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PRINCIPAL BLOCKS AND p-RADICAL GROUPS

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Abstract. Let G be a finite group and k a field of characteristic $p > 0$. In this paper, we obtain several equivalent conditions to determine whether the principal block B_0 of a finite p-solvable group G is p-radical, which means that B_0 has the property that $e_0(k_P)^G$ is semisimple as a kG-module, where P is a Sylow p-subgroup of G, k_P is the trivial kP-module, $(k_P)^G$ is the induced module, and e_0 is the block idempotent of B_0. We also give the complete classification of a finite p-solvable group G which has not more than three simple B_0-modules where B_0 is p-radical.

Keywords: principal block; p-radical group; p-radical block

MSC 2010: 20C05, 20C20

1. Introduction

Let G be a finite group and k a field of characteristic $p > 0$. Let P be a Sylow p-subgroup of G. In [15], Motose and Ninomiya first introduced the definition of p-radical groups. Namely, G is a p-radical group if the induced module $(k_P)^G$ of the trivial kP-module k_P is semisimple as a left kG-module. From the definition, it can be easily seen that G is a p-radical group if G is a p-group or a p'-group. In [19], Okuyama states that any p-radical group is p-solvable, so the discussion of p-radical groups can be carried out in finite p-solvable groups. In [12], Koshitani obtained a sufficient condition on p-radical groups, that is, if the vertex of V is contained in $\mathrm{Ker}(V)$ for any simple kG-module V then G is p-radical, where $\mathrm{Ker}(V) = \{x \in G\colon xv = v, \text{for any } v \in V\}$. Tsushima [23] discussed the relationship between p-nilpotent groups and p-radical groups and asserted that, if G has an abelian p-complement,

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then G is p-radical. However, we cannot say that any p-nilpotent group is p-radical. For instance, if $p = 3$ and $G = \mathrm{SL}(2,3)$, then G is 3-nilpotent but not 3-radical (see [15], Remark 2). Thus a much more profound theorem of Tsushima [23] claims that a p-nilpotent group G is p-radical if and only if $[O_{p'}(G), D] \cap C_{O_{p'}(G)}(D) = 1$ for any p-subgroup D of G. In order to prove this theorem, Tsushima generalized the concept of p-radical groups to a p-block form. Let B be a p-block of G and e_B be the block idempotent in kG corresponding to B. Then B is called a p-radical block if $e_B(k_P)^G$ is semisimple.

Evidently, G is p-radical if and only if every block of G is p-radical, so the study of p-radical blocks significantly assists the study of p-radical groups. In [5], Hida generalized Koshitani's result (see [12]) to a p-block version and claimed that, if the vertex of V is contained in $\mathrm{Ker}(V)$ for any simple kG-module V in a p-block B of G, then B is p-radical.

As we all know, the principal block of a group algebra plays an important role in the study of the theory of modular representations of finite groups. This prompts us to investigate when the principal block of a finite group is p-radical. We will give the definition of principal p-radical groups as follows:

Definition 1.1. G is called a principal p-radical group if the principal block of G is p-radical.

In this paper, we focus on the connection between p-radical groups and principal p-radical groups in finite p-solvable groups. Use the properties of p-constrained groups (see [6], Chapter 7, Theorem 13.6), the principal block B_0 of G is isomorphic with the group algebra $k(G/O_{p'}(G))$, which is due to [3]. We can prove the following theorem.

Theorem 1.2. *Let G be a finite group. Then the following results hold.*

(i) *If G is principal p-radical, and if $G/O_{p',p}(G)$ is principal p-radical, then G is p-solvable.*
(ii) *If G is p-solvable, then the following conditions are equivalent:*
  (a) *G is principal p-radical.*
  (b) *$G/O_{p'}(G)$ is p-radical.*
  (c) *$G/O_{p',p}(G)$ is p-radical.*
(iii) *G is a p-solvable and principal p-radical group if and only if every simple kG-module S in the principal block of G satisfies the following property* (**P**)*:*
  (**P**) *There exist a subgroup H of G and a simple kH-module U such that*
  (1) *$S = U^G$ and some vertex D of S is contained in* $\mathrm{Ker}(U)$,
  (2) *$H \cap P^g \in \mathrm{Syl}_p(H)$ for every $g \in G$.*

Corollary 1.3. *Let G be p-solvable and principal p-radical, and let S, H, U, and D be as in Theorem* 1.2 (iii). *Then*
(i) $(U_{H\cap N_G(D)})^{N_G(D)}$ *is the Green correspondent of S with respect to* $(G, D, N_G(D))$,
(ii) *if* $D \subset \mathrm{Ker}(S)$, *then* $S_{N_G(D)} = (U_{H\cap N_G(D)})^{N_G(D)}$ *is a simple $kN_G(D)$-module.*

Corollary 1.4. *If the vertex D of S is contained in* $\mathrm{Ker}(S)$ *for any simple kG-module S in the principal block of G, then G is principal p-radical, and $S_{N_G(D)}$ is a simple $kN_G(D)$-module.*

We can easily prove that every p-nilpotent group is principal p-radical. The class of p-radical groups is properly contained in the class of principal p-radical groups (see Remark 3.2). We also give an example (Example 3.7(ii)) to show that there exists a p-solvable group G such that G is not principal p-radical. This allows us to consider when p-solvable groups are principal p-radical. Let $l(B)$ be the number of non-isomorphic simple B-modules and B_0 be the principal block of G. We obtain the following theorem.

