_1 \\ &= ((x_0 \vee (x_1 \wedge z_1)) \wedge y) \vee z_1 \\ &= ((x_0 \vee z_1) \wedge x_1 \wedge y) \vee z_1 \\ &= (x_1 \wedge y) \vee z_1 \text{ as } x_0 \vee z_1 = d_a \geq x_1 \wedge y \\ &= t. \end{aligned}$$

So, $v \geq t$ and $a^\circ\varphi \cap (a\varphi \vee [y)) = [t)$.

(iii). From (ii) there exists $y_1 \in a\varphi$ such that $[y_1) \vee a^\circ\varphi = [y) \vee a^\circ\varphi$. Using Lemma 3.3 and modularity of $F(D)$, we get

$$\begin{aligned} &((a\varphi \cap b^\circ\varphi) \vee [y)) \cap (a^\circ\varphi \vee b\varphi \vee [y)) \\ &\quad = ((a\varphi \cap b^\circ\varphi) \cap (a^\circ\varphi \vee b\varphi \vee [y))) \vee [y) \\ &\quad = ((a\varphi \cap b^\circ\varphi) \cap (a^\circ\varphi \vee b\varphi \vee [y_1))) \vee [y) \\ &\quad = (b^\circ\varphi \cap (a\varphi \cap (a^\circ\varphi \vee b\varphi \vee [y_1)))) \vee [y) \\ &\quad = (b^\circ\varphi \cap ((a\varphi \cap (a^\circ\varphi \vee b\varphi)) \vee [y_1))) \vee [y) \\ &\quad = (b^\circ\varphi \cap ([d_a) \vee (a\varphi \cap b\varphi) \vee [y_1))) \vee [y) \\ &\quad = (b^\circ\varphi \cap (a\varphi \cap (b\varphi \vee [y_1 \wedge d_a)))) \vee [y) \\ &\quad = (a\varphi \cap [t_1)) \vee [y) \\ &\quad = [t_2) \vee [y) \\ &\quad = [t_2 \wedge y). \end{aligned}$$

where $t_1, t_2 \in D$ are such elements that $b^\circ\varphi \cap (b\varphi \vee [y_1 \wedge d_a)) = [t_1)$ (see the proof of (ii)), $a\varphi \cap [t_1) = [t_2)$ from (i). Thus $t = t_2 \wedge y$. $\square$

Theorem 3.5 *Let (K, D, φ) be a $\underline{K}_2$-triple. Then for any $a, b \in K$ and $x, y \in D$ there exists an element $t \in D$ such that*

$$(a^\circ\varphi \vee [x)) \cap (b^\circ\varphi \vee [y)) = (a \vee b)^\circ\varphi \vee [t).$$

Proof Let $a, b \in K$ and $x, y \in D$. It is enough to show that there is $t \in D$ such that

$$(a^\circ\varphi \vee [x)) \cap (b^\circ\varphi \vee [y)) \cap (a \wedge b)\varphi = [t)$$

because then

$$\begin{aligned}
[t) \vee (a \vee b)^\circ\varphi &= ((a^\circ\varphi \vee [x)) \cap (b^\circ\varphi \vee [y)) \cap (a \wedge b)\varphi) \vee (a \vee b)^\circ\varphi \\
&= (a^\circ\varphi \vee [x)) \cap (b^\circ\varphi \vee [y)) \cap ((a \wedge b)\varphi \vee (a \vee b)^\circ\varphi) \\
&= (a^\circ\varphi \vee [x)) \cap (b^\circ\varphi \vee [y))
\end{aligned}$$

by modularity of $F(D)$ and since $(a \vee b)\varphi \vee (a \vee b)^\circ\varphi = D$. In accordance with Lemma 3.4, we can suppose $x \in a\varphi$ and $y \in b\varphi$. Then by Lemma 3.3 and by modularity of $F(D)$,

