Hristo M. Manev
Almost Contact B-metric Manifoldsas Extensions of a 2-dimensional Space-form

Almost Contact B-metric Manifolds as Extensions of a 2-dimensional Space-form

Hristo M. MANEV

Medical University of Plovdiv, Faculty of Public Health,
Department of Medical Informatics, Biostatistics and Electronic Education,
15-A Vasil Aprilov Blvd., Plovdiv 4002
&
Paisii Hilendarski University of Plovdiv, Faculty of Mathematics and
Informatics, Department of Algebra and Geometry, 236 Bulgaria Blvd.,
Plovdiv 4027, Bulgaria
e-mail: hmanev@uni-plovdiv.bg

Abstract

The object of investigations are almost contact B-metric manifolds which are derived as a product of a real line and a 2-dimensional manifold equipped with a complex structure and a Norden metric. There are used two different methods for generation of the B-metric on the product manifold. The constructed manifolds are characterised with respect to the Ganchev–Mihova–Gribachev classification and their basic curvature properties.

Key words: Almost contact manifold, B-metric, cone, S^1-solvable extension, complex space-form, Norden metric.

2010 Mathematics Subject Classification: 53C15, 53C50

1 Introduction

The differential geometry of almost contact metric manifolds is well studied (e.g. [1]). Ganchev, Mihova, Gribachev begin investigations on the almost contact manifolds with B-metric in [3]. These manifolds are the odd-dimensional counterpart of almost complex manifolds with Norden metric (briefly, almost Norden

manifolds) [2, 4], where the almost complex structure acts as an anti-isometry regarding the metric. Further, almost contact B-metric manifolds of arbitrary odd dimension are investigated in many works, for example [11, 12, 13, 14, 15, 17].

An object of our special interest is the case of the lowest dimension 3 of the considered manifolds ([6, 7, 8, 9, 10]). In the present paper, there are used two different methods for construction of an almost contact B-metric manifold as a product of a real line and a 2-dimensional Norden manifold. The goal of this work is a characterisation of the obtained manifolds.

The paper is organized as follows. In Sect. 2 we recall some preliminary facts about almost contact B-metric manifolds and almost Norden manifolds. In Sect. 3 and Sect. 4 we study the considered manifolds derived from a space-form of real dimension 2 equipped with a complex structure and a Norden metric, which are constructed as its cone and its S^1-solvable extension, respectively.

2   Preliminaries

2.1   Almost contact B-metric manifolds

Let $(M, \varphi, \xi, \eta, g)$ be an *almost contact B-metric manifold*, where M is a $(2n+1)$-dimensional differentiable manifold with an almost contact structure (φ, ξ, η) consisting of an endomorphism φ of the tangent bundle, a Reeb vector field ξ, its dual contact 1-form η as well as M is equipped with a pseudo-Riemannian metric g of signature $(n + 1, n)$, called *B-metric*, such that:

$$\varphi\xi = 0, \quad \varphi^2 = -\mathrm{Id} + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1,$$
$$g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y),$$

where Id is the identity map ([3]). In the latter equality and further, x, y, z, w will stand for arbitrary elements of $\mathfrak{X}(M)$, the Lie algebra of tangent vector fields, or vectors in the tangent space T_pM of M at an arbitrary point p in M. Let us recall that these manifolds are the odd dimensional extension of the almost Norden manifolds and the case with indefinite metrics corresponding to almost contact metric manifolds.

The associated metric $\widetilde{g}$ of g on M is defined by $\widetilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y)$ and it is also a B-metric. The manifold $(M, \varphi, \xi, \eta, \widetilde{g})$ is also an almost contact B-metric manifold.

The Ganchev–Mihova–Gribachev classification of almost contact B-metric manifolds, consisting of eleven basic classes $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{11}$, is given in [3]. It is made with respect to the (0,3)-tensor F defined by

$$F(x, y, z) = g\big((\nabla_x \varphi)\, y, z\big),$$

where ∇ is the Levi-Civita connection of g and the following general properties are valid:

$$F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi),$$
$$F(x, \varphi y, \xi) = (\nabla_x \eta)(y) = g(\nabla_x \xi, y).$$

The intersection of the basic classes is the special class $\mathcal{F}_0$, determined by the condition $F(x, y, z) = 0$ and it is known as the class of the *cosymplectic B-metric manifolds*.

Let $\{e_i; \xi\}$ $(i = 1, 2, \ldots, 2n)$ be a basis of T_pM and let (g_{ij}) be the corresponding matrix of g and (g^{ij}) be its inverse matrix. The 1-forms θ, θ^*, ω associated with F, called Lee forms, are determined by:

$$\theta(z) = g^{ij}F(e_i, e_j, z), \quad \theta^*(z) = g^{ij}F(e_i, \varphi e_j, z), \quad \omega(z) = F(\xi, \xi, z).$$

In the present work, we consider the case of the lowest dimension of the considered manifolds, i.e. $\dim M = 3$.