Theorem 1.5. *Let G be a p-solvable group with $l(B_0) \leqslant 3$. Then G is principal p-radical except for the following situations:*

(i) $p = 3$ *and* $G/O_{p',p}(G) \cong \mathrm{SL}(2, 3)$,
(ii) $p = 2$ *and* $G/O_{p',p}(G) \cong M(3) \rtimes P$, *where P is $\mathbb{Z}_8$ or* SD_{16}, *where* $M(3) = \langle x, y\colon x^3 = y^3 = z^3 = 1,\ y^x = yz,\ z^x = z,\ z^y = z\rangle$, *the non-abelian* 3*-group which is of order* 27 *and has exponent* 3, *and* SD_{16} *denotes the semi-dihedral group of order* 16. *In particular, if $l(B_0) \leqslant 2$, then G is principal p-radical.*

All groups in this paper are finite and all modules are finitely generated left modules. Furthermore, $\mathbb{Z}_n$ denotes the cyclic group of order n. E_{p^n} is the elementary abelian group of order p^n. Further Q_8 denotes the quaternion group of order 8, S_n is the symmetric group of degree n. Let q be a prime. Following [21], page 229, we define $T_0(q)$ to be the subgroup of $\mathrm{GL}(2, q)$ consisting of the matrices $\left(\begin{smallmatrix} x & 0 \\ 0 & \pm x^{-1}\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & x \\ \pm x^{-1} & 0\end{smallmatrix}\right)$, $x \in \mathrm{GF}(q)$, $x \neq 0$. $\mathcal{J}(kG)$ denotes the Jacobson radical of the group algebra kG. Let B_0 be the principal block of G and e_0 be the block idempotent in kG corresponding to B_0. Let $\Delta(G)$ be the augmentation ideal of kG. For a subset S of G, $\widehat{S}$ denotes the sum of all elements of S. If T is a subset of kG, we write $r(T) = r_G(T)$ and $l(T) = l_G(T)$ for the right and left annihilators of T in kG. The notation and terminology undefined are standard, the reader is referred to [1] and [4].

Let $\mathcal{A}(G) = \{H \subset G\colon \mathcal{J}(B_0) \subset kG \cdot \mathcal{J}(kH)\}$, let $\mathcal{B}(G) = \{H \subset G\colon$ the induced kG-module e_0W^G is semisimple for every simple kH-module $W\}$ and let $\mathcal{C}(G) = \{H \subset G\colon H$ contains a Sylow p-subgroup of $G\}$.

Lemma 2.1. $\mathcal{A}(G) = \mathcal{B}(G) \subset \mathcal{C}(G)$.

Proof. Following [10], Lemma 1.5, $\mathcal{A}(G) = \mathcal{B}(G)$. We need only verify that $\mathcal{B}(G) \subset \mathcal{C}(G)$. If $H \in \mathcal{B}(G)$, by [11], Theorem 2.2, we may choose a simple kG-module V in B_0 such that some Sylow p-subgroup P is a vertex of V. Let W be a simple submodule of V_H. Since $e_0V = V$, we have $0 \neq \mathrm{Hom}_{kH}(W, V_H) \cong \mathrm{Hom}_{kG}(W^G, V) \cong \mathrm{Hom}_{kG}(e_0W^G, V)$. Hence, V is a direct summand of W^G since e_0W^G is semisimple. It follows that V is H-projective, and so $P \subset_G H$. Thus $H \in \mathcal{C}(G)$. □

Remark 2.2. For an arbitrary extension field K of k there holds $\mathcal{J}(KG) = K \otimes_k \mathcal{J}(kG)$ and $\mathcal{J}(kG) = \mathcal{J}(KG) \cap kG$ since $kG/\mathcal{J}(kG)$ is a separable algebra. Thus it easily holds that $\mathcal{J}(\widetilde{B}_0) \subset KG \cdot \mathcal{J}(KH)$ if and only if $\mathcal{J}(B_0) \subset kG \cdot \mathcal{J}(kH)$ for any subgroup H of G, where $\widetilde{B}_0$ and B_0 are the principal blocks of KG and kG, respectively. This means that $\mathcal{A}(G)$ is determined by G and p.

Lemma 2.3 ([8], Chapter 5, Lemma 4.3)**.** *Let P be a Sylow p-subgroup of G. Then*

(i) $\bigcap\limits_{x\in G} kG \cdot \Delta(P^x) = \bigcap\limits_{x\in G} \Delta(P^x) \cdot kG$ *is a nilpotent ideal of* kG,

(ii) $$\bigcap_{x\in G} kG \cdot \Delta(P^x) = \Big\{\sum_{x\in G} u_x x\colon \sum_{y\in S} u_{xy} = 0 \text{ for all } x \in G \text{ and all } S \in \mathrm{Syl}_p(G)\Big\}$$ $$= \Big\{\sum_{x\in G} u_x x\colon \sum_{y\in S} u_{yx} = 0 \text{ for all } x \in G \text{ and all } S \in \mathrm{Syl}_p(G)\Big\}.$$

In the following theorem, we give some characterizations for principal p-radical groups.

Theorem 2.4. *The following conditions are equivalent:*

(i) *G is a principal p-radical group.*

(ii) $\mathcal{A}(G) = \mathcal{B}(G) = \mathcal{C}(G)$.

(iii) $\mathcal{J}(B_0) \subset kG \cdot \Delta(P)$ *for some (and hence all)* $P \in \mathrm{Syl}_p(G)$.

(iv) $\mathcal{J}(B_0) \subset \bigcap\limits_{S\in\mathrm{Syl}_p(G)} kG \cdot \Delta(S)$.

(v) $l_G(\mathcal{J}(B_0)) \supset \sum\limits_{S\in\mathrm{Syl}_p(G)} \widehat{S} \cdot kG$.

(vi) $\mathcal{J}(B_0) \subset \Big\{\sum\limits_{x\in G} u_x x\colon \sum\limits_{y\in S} u_{xy} = 0$ *for all* $x \in G$ *and all* $S \in \mathrm{Syl}_p(G)\Big\}$.

P r o o f. The equivalence of (iv) and (vi) follows from Lemma 2.3.

(i) $\Leftrightarrow$ (iii): By Lemma 2.1, $e_0(k_P)^G$ is semisimple if and only if $\mathcal{J}(B_0) \subset kG\cdot\Delta(P)$ for $P \in \mathrm{Syl}_p(G)$.