$$\begin{aligned}
&(a^\circ\varphi \vee [x)) \cap (b^\circ\varphi \vee [y)) \cap (a \vee b)\varphi \\
&\quad = ((a^\circ\varphi \vee [x)) \cap (a\varphi \vee b\varphi)) \cap ((b^\circ\varphi \vee [y)) \cap (a\varphi \vee b\varphi)) \\
&\quad = ((a^\circ\varphi \cap (a\varphi \vee b\varphi)) \vee [x)) \cap ((b^\circ\varphi \cap (a\varphi \vee b\varphi)) \vee [y)) \\
&\quad = ((a^\circ\varphi \cap a\varphi) \vee (a^\circ\varphi \cap b\varphi) \vee [x)) \cap ((b^\circ\varphi \cap a\varphi) \vee (b^\circ\varphi \cap b\varphi) \vee [y)) \\
&\quad = ([d_a \wedge x) \vee (a^\circ\varphi \cap b\varphi)) \cap ([d_b \wedge y) \vee (b^\circ\varphi \cap a\varphi))
\end{aligned}$$

where d_a, d_b are as in the proof of Lemma 3.4. Denote $x_0 = x \wedge d_a$, $y_0 = y \wedge d_b$ and $x_0 \wedge y_0 = z$. We first show that

$$((a\varphi \cap b^\circ\varphi) \vee [z)) \cap ((a^\circ\varphi \cap b\varphi) \vee [z)) = [p),$$

for some $p \in D$. Since $a^\circ\varphi \vee b\varphi \supseteq a^\circ\varphi \cap b\varphi$, we can write

$$\begin{aligned}
&((a\varphi \cap b^\circ\varphi) \vee [z)) \cap ((a^\circ\varphi \cap b\varphi) \vee [z)) \\
&\quad = ((a\varphi \cap b^\circ\varphi) \vee [z)) \cap (a^\circ\varphi \vee b\varphi \vee [z)) \cap ((a^\circ\varphi \cap b\varphi) \vee [z)) \\
&\quad = [q) \cap ((a^\circ\varphi \cap b\varphi) \vee [z))
\end{aligned}$$

where $[q) = ((a\varphi \cap b^\circ\varphi) \vee [z)) \cap (a^\circ\varphi \vee b\varphi \vee [z))$, by Lemma 3.4 (iii). Evidently $[q) \supseteq [z)$. Hence by modularity we get

$$\begin{aligned}
&[q) \cap ((a^\circ\varphi \cap b\varphi) \vee [z)) \\
&\quad = ([q) \cap a^\circ\varphi \cap b\varphi) \vee [z) \\
&\quad = ([q) \cap (a^\circ \wedge b)\varphi) \vee [z) \\
&\quad = [t_1) \vee [z) \text{ where } [q) \cap (a^\circ \wedge b)\varphi = [t_1) \text{ by Lemma 4.3(i)} \\
&\quad = [t_1 \wedge z) \\
&\quad = [p) \text{ where } p = t_1 \wedge z.
\end{aligned}$$

Since $[p) \supseteq [z) \supseteq [x_0), [y_0)$ and $F(D)$ is modular, we have

$$\begin{aligned}
&([x_0) \vee (a^\circ\varphi \cap b\varphi)) \cap ([y_0) \vee (b^\circ\varphi \cap a\varphi)) \\
&\qquad = ([p) \cap ([x_0) \vee (a^\circ\varphi \cap b\varphi))) \cap ([p) \cap ([y_0) \vee (b^\circ\varphi \cap a\varphi))) \\
&\qquad = (([p) \cap (a^\circ\varphi \cap b\varphi)) \vee [x_0)) \cap (([p) \cap (b^\circ\varphi \cap a\varphi)) \vee [y_0)) \\
&\qquad = ([v) \vee [x_0)) \cap ([w) \vee [y_0)) \text{ for some } v, w \in D \\
&\qquad = [(u \wedge x_0) \vee (w \wedge y_0)) \\
&\qquad = [t),
\end{aligned}$$

where $[v) = [p) \cap a^\circ\varphi \cap b\varphi$, $[w) = [p) \cap b^\circ\varphi \cap a\varphi$ and $t = (u \wedge x_0) \vee (w \wedge y_0) \in D$. □

4 $\underline{K}_2$-construction

In this section we generalize the construction of [3, 4] from the so-called K_2-algebras to $\underline{K}_2$-algebras. Also we prove that there exists a one-to-one correspondence between $\underline{K}_2$-algebras and $\underline{K}_2$-quadruples.