The basis $\{e_1, e_2, e_3\}$ in any tangent space at an arbitrary point of M is called a φ-*basis* if the following equalities are valid

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad e_3 = \xi,$$
$$g(e_1, e_1) = -g(e_2, e_2) = g(e_3, e_3) = 1, \quad g(e_i, e_j) = 0, \ i \neq j. \tag{1}$$

According to [6], the components of F, θ, θ^*, ω, denoted by $F_{ijk} = F(e_i, e_j, e_k)$, $\theta_k = \theta(e_k)$, $\theta_k^* = \theta^*(e_k)$, $\omega_k = \omega(e_k)$, with respect to the given φ-basis are:

$$\theta_1 = F_{221} - F_{331}, \ \theta_2 = F_{222} - F_{332}, \ \theta_3 = F_{223} - F_{322},$$
$$\theta_1^* = F_{231} + F_{321}, \ \theta_2^* = F_{223} + F_{322}, \ \theta_3^* = F_{222} + F_{332}, \tag{2}$$
$$\omega_1 = 0, \qquad\qquad \omega_2 = F_{112}, \qquad \omega_3 = F_{113}.$$

Let us denote by F^s $(s = 1, 2, \ldots, 11)$ the components of F in the corresponding basic classes $\mathcal{F}_s$. In [6], F^s are obtained for $\dim M = 3$ and it is established that the class of 3-dimensional almost contact B-metric manifolds is

$$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}.$$

According to [14], the class of the normal almost contact B-metric manifolds is $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6$, since the Nijenhuis tensor of almost contact structure vanishes there. Therefore, we conclude that the class of the 3-dimensional normal almost contact B-metric manifolds is $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5$. The corresponding components of F for $x = x^i e_i$, $y = y^j e_j$, $z = z^k e_k$ are the following ones in compliance with [6]:

$$F^1(x, y, z) = \left(x^2\theta_2 - x^3\theta_3\right)\left(y^2z^2 + y^3z^3\right),$$
$$\theta_2 = F_{222} = F_{233}, \quad \theta_3 = -F_{322} = -F_{333};$$
$$F^4(x, y, z) = \tfrac{1}{2}\theta_1\left\{x^2\left(y^1z^2 + y^2z^1\right) - x^3\left(y^1z^3 + y^3z^1\right)\right\},$$
$$\tfrac{1}{2}\theta_1 = F_{212} = F_{221} = -F_{313} = -F_{331}; \tag{3}$$
$$F^5(x, y, z) = \tfrac{1}{2}\theta_1^*\left\{x^2\left(y^1z^3 + y^3z^1\right) + x^3\left(y^1z^2 + y^2z^1\right)\right\},$$
$$\tfrac{1}{2}\theta_1^* = F_{213} = F_{231} = F_{312} = F_{321}.$$

In [12], there are given fundamental facts of the conformal geometry of almost contact B-metric manifolds. We recall some of them which are in relation with the obtained results in this work.

The contactly conformal transformations are determined by

$$\bar{g} = e^{2u} \cos 2v \; g + e^{2u} \sin 2v \; \widetilde{g} + (1 - e^{2u} \cos 2v - e^{2u} \sin 2v)\eta \otimes \eta$$

and they form a group C. The minimal contactly conformal equivalent class with respect to C is contained in the most cramped class with non-zero Lee forms θ and θ^*, the class $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5$, whose is also the minimal contactly conformally invariant class with respect to C.

The classes $\mathcal{F}_1 \oplus \mathcal{F}_4$ and $\mathcal{F}_1 \oplus \mathcal{F}_5$ are locally contactly conformally invariant with respect to the transformations of the groups $C_{1,4}$ and $C_{1,5}$, respectively, which are contactly conformal transformations satisfying the conditions $\mathrm{d}u(\xi) = 0$ and $\mathrm{d}v(\xi) = 0$, respectively. Similarly, the classes $\mathcal{F}_1$, $\mathcal{F}_4$ and $\mathcal{F}_5$ are locally contactly conformally invariant with respect to the transformations of the groups C_1, C_4 and C_5, respectively. The latter groups are determined by the conditions:

$$C_1\colon \; du(\xi) = dv(\xi) = 0,$$
$$C_4\colon \; du \circ \varphi = dv \circ \varphi^2, \quad du(\xi) = 0,$$
$$C_5\colon \; du \circ \varphi = dv \circ \varphi^2, \quad dv(\xi) = 0.$$

The subsets $\mathcal{F}_i^0$ $(i = 1, 4, 5)$ are subclasses of $\mathcal{F}_i$ with closed Lee forms θ and θ^*. An $\mathcal{F}_0$-manifold is contactly conformally equivalent to a manifold from the classes $\mathcal{F}_1^0 \oplus \mathcal{F}_4^0$, $\mathcal{F}_1^0 \oplus \mathcal{F}_5^0$, $\mathcal{F}_1^0$, $\mathcal{F}_4^0$, $\mathcal{F}_5^0$ with respect to transformations belonging to the groups $C_{1,4}^0$, $C_{1,5}^0$, C_1^0, C_4^0, C_5^0, respectively. The latter subgroups are determined as follows:

$$C_{1,4} \supset C_{1,4}^0\colon \; d(du \circ \varphi) = 0, \qquad C_{1,5} \supset C_{1,5}^0\colon \; d(dv \circ \varphi) = 0,$$
$$C_1 \supset C_1^0\colon \; d(du \circ \varphi) = d(dv \circ \varphi) = 0,$$
$$C_4 \supset C_4^0\colon \; d(du \circ \varphi) = 0, \qquad\qquad C_5 \supset C_5^0\colon \; d(dv \circ \varphi) = 0.$$

In [11], it is defined the square norm of $\nabla\varphi$ as follows

$$\|\nabla\varphi\|^2 = g^{ij} g^{ks} g\big((\nabla_{e_i}\varphi)\, e_k, \big(\nabla_{e_j}\varphi\big)\, e_s\big). \tag{4}$$

If an almost contact B-metric manifold has a zero square norm of $\nabla\varphi$ it is called an *isotropic-cosymplectic B-metric manifold* ([11]). Obviously, the equality $\|\nabla\varphi\|^2 = 0$ is valid if $(M, \varphi, \xi, \eta, g)$ is an $\mathcal{F}_0$-manifold, but the inverse implication is not always true.