(ii) $\Rightarrow$ (iii): For any $P \in \mathrm{Syl}_p(G)$, since P is of p'-index, then by hypothesis $\mathcal{J}(B_0) \subset kG \cdot \Delta(P)$.

(iii) $\Rightarrow$ (iv): If $\mathcal{J}(B_0) \subset kG \cdot \Delta(P)$ for some $P \in \mathrm{Syl}_p(G)$, then, for all $x \in G$,

$$\mathcal{J}(B_0) = \mathcal{J}(B_0)^x \subset kG \cdot \Delta(P)^x = kG \cdot \Delta(P^x),$$

and hence $\mathcal{J}(B_0) \subset \bigcap\limits_{S\in\mathrm{Syl}_p(G)} kG \cdot \Delta(S)$.

(iv) $\Rightarrow$ (ii): Let $H \in \mathcal{C}(G)$. Then H contains a Sylow p-subgroup P of G. Let $G = \bigcup\limits_{i=1}^{n} x_i H$ be a left coset decomposition of G over H. Then we have $\mathcal{J}(B_0) \subset \bigcap\limits_{x\in G} kG \cdot \Delta(P^x) \subset \bigcap\limits_{x\in H} kG \cdot \Delta(P^x) = \bigcap\limits_{x\in H}\Bigl(\sum\limits_{i=1}^{n} x_i kH \cdot \Delta(P^x)\Bigr) = \sum\limits_{i=1}^{n} x_i\Bigl(\bigcap\limits_{x\in H} kH \times \Delta(P^x)\Bigr) \subset kG \cdot \Bigl(\bigcap\limits_{x\in H} kH \cdot \Delta(P^x)\Bigr) \subset kG \cdot \mathcal{J}(kH)$. Hence $H \in \mathcal{A}(G)$, and so the desired conclusion follows by virtue of Lemma 2.1.

(iv) $\Rightarrow$ (v): $l_G(\mathcal{J}(B_0)) \supset l_G\Bigl(\bigcap\limits_{S\in\mathrm{Syl}_p(G)} kG \cdot \Delta(S)\Bigr) = \sum\limits_{S\in\mathrm{Syl}_p(G)} r_G(kG \cdot \Delta(S)) = \sum\limits_{S\in\mathrm{Syl}_p(G)} r_G(l_G(\widehat{S})) = \sum\limits_{S\in\mathrm{Syl}_p(G)} r_G(l_G(\widehat{S} \cdot kG)) = \sum\limits_{S\in\mathrm{Syl}_p(G)} \widehat{S} \cdot kG$.

(v) $\Rightarrow$ (iv): $\mathcal{J}(B_0) = l_G(r_G(\mathcal{J}(B_0))) \subset l_G\Bigl(\sum\limits_{S\in\mathrm{Syl}_p(G)} \widehat{S} \cdot kG\Bigr) = \bigcap\limits_{S\in\mathrm{Syl}_p(G)} l_G(\widehat{S} \times kG) = \bigcap\limits_{S\in\mathrm{Syl}_p(G)} l_G(\widehat{S}) = \bigcap\limits_{S\in\mathrm{Syl}_p(G)} kG \cdot \Delta(S)$. Hence, the result follows. $\square$

Let N be a normal subgroup of G and let $\nu\colon\, G \to G/N$ be the natural homomorphism. Then $\nu^*\colon\, kG \to k(G/N)$ in the algebra homomorphism induced by ν and the kernel of this homomorphism is $\mathrm{Ker}(\nu^*) = kG \cdot \Delta(N)$. A group G is called p-constrained if $C_G(O_{p',p}(G)/O_{p'}(G)) \subset O_{p',p}(G)$. It is well-known (see [4], pages 268–270, or [6], Chapter 7, Definition 13.3) that any p-solvable group is p-constrained.

Lemma 2.5 ([3], Theorem 2.1)**.** *Let G be p-constrained. Then kG is indecomposable if and only if $O_{p'}(G) = 1$.*

Lemma 2.6 ([6], Chapter 7, Theorem 13.6)**.** *Let G be p-constrained, let $N = O_{p'}(G)$ and $e = |N|^{-1}\widehat{N}$. Then $kGe \cong k(G/N)$ is the principal block of G and $\nu^*(e) = 1$, where $\nu^*\colon\, kG \to k(G/N)$ induced by the natural homomorphism $\nu\colon\, G \to G/N$.*

We now state some preliminary results on p-radical groups.

Lemma 2.7 ([1], Chapter 6, Theorem 6.5)**.** *Assume that $N \lhd G$. Then the following statements hold.*

(i) *If G is p-radical, so are N and G/N.*
(ii) *If N is a p-group, then G is p-radical if and only if G/N is p-radical.*
(iii) *If G/N is a p'-group, then G is p-radical if and only if N is p-radical.*

Lemma 2.8 ([23], Theorem 2)**.** *Let G be a p-nilpotent group. Then G is p-radical if and only if $[O_{p'}(G), D] \cap C_{O_{p'}(G)}(D) = 1$ for any p-subgroup D of G.*

3. Proof of Theorem 1.2

The proof of (i) is inspired by [19]. Let S be a simple kG-module in B_0 and Q be its vertex. Let H be a subgroup of G. By [1], Chapter 3, Lemma 4.9; Chapter 2, Lemma 3.7, then

$$\mathrm{Tr}_H^G(\mathrm{Hom}_{kH}(S,S)) = \begin{cases} \mathrm{Hom}_{kG}(S,S), & Q \subset_G H, \\ 0, & \text{otherwise.} \end{cases}$$

Since G is principal p-radical, $e_0(k_P)^G$ is semisimple. From [23], Lemma 2, it follows that

$$\mathrm{Tr}_H^G(\mathrm{Hom}_{kH}(e_0(k_P)^G,S)) = \begin{cases} \mathrm{Hom}_{kG}(e_0(k_P)^G,S), & Q \subset_G H, \\ 0, & \text{otherwise.} \end{cases}$$