Definition 4.1 A $\underline{K}_2$-quadruple is (K, D, φ, γ) where

(i) (K, D, φ) is a $\underline{K}_2$-triple, and

(ii) γ is a monomial congruence on D, that is every γ class $[y]\gamma$ has a largest element $(\max[y]\gamma)$.

Let $L \in \underline{\mathbf{K}}_2$. Then $(L^{\circ\circ}, L^\vee, \varphi(L))$ is a K_2-triple. Let $\gamma(L)$ be the restriction of the congruence Φ on $L^\vee$. Since $\max[x]\gamma = x^{\circ\circ}$, for every $x \in L^\vee$. Then $\gamma(L)$ is a monomial congruence on $L^\vee$. We say that $(L^{\circ\circ}, L^\vee, \varphi(L), \gamma(L))$ is the quadruple associated with L and this quadruple is a $\underline{K}_2$-quadruple.

Theorem 4.2 *Let (K, D, φ, γ) be a $\underline{K}_2$-quadruple. Then*

$$L = \{(a, a^\circ\varphi \vee [x)) \colon a \in K, x \in D, \max[x]\gamma \in a^\circ\varphi\}$$

is a $\underline{K}_2$-algebra if we define

$$\begin{aligned}
(a, a^\circ\varphi \vee [x)) \wedge (b, b^\circ\varphi \vee [y)) &= (a \wedge b, (a^\circ\varphi \vee [x)) \vee (b^\circ\varphi \vee [y))), \\
(a, a^\circ\varphi \vee [x)) \vee (b, b^\circ\varphi \vee [y)) &= (a \vee b, (a^\circ\varphi \vee [x)) \cap (b^\circ\varphi \vee [y))), \\
(a, a^\circ\varphi \vee [x))^\circ &= (a^\circ, a\varphi), \\
1_L &= (1, [1)), \\
0_L &= (0, D).
\end{aligned}$$

Moreover, $L^{\circ\circ} \cong K$.

Proof Let $F_d(D)$ denote the dual lattice to the modular lattice $F(D)$ of all filters of D. Evidently, L is a subset of the direct product $K \times F_d(D)$. We show first that L is a sublattice of $K \times F_d(D)$. Let $(a, a^\circ\varphi \vee [x)), (b, b^\circ\varphi \vee [y)) \in L$. Then

$$(a, a^\circ\varphi \vee [x)) \wedge (b, b^\circ\varphi \vee [y)) = (a \wedge b, (a \wedge b)^\circ\varphi \vee [x \wedge y)) \in L,$$

because of φ is a lattice homomorphism and

$$\max[x \wedge y]\gamma = \max[x]\gamma \wedge \max[y]\gamma \in a^\circ\varphi \vee b^\circ\varphi = (a \wedge b)^\circ\varphi.$$

Moreover,

$$\begin{aligned}
&(a, a^\circ\varphi \vee [x)) \vee (b, b^\circ\varphi \vee [y)) \\
&\quad = (a \vee b, (a^\circ\varphi \vee [x)) \cap (b^\circ\varphi \vee [y))) \\
&\quad = (a \vee b, (a \vee b)^\circ\varphi \vee [t)) \text{ for some } t \in D, \text{ by Theorem 3.5.}
\end{aligned}$$

Now we prove that $\max[x]\gamma \in a^\circ\varphi$ and $\max[y]\gamma \in b^\circ\varphi$ implies $\max[t]\gamma \in (a \vee b)^\circ\varphi$. From the proof of Theorem 3.5, $t = (v \wedge x_0) \vee (w \wedge y_0)$ where $v \in a^\circ\varphi$, $w \in b^\circ\varphi$, $x_0 = x \wedge d_a$ and $y_0 = y \wedge d_a$. Then

$$t = (v \wedge x \wedge d_a) \vee (w \wedge y \wedge d_b) = (x \wedge v_0) \vee (y \wedge w_0)$$

where $v_0 = v \wedge d_a \in a^\circ\varphi$ and $w_0 = w \wedge d_b \in b^\circ\varphi$. Then

$$\max[t]\gamma \geq (\max[x]\gamma \wedge \max[v_0]\gamma) \vee (\max[y]\gamma \wedge [w_0]\gamma) \in a^\circ\varphi \cap b^\circ\varphi = (a \vee b)^\circ\varphi,$$