Let $R = [\nabla, \nabla] - \nabla_{[\,,\,]}$ be the curvature $(1,3)$-tensor of ∇ and the corresponding curvature $(0,4)$-tensor be denoted by the same letter: $R(x, y, z, w) = g(R(x,y)z, w)$. The following properties are valid in general:

$$R(x, y, z, w) = -R(y, x, z, w) = -R(x, y, w, z),$$
$$R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0. \tag{5}$$

Let the essential curvature-like tensors π_1 and π_2 (i.e. tensors generated by g and φ such that they possess the properties (5)) of types $(1,3)$ and $(0,4)$ are

defined by

$$
\begin{aligned}
\pi_1(x, y)z &= g(y, z)x - g(x, z)y, \\
\pi_2(x, y)z &= g(y, \varphi z)\varphi x - g(x, \varphi z)\varphi y, \\
\pi_1(x, y, z, w) &= g(y, z)g(x, w) - g(x, z)g(y, w), \\
\pi_2(x, y, z, w) &= g(y, \varphi z)g(x, \varphi w) - g(x, \varphi z)g(y, \varphi w).
\end{aligned} \tag{6}
$$

Let the Ricci tensor ρ and the scalar curvature τ for R and g as well as their associated quantities be defined as follows

$$
\begin{aligned}
\rho(y, z) &= g^{ij}R(e_i, y, z, e_j), & \rho^*(y, z) &= g^{ij}R(e_i, y, z, \varphi e_j), \\
\tau &= g^{ij}\rho(e_i, e_j), & \tau^* &= g^{ij}\rho^*(e_i, e_j), & \tau^{**} &= g^{ij}\rho^*(e_i, \varphi e_j).
\end{aligned} \tag{7}
$$

The sectional curvature of each non-degenerate 2-plane α in T_pM with respect to g and R has the following form

$$
k(\alpha; p) = \frac{R(x, y, y, x)}{g(x, x)g(y, y)}, \tag{8}
$$

where $\{x, y\}$ is an orthogonal basis of α.

A 2-plane α is called a *φ-holomorphic section* (respectively, a *ξ-section*) if $\alpha = \varphi\alpha$ (respectively, $\xi \in \alpha$).

2.2   Almost complex manifold with Norden metric

Let us remark that the $2n$-dimensional contact distribution $H = \ker(\eta)$ of $(M, \varphi, \xi, \eta, g)$ can be considered as an almost complex manifold N endowed with an almost complex structure $J = \varphi|_H$ and a metric $h = g|_H$, where $\varphi|_H$ and $g|_H$ are the restrictions of φ and g on H, respectively.

Let x', y', z', w' denote arbitrary vector fields or vectors in the contact distribution H of M. Since g is a B-metric of M, then h is a *Norden metric* on H, i.e. it is compatible with J as follows

$$
h(Jx', Jy') = -h(x', y').
$$

The associated Norden metric $\widetilde{h}$ of h is determined by

$$
\widetilde{h}(x', y') = h(x', Jy').
$$

Both metrics h and $\widetilde{h}$ have signature (n, n).

We recall that an $2n$-dimensional manifold N with almost complex structure J and Norden metric h is an almost Norden manifold (N, J, h). The manifold $(N, J, \widetilde{h})$ is also an almost Norden manifold.

In the present work we pay attention to the case of the lowest dimension. Let (N, J, h) be a 2-dimensional almost Norden manifold. It is known that such a manifold is a space-form, i.e. the manifold has constant sectional curvature k' and its curvature tensor has the form $R' = k'\,\pi_1'$, where π_1' is the

essential curvature-like tensor as π_1 in (6) but with respect to h. Moreover, according to [2], any 2-dimensional almost Norden manifold is integrable and it belongs to the basic class $\mathcal{W}_1$. In other words, this is the class of the considered manifolds which are conformally equivalent to Kähler–Norden manifolds, where conformal transformations of the metric are given by $\overline{h} = e^{2u}(\cos 2v\, h + \sin 2v\, \widetilde{h})$ for differentiable functions u, v on N. The Kähler–Norden manifolds are the most specialized case of the almost Norden manifolds, determined by $\nabla' J = 0$, where ∇' is the Levi-Civita connection of h. Their class is denoted by $\mathcal{W}_0$ and it is a subclass of $\mathcal{W}_1$. The fundamental tensor F' of (N, J, h) is defined by $F'(x', y', z') = h\big((\nabla'_{x'} J)\, y', z'\big)$ and for any $\mathcal{W}_1$-manifold it is determined as follows

$$F'(x', y', z') = \frac{1}{2}\{h(x', y')\theta'(z') + h(x', Jy')\theta'(Jz')$$
$$+ h(x', z')\theta'(y') + h(x', Jz')\theta'(Jy')\}. \tag{9}$$

A more wide subclass of $\mathcal{W}_1$ comparing with $\mathcal{W}_0$ contains the so-called *isotropic-Kähler–Norden manifolds* ([16]). They have a vanishing square norm of $\nabla' J$, i.e. $\|\nabla' J\|^2 = 0$, where this square norm is defined by

$$\|\nabla' J\|^2 = h^{ij} h^{ks} h\big((\nabla'_{e_i} J) e_k, (\nabla'_{e_j} J) e_s\big) \tag{10}$$

with respect to an arbitrary basis.