By Mackey decomposition theorem, we have that k_Q is a trivial source module of S. Let U be an indecomposable direct summand of S_P, it follows that

$$U|S_P|((k_Q)^G)_P = \bigoplus_{t\in Q\backslash G/P} (k^t_{Q^t\cap P})^P.$$

This implies that $U \cong (k_{Q^t\cap P})^P$ for some $t \in Q\backslash G/P$. By [1], Chapter 2, Lemma 2.5 and 3.5, we have $\mathrm{Tr}^G_{Q^t\cap P}(\mathrm{Hom}_{k(Q^t\cap P)}(e_0(k_P)^G, S)) \neq 0$. Hence $Q^x = Q^t \cap P$ for some $x \in G$. This implies $S_P = \oplus(k_{Q^x})^P$ for some x with $Q^x \subset P$. By [9], Corollary 3.6, there exists a block b of $Q^xC_G(Q^x)$ such that Q^x is a defect group of b and $b^G = B_0$. Since B_0 is the principal block of G, Brauer's third main theorem implies that b is the principal block of $Q^xC_G(Q^x)$. It follows that Q^x is the unique Sylow p-subgroup of $Q^xC_G(Q^x)$. Therefore, $Z(P) \subset Q^x$ as $Q^x \subset P$. This proves that $Z(P) \subset \mathrm{Ker}(S)$. By [1], Chapter 4, Lemma 4.12, $1 \neq Z(P) \subset O_{p',p}(G)$. Hence G is p-solvable by induction.

For (ii), (a) $\Rightarrow$ (b): Let $N = O_{p'}(G)$ and let $\overline{G} = G/N$. Let $\nu\colon G \to \overline{G}$ be the natural homomorphism, and $\nu^*\colon kG \to k\overline{G}$ the algebra homomorphism induced by ν. Since G is p-solvable, G is p-constrained. Thus, we have $B_0 \cong k\overline{G}$ and $\nu^*(e_0) = 1$ by Lemma 2.6. Let $P \in \mathrm{Syl}_p(G)$. If B_0 is p-radical, then $\mathcal{J}(B_0) \subset kG \cdot \Delta(P)$. It follows that $\nu^*(\mathcal{J}(B_0)) \subset \nu^*(kG \cdot \Delta(P))$. This implies that $\mathcal{J}(k\overline{G}) \subset k\overline{G} \cdot \Delta(\overline{P})$.

(b) $\Rightarrow$ (a): Assume that $\overline{G}$ is p-radical. Then $\mathcal{J}(k\overline{G}) \subset k\overline{G} \cdot \Delta(\overline{P})$, and thus

$$\mathcal{J}(B_0) = \mathcal{J}(kG)e_0 \subset kG \cdot \Delta(P) + \mathrm{Ker}(\nu^*) = kG \cdot \Delta(P) + kG(1 - e_0),$$

and

$$\mathcal{J}(B_0) \subset (kG \cdot \Delta(P) + kG(1 - e_0))e_0 \subset kG \cdot \Delta(P)e_0 \subset kG \cdot \Delta(P).$$

The result follows from Theorem 2.4.

(b) $\Leftrightarrow$ (c): The equivalence of (b) and (c) follows from Lemma 2.7.

For (iii), assume that G is a p-solvable and principal p-radical group. By [24], Theorem 3, then if S is a simple kG-module with $S \in B_0$, there exist a subgroup H of G and a simple kH-module U such that $S = U^G$ and $\dim_k(U)$ is a p'-number. By [23], Lemma 2, and Fong's dimension formula [2], Theorem (2B), we have $\dim_k(\mathrm{Hom}_{kG}(S, (k_P)^G)) = \dim_k(\mathrm{Hom}_{kG}(S, e_0(k_P)^G)) = \dim_k(S)_{p'}$, the p'-part of $\dim_k(S)$, since $e_0(k_P)^G$ is semisimple. The result follows from [13], Lemma 4.

Conversely, assume that every simple kG-module in B_0 satisfies the property (**P**). Then G is p-solvable by [13], Lemma 2. By hypothesis, there exist a subgroup H of G and a simple kH-module U such that $H \cap P^g \in \mathrm{Syl}_p(H)$ for every $g \in G$, $S = U^G$, and some vertex D of S is contained in $\mathrm{Ker}(U)$. We may assume without loss of generality that $D \subset P$. Since H is p-solvable, there exist a subgroup K of H and a simple kK-module W such that $U = W^H$ and $\dim_k(W)$ is a p'-number by [24], Theorem 3. For any $g \in G$, we can find $x \in H$ such that $H \cap P^g = (H \cap P)^x$. Since $\mathrm{Ker}(U) \subset \mathrm{Ker}(W)$, we have $D^x \subset \mathrm{Ker}(W) \cap (H \cap P)^x = \mathrm{Ker}(W) \cap H \cap P^g \subset K \cap H \cap P^g = K \cap P^g$. Further [16], Chapter 4, Lemma 3.4 and Theorem 7.8, imply that D is a vertex of W since $S = W^G$. By [16], Chapter 4, Theorem 7.5, then $D \in \mathrm{Syl}_p(K)$ because $\dim_k(W)$ is a p'-number. Hence $K \cap P^g \in \mathrm{Syl}_p(K)$. By [13], Lemma 4, we have $\dim_k(\mathrm{Hom}_{kG}(S, e_0(k_P)^G)) = \dim_k(\mathrm{Hom}_{kG}(S, (k_P)^G)) = \dim_k(S)_{p'}$. It follows that $\dim_k(\mathrm{Hom}_{kG}(S, \mathrm{Soc}(e_0(k_P)^G))) = \dim_k(S)_{p'}$ since $\mathrm{Hom}_{kG}(S, e_0(k_P)^G) = \mathrm{Hom}_{kG}(S, \mathrm{Soc}(e_0(k_P)^G))$. By [23], Lemma 2, and Fong's dimension formula [2], Theorem (2B), we have that $e_0(k_P)^G$ is semisimple, as required. $\square$