because of $\max[v_0]\gamma \geq v_0 \in a^\circ\varphi$ and $\max[w_0]\gamma \geq w_0 \in b^\circ\varphi$ implies $\max[v_0]\gamma \in a^\circ\varphi$ and $\max[w_0]\gamma \in b^\circ\varphi$, respectively. Then $(a \vee b, (a \vee b)^\circ\varphi \vee [t)) \in L$. Therefore L is a sublattice of $K \times F_d(D)$. Hence L is a modular lattice. The order of L is given by

$$(a, a^\circ\varphi \vee [x)) \leq (b, b^\circ\varphi \vee [y)) \text{ iff } a \leq b \text{ and } a^\circ\varphi \vee [x) \supseteq b^\circ\varphi \vee [y).$$

L is bounded and

$$(0, D) \leq (a, a^\circ\varphi \vee [x)) \leq (1, [1)).$$

In addition,

$$\begin{aligned}
(a, a^\circ\varphi \vee [x)) &\leq (a, a^\circ\varphi) = (a, a^\circ\varphi \vee [x))^{\circ\circ}, \\
((a, a^\circ\varphi \vee [x)) \wedge (b, b^\circ\varphi \vee [y)))^\circ &= (a, a^\circ\varphi \vee [x))^\circ \vee (b, b^\circ\varphi \vee [y))^\circ, \\
(1, [1))^\circ &= (0, D).
\end{aligned}$$

Then L is a modular *GMS*-algebra. Also we get

$$\begin{aligned}
&(a, a^\circ\varphi \vee [x)) \wedge (a, a^\circ\varphi \vee [x))^\circ \\
&\quad = (a \wedge a^\circ, a^\circ\varphi \vee [x) \vee a\varphi) \\
&\quad = (a \wedge a^\circ, a^\circ\varphi \vee a\varphi) \text{ as } [x) \subseteq a\varphi \vee a^\circ\varphi = D \\
&\quad = (a, a^\circ\varphi) \wedge (a^\circ, a\varphi) \\
&\quad = (a, a^\circ\varphi \vee [x))^{\circ\circ} \wedge (a, a^\circ\varphi \vee [x))^\circ,
\end{aligned}$$

and

$$(a, a^\circ\varphi \vee [x)) \wedge (a, a^\circ\varphi \vee [x))^\circ \leq (b, b^\circ\varphi \vee [y)) \vee (b, b^\circ\varphi \vee [y))^\circ.$$

Hence $L \in \underline{\mathbf{K}}_2$. Now,

$$L^{\circ\circ} = \{(a, a^\circ\varphi \vee [x))^{\circ\circ} \colon (a, a^\circ\varphi \vee [x)) \in L\} = \{(a, a^\circ\varphi) \colon a \in K\} \cong K$$

under the isomorphism $(a, a^\circ\varphi) \mapsto a$. Then $L^{\circ\circ}$ is a Kleene algebra. Therefore L is a $\underline{K}_2$-algebra. □

Corollary 4.3 *From Theorem 4.2, we have*

(1) $L^\vee = \{(a, a^\circ\varphi \vee [x)) \in L \colon a \in K^\vee, x \in D\}$,

(2) $D(L) = \{(1, [x)) \colon x \in [1]\gamma, x \in D\}$.

Corollary 4.4 *Let* (K, D, φ, γ) *be a* $\underline{K}_2$*-quadruple. Then*

(1) If D is a distributive lattice, then L described by Theorem 4.2 is a K_2-algebra;

(2) If K is a Boolean algebra and $\gamma = \iota$, then L described by Theorem 4.2 is a modular S-algebra;

(3) If K is a Boolean algebra, D is a distributive lattice and $\gamma = \iota$, then L described by Theorem 4.2 is a Stone algebra.

We say that $L \in \underline{\mathbf{K}}_2$ from Theorem 4.2 is associated with the $\underline{K}_2$-quadruple (K, D, φ, γ) and the construction of L described in Theorem 4.2 will be called a $\underline{K}_2$-construction.