3   The cone over a 2-dimensional complex space-form with Norden metric

In this section, we consider the cone over N, $\mathcal{C}(N) = \mathbb{R}^+ \times N$, where $\mathbb{R}^+$ is the set of positive reals. We equip it with a metric g defined by

$$g\left(\left(x', a\tfrac{\mathrm{d}}{\mathrm{d}t}\right), \left(y', b\tfrac{\mathrm{d}}{\mathrm{d}t}\right)\right) = t^2\, h(x', y') + ab, \tag{11}$$

where t is the coordinate on $\mathbb{R}^+$ and a, b are differentiable functions on $\mathcal{C}(N)$.

We introduce an almost contact structure on the cone by

$$\varphi|_H = J, \quad \xi = \tfrac{\mathrm{d}}{\mathrm{d}t}, \quad \eta = \mathrm{d}t, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{12}$$

Obviously, the manifold $(\mathcal{C}(N), \varphi, \xi, \eta, g)$ is an almost contact B-metric manifold.

Using the general Koszul formula

$$2g(\nabla_x y, z) = xg(y, z) + yg(z, x) - zg(x, y)$$
$$+ g([x, y], z) - g([y, z], x) + g([z, x], y), \tag{13}$$

(11) and (12), we obtain the following equalities for the Levi-Civita connection ∇ of the B-metric g on $\mathcal{C}(N)$:

$$g\left(\nabla_{x'} y', z'\right) = t^2\, h\left(\nabla'_{x'} y', z'\right), \quad g\left(\nabla_\xi y', z'\right) = t\, h\left(y', z'\right),$$
$$g\left(\nabla_{x'} y', \xi\right) = -t\, h\left(x', y'\right), \qquad g\left(\nabla_{x'} \xi, z'\right) = t\, h\left(x', z'\right). \tag{14}$$

Further, using (14), we find the following formulae for the covariant derivatives with respect to ∇

$$\nabla_{x'}y' = \nabla'_{x'}y' - \frac{1}{t}g\left(x',y'\right)\xi, \quad \nabla_{\xi}y' = \frac{1}{t}y', \quad \nabla_{x'}\xi = \frac{1}{t}x'. \qquad (15)$$

Bearing in mind (5) and (15), we obtain

$$R(x',y')z' = \frac{1}{t^2}(k'-1)\pi_1(x',y')z',$$
$$R(x',y')\xi = R(x',\xi)z' = R(\xi,y')z' = R(x',\xi)\xi = R(\xi,y')\xi = 0.$$

Therefore, by direct computations, we get the following

Proposition 3.1 *The following equalities for the curvature tensor R of $(\mathcal{C}(N), \varphi, \xi, \eta, g)$ are valid:*

$$R(x',y',z',w') = \frac{1}{t^2}(k'-1)\pi_1(x',y',z',w'),$$
$$R(\xi,y',z',w') = R(x',\xi,z',w') = R(x',y',\xi,w') \qquad (16)$$
$$= R(x',y',z',\xi) = R(\xi,y',z',\xi) = 0.$$

According to (1) and (11), we obtain the components $h_{ij} = h(e_i, e_j)$ and $g_{ij} = g(e_i, e_j)$ and the non-zero of them are

$$h_{11} = -h_{22} = \frac{1}{t^2}, \quad g_{11} = -g_{22} = g_{33} = 1. \qquad (17)$$

Using (13), (15) and (17), we get

$$\begin{aligned}
\nabla_{e_1}e_1 &= \nabla'_{e_1}e_1 - \tfrac{1}{t}e_3, & \nabla_{e_1}e_2 &= \nabla'_{e_1}e_2, & \nabla_{e_1}e_3 &= \tfrac{1}{t}e_1, \\
\nabla_{e_2}e_1 &= \nabla'_{e_2}e_1, & \nabla_{e_2}e_2 &= \nabla'_{e_2}e_2 + \tfrac{1}{t}e_3, & \nabla_{e_2}e_3 &= \tfrac{1}{t}e_2, \qquad (18) \\
\nabla_{e_3}e_1 &= \tfrac{1}{t}e_1, & \nabla_{e_3}e_2 &= \tfrac{1}{t}e_2, & \nabla_{e_3}e_3 &= 0.
\end{aligned}$$

By virtue of (4), (12), (17) and (18), we obtain the value of the square norm of $\nabla\varphi$ as follows

$$\|\nabla\varphi\|^2 = 2\{(\theta_1)^2 - (\theta_2)^2\} - \frac{4}{t^2}, \qquad \|\nabla'J\|^2 = 2t^2\{(\theta_1)^2 - (\theta_2)^2\}.$$

The latter equalities imply

$$\|\nabla\varphi\|^2 = \frac{1}{t^2}\{\|\nabla'J\|^2 - 4\}$$

and the truthfulness of the following

Theorem 3.2   *1. The manifold $(\mathcal{C}(N), \varphi, \xi, \eta, g)$ is an isotropic-cosymplectic B-metric manifold if and only if the square norm of $\nabla'J$ on (N, J, h) is*

$$\|\nabla'J\|^2 = 4.$$

2. *The manifold (N, J, h) is an isotropic-Kähler-Norden manifold if and only if the square norm of $\nabla \varphi$ on $(\mathcal{C}(N), \varphi, \xi, \eta, g)$ is*

$$\|\nabla\varphi\|^2 = -\frac{4}{t^2}.$$

Taking into account (12), (17) and (18), we compute the components F_{ijk} of F and get

$$\begin{aligned} F_{111} = F_{122} = \theta_1', \quad F_{211} = F_{222} = -\theta_2', \\ F_{123} = F_{132} = F_{213} = F_{231} = \tfrac{1}{t}, \end{aligned} \tag{19}$$

where $\theta_i' = \theta'(e_i)$ for $i = 1, 2$, as well as the other components of F are zero.