P r o o f of Corollary 1.3. For (i), let $\widetilde{S} = (U_{H \cap N_G(D)})^{N_G(D)}$. By Mackey decomposition theorem,

$$\widetilde{S}_D = \bigoplus_{t \in H \cap N_G(D) \backslash N_G(D)/D} (U^t_{D \cap H^t \cap N_G(D)^t})^D = \bigoplus_t (U^t_{D \cap H^t})^D.$$

Since $D \lhd N_G(D)$ and $D \subset \mathrm{Ker}(U)$, we have $D \subset \mathrm{Ker}(\widetilde{S})$. By [16], Chapter 4, Lemma 3.4 and Theorem 7.8, then D is a vertex of every indecomposable direct summand of $\widetilde{S}$ since D is a vertex of S and $\widetilde{S} | S_{N_G(D)}$. It can be easily proved that $\widetilde{S}$ is the Green correspondent of S with respect to $(G, D, N_G(D))$ by Green's theorem [16], Chapter 4, Theorem 4.3.

For (ii), by Green's theorem, we have $S_{N_G(D)} = \widetilde{S} \oplus (\oplus U_i)$ for indecomposable $kN_G(D)$-module U_i such that U_i is $\mathcal{X}$-projective for all i, where $\mathcal{X} = \{H\colon H$ is a subgroup of $D^x \cap N_G(D)$ for some $x \in G - N_G(D)\}$. By [1], Chapter 3, Lemma 4.12, there exist no such U_i's since $D \subset \mathrm{Ker}(S)$. Hence $S_{N_G(D)} = (U_{H \cap N_G(D)})^{N_G(D)}$. Since G is principal p-radical, it follows by Mackey decomposition theorem that S is a trivial source module. The result follows from [20], Lemma 2.2. □

Remark 3.1. Using Theorem 1.2 (ii), obviously, if $l_p(G) = 1$, then G is principal p-radical.

Remark 3.2. Note that every p-radical group is principal p-radical. It is therefore appropriate to ask, whether any principal p-radical group is p-radical. The answer is no. The following example is due to Saksonov [22]. If $p = 3$ and $G = \mathrm{SL}(2,3)$, then G is 3-nilpotent but not 3-radical (see [15], Remark 2). But, by Remark 3.1, G is principal 3-radical.

Following Theorem 1.2, we can formulate several sufficient conditions for principal p-radical groups.

Proof of Corollary 1.4. The results follow from Theorem 1.2 (iii) and Corollary 1.3. □

Corollary 3.3. *If all simple kG-modules belonging to B_0 have k-dimension 1, then G is principal p-radical.*

Proof. This follows by [14], Theorem 6, and Remark 3.2. □

Corollary 3.4. *If the principal block B_0 of G satisfies $\mathcal{J}(B_0)^2 = 0$, then G is principal p-radical.*

Proof. The result follows by [26], Theorem. □

Assuming that G is a p-solvable and principal p-radical group, we give a group theoretical characterization of G.

Proposition 3.5. *If G is a p-solvable and principal p-radical group, then $k(G/O_{p'}(G))$ has no blocks of defect zero if and only if each pair of Sylow p-subgroups of $G/O_{p'}(G)$ has a nontrivial intersection. In particular, if $O_{p'}(G) = 1$, then the conclusion holds for G.*

Proof. By Theorem 1.2 (ii), we have that $\overline{G} = G/O_{p'}(G)$ is p-radical. From [15], Theorem 10, it follows that $k\overline{G}$ has no blocks of defect zero if and only if each pair of Sylow p-subgroups of $\overline{G}$ has a nontrivial intersection. The proof is completed. $\square$

Remark 3.6. Assume that G is a p-solvable and principal p-radical group. It is not necessarily true that each pair of Sylow p-subgroups of G has nontrivial intersection. For example, if $p = 2$ and $G = S_3$, then G is principal 2-radical by Theorem 1.2 (ii) since G is 2-nilpotent. But the intersection of different Sylow 2-subgroups is trivial.

Let $\mathcal{D}$ be the set of all p-nilpotent groups, $\mathcal{E}$ be the set of all p-solvable groups, $\mathcal{F}$ be the set of all p-constrained groups, and $\mathcal{G}$ be the set of all principal p-radical groups. We have the following example.

Example 3.7. (i) $\mathcal{D} \subsetneq \mathcal{G}$: By Remark 3.1, we just need to find a group G with $G \in \mathcal{G} - \mathcal{D}$. Let $p = 2$ and $G = G_{48}$ (see [7], Chapter 12, Definition 8.4, and Lemma 4.7). Then G is 2-solvable but not 2-nilpotent. Since $O_2(G) \cong Q_8$ and $G/O_2(G) \cong S_3$, we have that G is 2-radical by Lemma 2.7. Therefore, $G \in \mathcal{G}$.

(ii) $\mathcal{E} \nsubseteq \mathcal{G}$, $\mathcal{F} \nsubseteq \mathcal{G}$: Let $p = 3$ and $G = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathrm{SL}(2,3)$, where the semidirect product is with respect to the canonical homomorphism $\mathrm{SL}(2,3) \subset \mathrm{GL}(2,3) \cong \mathrm{Aut}(\mathbb{Z}_3 \times \mathbb{Z}_3)$. Thus G is 3-solvable and $O_{3'}(G) = 1$. By Theorem 1.2 (ii), if G is principal 3-radical, then G is 3-radical. From Lemma 2.7, we have that $\mathrm{SL}(2,3)$ is 3-radical. That is a contradiction (see Remark 3.2).

Proposition 3.8. *If $G \in \mathcal{F}$ and $G \notin \mathcal{E}$, then*

(i) $G/O_{p'}(G) \notin \mathcal{G}$,

(ii) $G \notin \mathcal{G}$.