Theorem 4.5 *Let $L \in \underline{\mathbf{K}}_2$. Let $(L^{\circ\circ}, L^\vee, \varphi(L), \gamma(L))$ be the $\underline{K}_2$-quadruple associated with L. Then L_1 associated with $(L^{\circ\circ}, L^\vee, \varphi(L), \gamma(L))$ is isomorphic to L.*

Proof For every $x \in L$, $x = x^{\circ\circ} \wedge (x \vee x^\circ)$ and by modularity of $F(L)$, we observe

$$x^\circ\varphi(L) \vee [x \vee x^\circ) = ([x^{\circ\circ}) \cap L^\vee) \vee [x \vee x^\circ) = L^\vee \cap ([x^{\circ\circ}) \vee [x \vee x^\circ)) = L^\vee \cap [x).$$

We shall prove that the mapping $f \colon L \to L_1$ defined by

$$xf = (x^{\circ\circ}, x^\circ\varphi(L) \vee [x \vee x^\circ)) = (x^{\circ\circ}, L^\vee \cap [x))$$

is the described isomorphism. Obviously $xf \in L_1$, since $\max[x \vee x^\circ]\gamma(L) = (x \vee x^\circ)^{\circ\circ} = x^{\circ\circ} \vee x^\circ \in [x^{\circ\circ}) \cap L^\vee = x^\circ\varphi(L)$. For every $x, y \in L$,

$$\begin{aligned}
(x \wedge y)f &= ((x \wedge y)^{\circ\circ}, (x \wedge y)^\circ\varphi(L) \vee [(x \wedge y) \vee (x \wedge y)^\circ)) \\
&= ((x \wedge y)^{\circ\circ}, [x \wedge y) \cap L^\vee), \\
xf \wedge yf &= (x^{\circ\circ}, x^\circ\varphi(L) \vee [x \vee x^\circ)) \wedge (y^{\circ\circ}, y^\circ\varphi(L) \vee [y \vee y^\circ)) \\
&= (x^{\circ\circ} \wedge y^{\circ\circ}, x^\circ\varphi(L) \vee [x \vee x^\circ) \vee y^\circ\varphi(L) \vee [y \vee y^\circ)).
\end{aligned}$$

Since $x = x^{\circ\circ} \wedge (x \vee x^\circ)$, $y = y^{\circ\circ} \wedge (y \vee y^\circ)$ and $\varphi(L)$ is a polarity (see Lemma 3.1), then by modularity of $F(L)$, we have

$$\begin{aligned}
&x^\circ\varphi(L) \vee [x \vee x^\circ) \vee y^\circ\varphi(L) \vee [y \vee y^\circ)\\
&\quad = (x \wedge y)^\circ\varphi(L) \vee [(x \vee x^\circ) \wedge (y \vee y^\circ))\\
&\quad = ([(x \wedge y)^{\circ\circ}) \cap L^\vee) \vee [(x \vee x^\circ) \wedge (y \vee y^\circ))\\
&\quad = L^\vee \cap ([(x \wedge y)^{\circ\circ}) \vee [(x \vee x^\circ) \wedge (y \vee y^\circ))\\
&\quad = L^\vee \cap [x^{\circ\circ} \wedge y^{\circ\circ} \wedge (x \vee x^\circ) \wedge (y \vee y^\circ))\\
&\quad = L^\vee \cap [x \wedge y).
\end{aligned}$$

Then $(x \wedge y)f = xf \wedge yf$. Also,

$$\begin{aligned}
(x \vee y)f &= ((x \vee y)^{\circ\circ}, [x \vee y) \cap L^\vee)\\
&= (x^{\circ\circ} \vee y^{\circ\circ}, [x) \cap [y) \cap L^\vee)\\
&= (x^{\circ\circ} \vee y^{\circ\circ}, ([x) \cap L^\vee) \cap ([y) \cap L^\vee))\\
&= (x^{\circ\circ}, [x) \cap L^\vee) \vee (y^{\circ\circ}, [y) \cap L^\vee)\\
&= xf \vee yf
\end{aligned}$$

and $0f = (0, L^\vee)$, $1f = (1, [1))$. Then f is a (0,1)-lattice homomorphism.
Now,