Using (2), we obtain the components of the Lee forms of $(\mathcal{C}(N), \varphi, \xi, \eta, g)$ and the non-zero of them are:

$$\theta_1 = \theta_1', \quad \theta_2 = \theta_2', \quad \theta_1^* = -\theta_2', \quad \theta_2^* = \theta_1', \quad \theta_3^* = \frac{2}{t}.$$

Bearing in mind (3) and (19), we establish the equality

$$F(x, y, z) = (F^1 + F^5)(x, y, z),$$

where F^1 and F^5 are the components of F in the basic classes $\mathcal{F}_1$ and $\mathcal{F}_5$, respectively. The non-zero components of F^1 and F^5 by means of (3) and (9) are the following

$$\begin{aligned} F_{111}^1 = F_{122}^1 = \theta_1, \quad F_{211}^1 = F_{222}^1 = -\theta_2, \\ F_{123}^5 = F_{132}^5 = F_{213}^5 = F_{231}^5 = \tfrac{1}{2}\theta_3^*. \end{aligned}$$

Therefore, we establish the truthfulness of the following

Theorem 3.3 *The manifold $(\mathcal{C}(N), \varphi, \xi, \eta, g)$*

1. *belongs to $\mathcal{F}_1 \oplus \mathcal{F}_5$,*

2. *belongs to $\mathcal{F}_5$ if and only if (N, J, h) is a $\mathcal{W}_0$-manifold,*

3. *could not belongs to $\mathcal{F}_1$.*

The class $\mathcal{F}_1 \oplus \mathcal{F}_5$ is a subclass of the class $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5$ of the 3-dimensional normal almost contact B-metric manifolds.

Using (16), (17) and (18), we compute the components $R_{ijk\ell} = R(e_i, e_j, e_k, e_\ell)$ of the curvature tensor R. The non-zero ones of them are determined by (5) and the following

$$R_{1212} = \frac{1}{t^2}(k' - 1). \tag{20}$$

Theorem 3.4 *The manifold $(\mathcal{C}(N), \varphi, \xi, \eta, g)$ is flat if and only if $k' = 1$.*

Bearing in mind (8), (17) and (20), we compute the basic sectional curvatures $k_{ij} = k(e_i, e_j)$ as follows

$$k_{12} = \frac{1}{t^2}(k' - 1), \qquad k_{13} = k_{23} = 0. \tag{21}$$

Taking into account (7), (17) and (20), we obtain the basic components $\rho_{jk} = \rho(e_j, e_k)$ and $\rho^*_{jk} = \rho^*(e_j, e_k)$ of the Ricci tensor ρ and its associated tensor ρ^*, respectively, as well as the values of the scalar curvature τ and its associated quantities τ^*, τ^{**}. The non-zero ones of them are:

$$\rho_{11} = -\rho_{22} = \rho^*_{12} = \rho^*_{21} = \tfrac{1}{2}\tau = \tfrac{1}{2}\tau^{**} = \tfrac{1}{t^2}(k' - 1). \tag{22}$$

By virtue of (21) and (22), we conclude the following

Proposition 3.5 *For the manifold* $(\mathcal{C}(N), \varphi, \xi, \eta, g)$

1. *the sectional curvatures of the ξ-sections vanish,*
2. $\tau^* = 0$,
3. $\tau^{**} = \tau$.

Proposition 3.6 *The following assertions for* $(\mathcal{C}(N), \varphi, \xi, \eta, g)$ *are valid:*

1. $k' < 1$ *if and only if* $\tau < 0$;
2. $k' = 1$ *if and only if* $\tau = 0$;
3. $k' > 1$ *if and only if* $\tau > 0$.

4   The S^1-solvable extension of a 2-dimensional complex space-form with Norden metric

In the present section, let us introduce a warped product 3-dimensional manifold $S^1(N) = \mathbb{R}^+ \times_{t^2} N$ as follows. Let $\mathrm{d}t$ be the coordinate 1-form on $\mathbb{R}^+$ and let us define an almost contact B-metric structure on $S^1(N)$ as follows

$$\varphi|_H = J, \quad \xi = \tfrac{\mathrm{d}}{\mathrm{d}t}, \quad \eta = \mathrm{d}t, \quad \eta \circ \varphi = 0, \quad g = \mathrm{d}t^2 + \cos 2t\, h - \sin 2t\, \widetilde{h}. \tag{23}$$

In [5], it is proved that the warped product manifold $S^1(N)$ equipped with the almost contact B-metric structure defined in (23) is an almost contact B-metric manifold. The constructed manifold $(S^1(N), \varphi, \xi, \eta, g)$ in this manner is called *an S^1-solvable extension of* (N, J, h) in [5].