Proof. (i) Assume that $G/O_{p'}(G) \in \mathcal{G}$. Since $G \in \mathcal{F}$, we have that $G/O_{p'}(G) \in \mathcal{F}$. Thus $G/O_{p'}(G)$ is p-radical by Theorem 1.2 (ii). From [19], Theorem 1, it follows that $G/O_{p'}(G)$ is p-solvable. Therefore $G \in \mathcal{E}$ is a contradiction.

(ii) The required assertion is a consequence of (i) and Theorem 1.2 (ii). $\square$

4. Proof of Theorem 1.5

The proof of Theorem 1.5 relies on Ninomiya's classification theorem (see [18], Theorem A and Theorem B, and [17], Theorem). By Lemma 2.6, $B_0 \cong k(G/O_{p'}(G))$. It follows that $l(B_0)$ is equal to the number of p-regular classes of $G/O_{p',p}(G)$. Thus, following Theorem 1.2 (ii), we just need to determine whether $G/O_{p',p}(G)$ is p-radical. Consider the case when $l(B_0) \leqslant 2$, and we have the following lemma.

Lemma 4.1. *If $l(B_0) \leqslant 2$, then G is principal p-radical.*

P r o o f. If $l(B_0) = 1$, then the conclusion follows directly by Corollary 3.3. If $l(B_0) = 2$, by [18], Theorem A, $G/O_{p',p}(G)$ has an abelian p-complement. Thus $G/O_{p',p}(G)$ is p-radical from [23], Proposition 2. This proves our conclusion. $\square$

In the proof of Lemma 4.1, we can see that if $l(B_0) \leqslant 3$ and $G/O_{p',p}(G)$ has an abelian p-complement, then G is principal p-radical. Using [18], Theorem B, and [17], Theorem, we shall consider the following ten cases:

(I) $p = 3$ and $G_1 = \mathrm{SL}(2,3)$,

(II) $p = 2$ and $G_2 = M(3) \rtimes P$, where P is $\mathbb{Z}_8$ or SD_{16},

(III) $p \neq 2$ and $G_3 = \mathbb{Z}_r \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_{p^n})$, where $r = 2p^n + 1$ is a prime,

(IV) $p \neq 2,3$ and $G_4 = E_{3^l} \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_{p^n})$, where $3^l = 2p^n + 1$,

(V) $p = 2$ and $G_5 = E_{5^2} \rtimes H$, where $H = \langle w, a \rangle$; $w^3 = a^8 = 1$, $a^{-1}wa = w^{-1}$,

(VI) $p = 2$ and $G_6 = E_{5^2} \rtimes H$, where $H = \langle w, a, b \rangle$; $w^3 = a^8 = b^2 = 1$, $a^{-1}wa = w$, $b^{-1}wb = w^{-1}$, $b^{-1}ab = a^5$,

(VII) $p = 2$ and $G_7 = E_{3^4} \rtimes H$, where $H = \langle w, a, b \rangle$; $w^5 = a^8 = 1$, $b^4 = a^4$, $a^{-1}wa = w$, $b^{-1}wb = w^2$, $b^{-1}ab = a^3$,

(VIII) $p = 2$ and $G_8 = E_{3^4} \rtimes H$, where $H = \langle w, a, b \rangle$; $w^5 = a^{16} = b^4 = 1$, $a^{-1}wa = w$, $b^{-1}wb = w^2$, $b^{-1}ab = a^{11}$,

(IX) $p = 2$ and $G_9 = E_{7^2} \rtimes T$, where $T = \langle R, w, x \rangle$; $Q_8 \cong R \lhd T$, $w^3 = x^4 = 1$, $x^2 \in R$, $x^{-1}wx = w^{-1}$,

(X) $p = 2$ and $G_{10} = E_{5^2} \rtimes T$, where $T = \langle R, w, x \rangle$; $T_0(5) \cong R \lhd T$, $w^3 = x^8 = 1$, $x^2 \in R$, $x^{-1}wx = w^{-1}$.

Therefore, we shall check the situations (I)–(X) and determine whether G_i ($i = 1, \ldots, 10$) is p-radical. For this purpose, we have to prove the following lemma.

Lemma 4.2. *Let G be a p-solvable group of p-length 1. Then G is p-radical if and only if $[O_{p'}(G), D] \cap C_{O_{p'}(G)}(D) = 1$ for any p-subgroup D of G.*

P r o o f. Since $l_p(G) = 1$, then $O_{p',p}(G)$ is p-nilpotent and is of p'-index in G. It follows that $O_{p'}(O_{p',p}(G)) = O_{p'}(G)$. By Lemma 2.7, we have that G is p-radical if and only if $O_{p',p}(G)$ is p-radical. The proof of the lemma is completed by Lemma 2.8. $\square$