$$\begin{aligned}
(xf)^\circ &= (x^{\circ\circ}, x^\circ\varphi(L) \vee [x \vee x^\circ))^\circ\\
&= (x^\circ, x^{\circ\circ}\varphi(L))\\
&= (x^\circ, [x^\circ) \cap L^\vee)\\
&= x^\circ f,
\end{aligned}$$

hence f is a homomorphism of $\underline{K}_2$-algebras.
Now assume $x_1 f = x_2 f$. Then $(x_1^{\circ\circ}, [x_1) \cap L^\vee) = (x_2^{\circ\circ}, [x_2) \cap L^\vee)$. It follows that $x_1^{\circ\circ} = x_2^{\circ\circ}$ and $[x_1) \cap L^\vee = [x_2) \cap L^\vee$. Now

$$\begin{aligned}
[x_1) &= [x_1^{\circ\circ} \wedge (x_1 \vee x_1^\circ))\\
&= [x_1^{\circ\circ}) \vee [x_1 \vee x_1^\circ)\\
&= [x_1^{\circ\circ}) \vee (L^\vee \cap [x_1 \vee x_1^\circ)) \text{ as } x_1 \vee x_1^\circ \in L^\vee\\
&= [x_1^{\circ\circ}) \vee (L^\vee \cap [x_1) \cap [x_1^\circ))\\
&= [x_2^{\circ\circ}) \vee (L^\vee \cap [x_2) \cap [x_2^\circ))\\
&= [x_2^{\circ\circ}) \vee (L^\vee \cap [x_2 \vee x_2^\circ))\\
&= [x_2^{\circ\circ}) \vee [x_2 \vee x_2^\circ) \text{ as } x_2 \vee x_2^\circ \in L^\vee\\
&= [x_2^{\circ\circ} \wedge (x_2 \vee x_2^\circ))\\
&= [x_2).
\end{aligned}$$

Consequently, $x_1 = x_2$ and f is injective. It remains to prove that f is surjective. Let $(x^{\circ\circ}, x^\circ\varphi(L) \vee [z)) \in L_1$, that is $z^{\circ\circ} = \max[z]\gamma(L) \in x^\circ\varphi(L) = [x^{\circ\circ}) \cap L^\vee$. Then by modularity of $F(L)$ we get

$$(x^{\circ\circ}, x^\circ\varphi(L) \vee [z)) = (x^{\circ\circ}, ([x^{\circ\circ}) \cap L^\vee) \vee [z)) = (x^{\circ\circ}, L^\vee \cap [x^{\circ\circ} \wedge z)).$$

Set $h = x^{\circ\circ} \wedge z$. Then $h^{\circ\circ} = x^{\circ\circ} \wedge z^{\circ\circ} = x^{\circ\circ}$ and consequently

$$(x^{\circ\circ}, x^{\circ}\varphi(L) \vee [z)) = (h^{\circ\circ}, [h) \cap L^{\vee}) = (h^{\circ\circ}, h^{\circ}\varphi(L) \vee [h \vee h^{\circ})) = hf.$$

Thus f is an isomorphism. □

5 Isomorphisms

In this section we define an isomorphism between two $\underline{K}_2$-quadruples and we show that two $\underline{K}_2$-algebras are isomorphic if and only if their associated $\underline{K}_2$-quadruples are isomorphic.

Definition 5.1 An isomorphism of the $\underline{K}_2$-quadruples (K, D, φ, γ) and $(K_1, D_1, \varphi_1, \gamma_1)$ is a pair (f, g), where f is an isomorphism of K and K_1, g is an isomorphism of D and D_1 such that $x \equiv y(\gamma)$ iff $xg \equiv yg(\gamma_1)$ for all $x, y \in D$ and the diagram

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, "f"'] & F(D) \arrow[d, "F(g)"] \\ K_1 \arrow[r, "\varphi_1"'] & F(D_1) \end{tikzcd}$$

commutes ($F(g)$ stands for the isomorphism of $F(D)$ and $F(D_1)$ induced by g).