Using (13) and (23), we compute the components of the covariant derivative ∇ as follows:

$$\nabla_{x'} y' = \nabla'_{x'} y' - \tfrac{1}{2} \sin 2t \{ g\,(x', y')\, \theta'^{\sharp} + g\,(x', Jy')\, J\theta'^{\sharp} \} + g\,(x', Jy')\, \xi,$$
$$\nabla_{\xi} y' = -Jy', \quad \nabla_{x'} \xi = -Jx', \quad \nabla_{\xi} \xi = 0. \tag{24}$$

In the latter equalities and further, we denote by $\theta'^{\sharp}$ the dual vector of θ' with respect to h. Analogously, $\theta^{\sharp}$ stands for the dual vector of θ with respect to g.

Using (24), we obtain

$$
\begin{aligned}
R(x',y')z' = {}& k'\{\cos 2t\,\pi_1(x',y')z' - \sin 2t\,J\pi_2(x',y')z'\} - \pi_2(x',y')z' \\
& + 2\sin 2t\,(\pi_1+\pi_2)(x',y',z',\theta'^\sharp)J\theta^\sharp \\
& - \tfrac{1}{2}\sin 2t\,\{g\,(y',z')\,\nabla_{x'}\theta'^\sharp - g\,(x',z')\,\nabla_{y'}\theta'^\sharp \\
& + g\,(y',Jz')\,\nabla_{x'}J\theta'^\sharp - g\,(x',Jz')\,\nabla_{y'}J\theta'^\sharp\} \\
& - \tfrac{1}{2}\{\cos 2t\,(\pi_1+\pi_2)(x',y',z',\theta'^\sharp) \\
& + \sin 2t\,(\pi_1+\pi_2)(x',y',z',J\theta'^\sharp)\}\xi, \\
R(x',y')\xi = {}& -\tfrac{1}{2}(\pi_1'+\pi_2')(x',y')\theta'^\sharp, \\
R(\xi,y')z' = {}& -\tfrac{1}{2}(\pi_1'+\pi_2')(\theta'^\sharp,y')z' \\
& - \cos 2t\,\{h\,(y',z')\,(\theta_1'e_1 - \theta_2'e_2) + h\,(y',Jz')\,(\theta_2'e_1 + \theta_1'e_2)\} \\
& + g\,(y',z')\,\xi, \\
R(x',\xi)z' = {}& -\tfrac{1}{2}(\pi_1'+\pi_2')(x',\theta'^\sharp)z' \\
& + \cos 2t\,\{h\,(x',z')\,(\theta_1'e_1 - \theta_2'e_2) + h\,(x',Jz')\,(\theta_2'e_1 + \theta_1'e_2)\} \\
& + g\,(x',z')\,\xi.
\end{aligned}
\tag{25}
$$

The equalities (23) and (25) imply the following

Proposition 4.1 *The following equalities for the curvature tensor R of $(S^1(N),$ $\varphi,\xi,\eta,g)$ are valid:*

$$
\begin{aligned}
R(x',y',z',w') = {}& k'\{\cos 2t\,\pi_1(x',y',z',w') - \sin 2t\,\pi_2(x',y',z',\varphi w')\} \\
& - \pi_2(x',y',z',w') \\
& + 4\sin 2t\,\{\cos 2t\,(\pi_1+\pi_2)(x',y',z',\theta^\sharp) \\
& - \sin 2t\,(\pi_1+\pi_2)(x',y',z',\varphi\theta^\sharp)\}\theta(\varphi w') \\
& - \sin 2t\,\{g\,(y',z')\,[\cos 2t\,(\nabla_{x'}\theta)w' + \sin 2t\,(\nabla_{x'}\theta^*)w'] \\
& - g\,(x',z')\,[\cos 2t\,(\nabla_{y'}\theta)w' + \sin 2t\,(\nabla_{y'}\theta^*)w'] \\
& + g\,(y',Jz')\,[\sin 2t\,(\nabla_{x'}\theta)w' - \cos 2t\,(\nabla_{x'}\theta^*)w'] \\
& - g\,(x',Jz')\,[\sin 2t\,(\nabla_{y'}\theta)w' - \cos 2t\,(\nabla_{y'}\theta^*)w']\}, \\
R(x',y',z',\xi) = {}& -(\pi_1+\pi_2)(x',y',z',\theta^\sharp), \\
R(x',y',\xi,w') = {}& -(\pi_1+\pi_2)(x',y',\theta^\sharp,w'), \\
R(x',\xi,z',w') = {}& -(\pi_1+\pi_2)(x',\theta^\sharp,z',w'), \\
R(\xi,y',z',w') = {}& -(\pi_1+\pi_2)(\theta^\sharp,y',z',w'), \\
R(\xi,y',z',\xi) = {}& g\,(y',z')\,.
\end{aligned}
\tag{26}
$$

Using (1) and (23), we obtain the components g_{ij} and h_{ij} as follows

$$
\begin{aligned}
& g_{11} = -g_{22} = g_{33} = 1,\; g_{12} = g_{21} = 0, \\
& h_{11} = -h_{22} = \cos 2t,\quad h_{12} = h_{21} = -\sin 2t.
\end{aligned}
\tag{27}
$$