Lemma 4.3. *G_1 and G_2 are not p-radical.*

P r o o f. By Remark 3.2, $G_1 = \mathrm{SL}(2,3)$ is not 3-radical. For Case (II), since $\mathrm{Aut}(M(3)) \cong E_{3^2} \rtimes \mathrm{GL}(2,3)$, we may regard P as a subgroup of $\mathrm{GL}(2,3)$ (see the proof of Proposition 3.3 in [18]). Let $M(3) = \langle x, y \colon x^3 = y^3 = z^3 = 1,\ y^x = yz,\ z^x = z,\ z^y = z \rangle$. Then there exists $t \in P$ such that $t^{-1}zt = z$. In fact, since all cyclic subgroups of order 8 or all semi-dihedral subgroups of order 16 in $\mathrm{GL}(2,3)$ are conjugate to each other, we may without loss of generality assume that $P = \left\langle \left(\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right) \right\rangle$ or $\left\langle \left(\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right) \right\rangle$, and we can choose $t = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right) \in P$ with $t^{-1}zt = z$. Let $H = M(3) \rtimes \langle t \rangle$. Then $H \lhd G_2$, and if G_2 is 2-radical, then H is 2-radical from Lemma 2.7. Since $H' = M(3)$, we know that $H = H'\langle t \rangle$. Moreover, since $z \in N_H(\langle t \rangle)$, H is not a Frobenius group with $\langle t \rangle$ as a complement. Therefore we have $\mathcal{J}(kH) \nsubseteq Z(kH)$ from [25], Theorem A. Using the definition of p-radical groups, it suffices to show that $I = \bigcap\limits_{h \in H} kH \cdot \Delta(\langle t \rangle^h) \subset Z(kH)$. Let U be the set of all elements of H of order 2. Since $I \subset kH(u-1)$ for any $u \in U$, we have $I(u-1) \subset kH(u-1)^2 = 0$. Obviously, $\{t, xt, yt\} \subset U$ and $H = \langle U \rangle$. It follows that $\Delta(H) = \sum\limits_{u \in U} (u-1)kH$. This implies $I \cdot \Delta(H) = 0$, and thus $I \subset l_H(\Delta(H)) = k\widehat{H} \subset Z(kH)$. This leads to a contradiction. $\square$

Lemma 4.4. *G_3 and G_4 are p-radical.*

Proof. For Case (III), let $G_3 = \langle a \rangle \rtimes (\langle b \rangle \times \langle c \rangle)$, where $\langle a \rangle \cong \mathbb{Z}_r$, $\langle b \rangle \cong \mathbb{Z}_2$, $\langle c \rangle \cong \mathbb{Z}_{p^n}$. Then we have $O_{p'}(G_3) = \langle a \rangle \rtimes \langle b \rangle$, and $\langle a \rangle \rtimes \langle c \rangle$ is a Frobenius group with $\langle c \rangle$ as a complement from the proof of Theorem in [17]. By Sylow's theorem, we may without loss of generality choose a nontrivial subgroup D of $\langle c \rangle$. Then $\langle b \rangle \subset C_{O_{p'}(G_3)}(D)$. For any $1 \neq u \in D$, since $\langle a \rangle \rtimes \langle c \rangle$ is a Frobenius group, this implies $C_{O_{p'}(G_3)}(D) \subset C_{O_{p'}(G_3)}(u) = \langle b \rangle$. It follows that $C_{O_{p'}(G_3)}(D) = \langle b \rangle$. Assume that $b \in [O_{p'}(G_3), D]$, then there exist $a^i b^j \in O_{p'}(G_3)$ and $c^k \in D$ such that $b = [a^i b^j, c^k]$. It follows that $b = b^{-j} a^{-i} c^{-k} a^i b^j c^k = (a^{-i}(a^i)^{c^k})^{b^j} \in \langle a \rangle$. This leads to a contradiction. Since G_3 is p-nilpotent, by Lemma 4.2, we have that G_3 is p-radical.

For Case (IV), the proof is similar and therefore will be omitted. $\square$

Lemma 4.5. *G_5 and G_6 are p-radical.*

Proof. By the proof of Theorem in [17], we can see $H \subset \mathrm{GL}(2,5)$. Choose $w = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 2 \end{smallmatrix}\right)$ and $P = \langle a, b \rangle \in \mathrm{Syl}_2(G_6)$, where $a = \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right)$. Then $G_5 = E_{5^2} \rtimes \langle w, ab \rangle$ and $G_6 = E_{5^2} \rtimes \langle w, a, b \rangle$. It follows that $G_5 \lhd G_6$. It suffices to show that G_6 is 2-radical from Lemma 2.7. Set $E_{5^2} = \langle x \rangle \times \langle y \rangle$. Obviously, we have $O_{2'}(G_6) = E_{5^2} \rtimes \langle w \rangle$. We can easily find that G_6 has exactly three elements of order 2, which are a^4, b, $a^4 b$. For any nontrivial subgroup D of P, by Lemma 4.2, we will show this result in three steps.

Step 1. If any two of a^4, b, $a^4 b$ are contained in D, then $C_{O_{2'}(G_6)}(D) = 1$.

Without loss of generality, assume that $a^4, b \in D$. We can easily check that $C_{O_{2'}(G_6)}(D) \subset \langle w \rangle \cap \langle x \rangle = 1$.

Step 2. If $D = \langle a^4 \rangle$ or $\langle a^4 b \rangle$ or $\langle b \rangle$, then $[O_{2'}(G_6), D] \cap C_{O_{2'}(G_6)}(D) = 1$.

Assume that $D = \langle a^4 \rangle$. From Step 1, we have $C_{O_{2'}(G_6)}(D) \subset \langle w \rangle$ and $[O_{2'}(G_6), a^4] \subset E_{5^2}$. This proves our conclusion. For the rest of these situations, the proof is similar.

Step 3. If $|D| \geqslant 4$, then the conclusion of Step 2 still holds.

By Step 1, we need only verify that this result holds for $\mathbb{Z}_4 \subset D$. After a simple calculation, we deduce that G_6 has exactly two cyclic subgroups of order 4, which are $\langle a^2 \rangle, \langle a^2 b \rangle$. This implies that a^4 is contained in two subgroups. If there is no element of the form $a^l b$ in D, then $[O_{2'}(G_6), D] \subset E_{5^2}$. Since $C_{O_{2'}(G_6)}(D) \subset \langle w \rangle$, the conclusion is proved. If there exists $a^l b \in D$, then $C_{O_{2'}(G_6)}(D) \subset C_{O_{2'}(G_6)}(a^l b) \subset E_{5^2}$. Thus $C_{O_{2'}(G_6)}(D) = 1$, as required. □

Using the method of Lemma 4.5, we can obtain the following lemma.