Theorem 5.2 *Let $L, M \in \underline{\mathbf{K}}_2$. Then $L \cong M$ if and only if*

$$(L^{\circ\circ}, L^{\vee}, \varphi(L), \gamma(L)) \cong (M^{\circ\circ}, M^{\vee}, \varphi(M), \gamma(M)).$$

Proof Let $\theta \colon L \to M$ be an isomorphism. We have two isomorphisms, $f \colon L^{\circ\circ} \to M^{\circ\circ}$ defined by $xf = x\theta$ and $g \colon L^{\vee} \to M^{\vee}$ defined by $xg = x\theta$. Now define $F(g) \colon F(L^{\vee}) \to F(M^{\vee})$ by $AF(g) = \{a\theta \colon a \in A\}$.

For every $a \in L^{\circ\circ}$, we have

$$\begin{aligned} (af)\varphi(M) &= (a\theta)\varphi(M) = [(a\theta)^{\circ}) \cap M^{\vee}, \\ a\varphi(L)F(g) &= ([a^{\circ}) \cap L^{\vee})F(g) = \{y\theta \colon y \in [a^{\circ}) \cap L^{\vee}\} = [(a\theta)^{\circ}) \cap M^{\vee}. \end{aligned}$$

For $x, y \in L^{\vee}$, $x \equiv y(\gamma(L))$ iff $x^{\circ\circ} = y^{\circ\circ}$ iff $x^{\circ\circ}\theta = y^{\circ\circ}\theta$ iff $(xg)^{\circ\circ} = (x\theta)^{\circ\circ} = x^{\circ\circ}\theta = y^{\circ\circ}\theta = (y\theta)^{\circ\circ} = (yg)^{\circ\circ}$. Hence $xg \equiv yg(\gamma(M))$. Then (f, g) is a $\underline{K}_2$-quadruple isomorphism. Conversely, we have to show that the isomorphism (f, g) of $\underline{K}_2$-quadruples $(L^{\circ\circ}, L^{\vee}, \varphi(L), \gamma(L))$ and $(M^{\circ\circ}, M^{\vee}, \varphi(M), \gamma(M))$ implies the existence of an isomorphism $h \colon L \to M$, between $\underline{K}_2$-algebras L, M constructed by $\underline{K}_2$-construction. We claim that

$$(a, a^{\circ}\varphi(L) \vee [x))h = (af, (af)^{\circ}\varphi(M) \vee [xg))$$

is the desired isomorphism. Firstly we note that

$$(\max[x]\gamma(L))g = \max[xg]\gamma(M) \text{ for all } x \in L^{\vee}$$

Then

$$\max[xg]\gamma(M) = (\max[x]\gamma(L))g \in (a^\circ\varphi(L))F(g) = (af)^\circ\varphi(M)$$

as $\max[x]\gamma(L) \in a^\circ\varphi(L)$. Hence h is well defined.

Since f and $F(g)$ are isomorphisms, then we get

$$\begin{aligned}
&(a, a^\circ\varphi(L) \vee [x)) \leq (b, b^\circ\varphi(L) \vee [y)) \\
&\qquad \Leftrightarrow a \leq b, a^\circ\varphi(L) \vee [x) \supseteq b^\circ\varphi(L) \vee [y) \\
&\qquad \Leftrightarrow af \leq bf, (a^\circ\varphi(L) \vee [x))F(g) \supseteq (b^\circ\varphi(L) \vee [y))F(g) \\
&\qquad \Leftrightarrow af \leq bf, (a^\circ\varphi(L))F(g) \vee [x)F(g) \supseteq (b^\circ\varphi(L))F(g) \vee [y)F(g) \\
&\qquad \Leftrightarrow af \leq bf, (af)^\circ\varphi(M) \vee [xg) \supseteq (bf)^\circ\varphi(M) \vee [yg) \\
&\qquad \Leftrightarrow (af, (af)^\circ\varphi(M) \vee [xg)) \leq (bf, (bf)^\circ\varphi(M) \vee [yg)) \\
&\qquad \Leftrightarrow (a, a^\circ\varphi(L) \vee [x))h \leq (b, b^\circ\varphi(L) \vee [y))h.
\end{aligned}$$

Thus, since h is a bijection, h is an isomorphism. □

In a subsequent paper, we shall consider homomorphisms, subalgebras and congruence pairs of $\underline{K}_2$-algebras.

Acknowledgement The author would like to thank the referee for his/her useful comments and valuable suggestions given to this paper.

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