Taking into account (13), (24) and (27), we get

$$
\begin{aligned}
&\nabla_{e_1} e_1 = \nabla'_{e_1} e_1 - \tfrac{1}{2}\sin 2t\,\theta'^{\sharp}, &&\nabla_{e_1} e_2 = \nabla'_{e_1} e_2 + \tfrac{1}{2}\sin 2t\,J\theta'^{\sharp} - \xi, \\
&\nabla_{e_2} e_1 = \nabla'_{e_2} e_1 + \tfrac{1}{2}\sin 2t\,J\theta'^{\sharp} - \xi, \ \nabla_{e_2} e_2 = \nabla'_{e_2} e_2 + \tfrac{1}{2}\sin 2t\,\theta'^{\sharp}, \\
&\nabla_{e_1} e_3 = \nabla_{e_3} e_1 = -e_2, &&\nabla_{e_2} e_3 = \nabla_{e_3} e_2 = e_1, \qquad \nabla_{e_3} e_3 = 0.
\end{aligned}
\tag{28}
$$

Bearing in mind (4), (10), (23), (27) and (28), we obtain the values of the square norms of $\nabla\varphi$ and $\nabla' J$ as follows

$$
\begin{aligned}
\|\nabla\varphi\|^2 &= 2\{(\theta'_1)^2 - (\theta'_2)^2\} + 4, \\
\|\nabla' J\|^2 &= (1 + \cos 4t)\{(\theta'_1)^2 - (\theta'_2)^2\} - 2\sin 4t\,\theta'_1\theta'_2.
\end{aligned}
$$

The latter equalities imply the truthfulness of the following

Theorem 4.2 *The manifold* $(S^1(N), \varphi, \xi, \eta, g)$ *is an isotropic-cosymplectic B-metric manifold if and only if it is valid*

$$
\|\nabla' J\|^2 = -2(1 + \cos 4t + \sin 4t\,\theta'_1\theta'_2).
$$

By virtue of (23), (27) and (28), we compute the components F_{ijk} of F. The non-zero ones of them are

$$
\begin{aligned}
F_{111} &= F_{122} = \cos 2t\,\theta'_1 - \sin 2t\,\theta'_2, \\
F_{211} &= F_{222} = -\sin 2t\,\theta'_1 - \cos 2t\,\theta'_2, \\
F_{131} &= F_{113} = -F_{232} = -F_{223} = -1.
\end{aligned}
\tag{29}
$$

Using (2), we obtain the components of the Lee forms of $(S^1(N), \varphi, \xi, \eta, g)$ and the non-zero of them are:

$$
\theta_1 = \theta_2^* = \cos 2t\,\theta'_1 - \sin 2t\,\theta'_2, \quad \theta_2 = -\theta_1^* = \sin 2t\,\theta'_1 + \cos 2t\,\theta'_2, \quad \theta_3 = -2.
$$

Bearing in mind (3) and (29), we establish the equality

$$
F(x', y', z') = (F^1 + F^4)(x', y', z'),
$$

where F^1 and F^4 are the components of F in the basic classes $\mathcal{F}_1$ and $\mathcal{F}_4$, respectively. The non-zero components of F^1 and F^4 are

$$
F^1_{111} = F^1_{122} = \theta_1, \quad F^1_{211} = F^1_{222} = -\theta_2, \quad F^4_{131} = F^4_{113} = -F^4_{232} = -F^4_{223} = \tfrac{1}{2}\theta_3.
$$

Therefore, we establish the validity of the following

Theorem 4.3 *The manifold* $(S^1(N), \varphi, \xi, \eta, g)$

1. *belongs to* $\mathcal{F}_1 \oplus \mathcal{F}_4$,
2. *belongs to* $\mathcal{F}_4$ *if and only if* (N, J, h) *is a Kähler–Norden manifold,*
3. *could not belongs to* $\mathcal{F}_1$.

The class $\mathcal{F}_1 \oplus \mathcal{F}_4$ is a subclass of the class $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5$ of the 3-dimensional normal almost contact B-metric manifolds.

Bearing in mind (26), (27) and (28), we calculate the components $R_{ijk\ell}$ of R. The non-zero of them are determined by (5) and the following

$$R_{1212} = k' \cos 2t - 1 + \sin 2t \, \cos 2t \, \{(\nabla_{e_2}\theta)e_2 - (\nabla_{e_1}\theta^*)e_2 + 8\theta_1\theta_2\}$$
$$+ \sin^2 2t \, \{(\nabla_{e_2}\theta^*)e_2 + (\nabla_{e_1}\theta)e_2 + 8(\theta_1)^2\}, \qquad (30)$$
$$R_{1213} = 2\theta_2, \quad R_{1223} = 2\theta_1, \quad R_{3113} = -R_{3223} = 1.$$

Bearing in mind (8), (27) and (30), we compute the basic sectional curvatures k_{ij} as follows

$$k_{12} = R_{1212}, \quad k_{13} = k_{23} = 1. \qquad (31)$$

Using (7), (27) and (30), we get the basic components ρ_{jk} and ρ_{jk}^* as well as the values τ, τ^* and τ^{**}:

$$
\begin{aligned}
&\rho_{11} = -\rho_{22} = R_{1212} + 1, \, \rho_{33} = 2, &&\rho_{11}^* = \rho_{22}^* = \rho_{33}^* = 0, \\
&\rho_{12} = \rho_{21} = 0, &&\rho_{13} = \rho_{31} = -2\theta_1, \, \rho_{23} = \rho_{32} = -2\theta_2 \\
&\rho_{12}^* = \rho_{21}^* = R_{1212}, &&\rho_{13}^* = \rho_{31}^* = 2\theta_2, \quad \rho_{23}^* = \rho_{32}^* = -2\theta_1, \\
&\tau = 2R_{1212} + 4, &&\tau^* = 0, \qquad\quad \tau^{**} = 2R_{1212}.
\end{aligned}
\qquad (32)
$$

By virtue of the latter equalities for ρ, ρ^* and (27) for g_{ij}, we get

Theorem 4.4 *For the manifold* $(S^1(N), \varphi, \xi, \eta, g)$ *the following assertions are equivalent:*

1. (N, J, h) *is a Kähler–Norden manifold;*
2. $\rho = k' \cos 2t \, g + (2 - k' \cos 2t)\eta \otimes \eta$;
3. $\rho^* = (1 - k' \cos 2t)(\widetilde{g} - \eta \otimes \eta)$.