Lemma 4.6. *G_7 and G_8 are p-radical.*

Sketch of proof. By the proof of Theorem in [17], we have $H \subset \mathrm{GL}(4, 3)$. Set

$$w = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 2 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 2 & 0 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 2 \end{pmatrix}.$$

Then $G_7 = E_{3^4} \rtimes \langle w, a^2, ab \rangle$ and $G_8 = E_{3^4} \rtimes \langle w, a, b \rangle$. Hence $G_7 \lhd G_8$. We need only verify that G_8 is 2-radical by Lemma 2.7. Note that G_8 has exactly three elements of order 2, which are a^8, b^2, $a^8 b^2$. For any nontrivial subgroup D of $\langle a, b \rangle$, we get $C_{O_{2'}(G_8)}(D) = 1$ if any two of a^8, b^2, $a^8 b^2$ are contained in D. And we can also deduce that $[O_{2'}(G_8), D] \cap C_{O_{2'}(G_8)}(D) = 1$, where $D = \langle a^8 \rangle$ or $\langle a^8 b^2 \rangle$ or $\langle b^2 \rangle$. Therefore, we may assume that $D \supset \mathbb{Z}_4$. If $a^8 \in D$, then $a^8 \in \langle a^4 \rangle$ or $\langle a^4 b^2 \rangle$ or $\langle a^{12} b^2 \rangle$. Imitating the proof of Step 3 in the above lemma, we can obtain $[O_{2'}(G_8), D] \cap C_{O_{2'}(G_8)}(D) = 1$. Similarly, for the case $b^2 \in D$ or $a^8 b^2 \in D$, the conclusion holds for G_8. □

Now, if we can show that G_9 and G_{10} are p-radical, then Theorem 1.5 is completed by Theorem 1.2 (ii). Note that G_i ($i = 1, \ldots, 8$) are of p-length 1 in Cases (I)–(VIII), respectively, thus we can use Lemma 4.2 to prove the required conclusion. But G_9 and G_{10} are of 2-length 2, so Lemma 4.2 is inappropriate for Cases (IX) and (X). This forces us back to the definition of p-radical blocks to prove our results. We have the following lemma.

Lemma 4.7. *G_9 and G_{10} are p-radical.*

Proof. For Case (IX), by the proof of Proposition 6.2 in [18], we have $O_{2'}(G_9) = E_{7^2}$ and T is a group G_{48} given in [7], Chapter 12, Definition 8.4. Since $T/O_2(T) \cong S_3$, this implies T is 2-radical from Lemma 2.7. Let B be a block of G_9. Then there exists a block b of $O_{2'}(G_9)$ which is covered by B. We continue the proof by the following steps.

Step 1. If b is the principal block of $O_{2'}(G_9)$, then $\mathcal{J}(B) \subset kG_9 \cdot \Delta(P)$, where $P \in \mathrm{Syl}_2(G_9)$.

Let $\chi \in \mathrm{Irr}(B)$. Then the principal character of $O_{2'}(G_9)$ is a constituent of $\chi_{O_{2'}(G_9)}$ by [1], Chapter 5, Lemma 2.3. This implies $O_{2'}(G_9) \subset \mathrm{Ker}(\chi)$, and it follows that $O_{2'}(G_9) \subset \mathrm{Ker}(B)$. Hence B is the principal block of $G_9/O_{2'}(G_9)$ by Lemma 2.5. Since T is 2-radical, let $\overline{G_9} = G_9/O_{2'}(G_9)$, and then $\mathcal{J}(k\overline{G_9}) \subset k\overline{G_9} \cdot \Delta(\overline{P})$. The conclusion follows directly by the proof of Theorem 1.2 (ii).

Step 2. If b is not the principal block of $O_{2'}(G_9)$, then the conclusion of Step 1 still holds.

Obviously, b contains a unique irreducible character μ which is not the principal character of $O_{2'}(G_9)$. Let $T(b)$ be the inertia group of b in G_9. Then $T(\mu) = T(b)$ contains a defect group D of B by [1], Chapter 5, Corollary 2.6. Since $O_{2'}(G_9) = E_{7^2}$ is abelian, μ is a linear character and $O_{2'}(G_9)/\mathrm{Ker}(\mu)$ is cyclic. Hence D centralizes $O_{2'}(G_9)/\mathrm{Ker}(\mu) \neq 1$. This implies $O_{2'}(G_9) = C_{O_{2'}(G_9)}(D) \times [O_{2'}(G_9), D] = C_{O_{2'}(G_9)}(D) \cdot \mathrm{Ker}(\mu) \supsetneq \mathrm{Ker}(\mu)$. It follows that $C_{O_{2'}(G_9)}(D) \neq 1$, and thus there exists an element $u \in O_{2'}(G_9)^{\#}$ such that $D \subset C_{G_9}(u)$. Since $T \subset \mathrm{GL}(2,7)$ and T acts transitively on $O_{2'}(G_9)^{\#}$, we have $C_T(u) = 1$ by $|T| = |G_{48}| = 48$. This implies $C_{G_9}(u) = E_{7^2}$. Note that T has a unique element x^2 of order 2 and $x^2 \in Z(T)$. It follows that $E_{7^2} \rtimes \langle x^2 \rangle \lhd G_9$. Moreover, $D \subset E_{7^2} \rtimes \langle x^2 \rangle$. We can see that $E_{7^2} \rtimes \langle x^2 \rangle$ is 2-radical by [23], Proposition 2. Following [1], Chapter 6, Theorem 2.3, $\mathcal{J}(B) = B \cdot \mathcal{J}(k(E_{7^2} \rtimes \langle x^2 \rangle)) \subset kG_9 \cdot \Delta(\langle x^2 \rangle) \subset kG_9 \cdot \Delta(P)$, where $P \in \mathrm{Syl}_2(G_9)$.

Consequently, we have $\mathcal{J}(B) \subset kG_9 \cdot \Delta(P)$ for any block B of G_9. Therefore, G_9 is 2-radical.

For Case (X), G_{10} satisfies all crucial conditions which are used to prove that G_9 is 2-radical. Therefore, we can prove similarly that G_{10} is 2-radical. $\square$

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