Let us remark that the assertion (1) in the latter theorem is equivalent to the assertion that $(S^1(N), \varphi, \xi, \eta, g)$ is an $\mathcal{F}_4$-manifold, according to (2) in Theorem 4.3.

The assertion (2) in the latter theorem shows that $(S^1(N), \varphi, \xi, \eta, g)$ in this case is an η-Einstein manifold, according to [10]; whereas the assertion (3) in the latter theorem presents ρ^* as proportional to g^*, which is defined by $g^*(x, y) = g(x, \varphi y)$.

By virtue of (31) and (32), we conclude the following

Proposition 4.5 *For the manifold* $(S^1(N), \varphi, \xi, \eta, g)$

1. *the sectional curvatures of the ξ-sections are constant,*
2. $\tau^* = 0$,
3. $\tau^{**} = \tau - 4$.

Taking into account the first equality of (30) and the equalities in the last line of (32), we get the following corollaries.

Corollary 4.6 *If* (N, J, h) *is a Kähler–Norden manifold, then*

$$\tau = 2(k' \cos 2t + 1), \qquad \tau^{**} = 2(k' \cos 2t - 1).$$

Corollary 4.7 *If* (N, J, h) *is a Kähler–Norden manifold, then*

1. $k' < 0$ *if and only if* $2(k' + 1) < \tau < 2, \ 2(k' - 1) < \tau^{**} < -2$;
2. $k' = 0$ *if and only if* $\tau = 2, \ \tau^{**} = -2$;
3. $k' > 0$ *if and only if* $2 < \tau < 2(k' + 1), \ -2 < \tau^{**} < 2(k' - 1)$.

References

[1] Blair, D. E.: Riemannian Geometry of Contact and Symplectic Manifolds. Progress in Mathematics **203**, *Birkhäuser*, Boston, 2002.

[2] Ganchev, G., Borisov, A.: *Note on the almost complex manifolds with a Norden metric.* C. R. Acad. Bulg. Sci. **39** (1986), 31–34.

[3] Ganchev, G., Mihova, V., Gribachev, K.: *Almost contact manifolds with B-metric.* Math. Balkanica (N.S.) **7** (1993), 261–276.

[4] Gribachev, K., Mekerov, D., Djelepov, G.: *Generalized B-manifolds.* C. R. Acad. Bulg. Sci. **38** (1985), 299–302.

[5] Ivanov, S., Manev, H., Manev, M.: *Sasaki-like almost contact complex Riemannian manifolds.* arXiv.org arXiv:1402.5426 (2014), 1–19.

[6] Manev, H.: *On the structure tensors of almost contact B-metric manifolds.* Filomat **29** (2015), 427–436.

[7] Manev, H.: *Almost contact B-metric structures and the Bianchi classification of the three-dimensional Lie algebras.* arXiv.org arXiv:1412.8142 (2014), 1–12.

[8] Manev, H.: *Matrix Lie groups as 3-dimensional almost contact B-metric manifolds.* Facta Universitatis (Niš), Ser. Math. Inform **30**, 3 (2015), 341–351, arXiv:1503.00312.

[9] Manev, H.: *Space-like and time-like hyperspheres in real pseudo-Riemannian 4-spaces with almost contact B-metric structures.* arXiv.org arXiv:1504.00267 (2015), 1–8.

[10] Manev, H., Mekerov, D.: *Lie groups as 3-dimensional almost contact B-metric manifolds.* J. Geom. **106** (2015), 229–242.

[11] Manev, M.: *Natural connection with totally skew-symmetric torsion on almost contact manifolds with B-metric.* Int. J. Geom. Methods Mod. Phys. **9**, 1250044 (2012), 1–20.

[12] Manev, M., Gribachev, K.: *Contactly conformal transformations on almost contact manifolds with B-metric.* Serdica Math. J. **19** (1993), 287–299.

[13] Manev, M., Gribachev, K.: *Conformally invariant tensors on almost contact manifolds with B-metric.* Serdica Math. J. **20** (1994), 133–147.

[14] Manev, M., Ivanova, M.: *Canonical type connections on almost contact manifold with B-matric.* Ann. Global Anal. Geom. **43**, 4 (2013), 397–408.

[15] Manev, M., Ivanova, M.: *A classification of the torsion tensors on almost contact manifolds with B-metric.* Cent. Eur. J. Math. **12** (2014), 1416–1432.

[16] Mekerov, D., Manev, M.: *On the geometry of quasi-Kähler manifolds with Norden metric.* Nihonkai Math. J. **16** (2005), 89–93.

[17] Nakova, G., Gribachev, K.: *Submanifolds of some almost contact manifolds with B-metric with codimension two, I.* Math. Balkanica (N.S.) **11** (1997), 255–267.