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Lifts of Foliated Linear Connections to the Second Order Transverse Bundles

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Abstract

The second order transverse bundle $T^2_{\mathrm{tr}}M$ of a foliated manifold M carries a natural structure of a smooth manifold over the algebra $\mathbb{D}^2$ of truncated polynomials of degree two in one variable. Prolongations of foliated mappings to second order transverse bundles are a partial case of more general $\mathbb{D}^2$-smooth foliated mappings between second order transverse bundles. We establish necessary and sufficient conditions under which a $\mathbb{D}^2$-smooth foliated diffeomorphism between two second order transverse bundles maps the lift of a foliated linear connection into the lift of a foliated linear connection.

Key words: Foliation, transverse bundle, second order transverse bundle, projectable linear connection, Lie derivative, Weil bundle.

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1 Introduction

Transverse Weil bundle $T^{\mathbb{A}}_{\mathrm{tr}}M$ of a foliated manifold M defined by a Weil algebra $\mathbb{A}$ [7, 8] carries a natural structure of a smooth manifold over $\mathbb{A}$ [8]. This makes it possible to apply methods of the theory of manifolds over algebras to the study of geometry of $T^{\mathbb{A}}_{\mathrm{tr}}M$. The second order transverse bundle $T^2_{\mathrm{tr}}M$ of a foliated manifold M is naturally equivalent to the Weil bundle $T^{\mathbb{D}^2}_{\mathrm{tr}}M$ defined by the algebra $\mathbb{D}^2$ of truncated polynomials of degree two in one variable. In this paper, we study the behavior of lifts of foliated connections (lifted connections) on second order transverse bundles under $\mathbb{D}^2$-smooth diffeomorphisms preserving the lifted foliations and establish conditions, in terms of transverse

Lie derivatives, under which such a diffeomorphism maps a lifted connection into a lifted one. Another way to obtain conditions under which a $\mathbb{D}^2$-smooth diffeomorphism maps a lifted connection into a lifted one is to generalize the notion of a Lie jet with respect to a field of $\mathbb{A}$-velocities [10].

We define the lift of a foliated connection applying to the connection object the functor T^2_{tr} which is viewed as the functor of $\mathbb{D}^2$-prolongation. Lifts of linear connections to higher order tangent bundles and to Weil bundles were introduced by A. Morimoto [5, 6]. A. P. Shirokov [1] applied theory of manifolds over algebras to the definition and study of these lifts. $\mathbb{D}^2$-smooth linear connections on second order tangent bundles studied in [2]. Applying A. Morimoto's approach, R. Wolak [12] constructed lifts of linear connections in transverse bundles $T_{\mathrm{tr}}M$ to higher order transverse bundles. V. V. Vishnevskii [11] applied methods used by A. P. Shirokov and A. Morimoto to the study of lifts of projectable linear connections on manifolds fibered by a sequence of submersions.

2 $\mathbb{D}^2$-smooth structure on the second order transverse bundle

The projection $p\colon \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m \ni \{x^i, y^\alpha\} \mapsto \{x^i\} \in \mathbb{R}^n$, where the indices $i, j, \dots$ and $\alpha, \beta, \dots$ run, respectively, through the sets of values $\{1, \dots, n\}$ and $\{n+1, \dots, n+m\}$, defines the model codimension n foliation $\mathcal{F}_{n,m}$ on the space $\mathbb{R}^{n+m}$ representing it as a union of m-dimensional leaves. A diffeomorphism $f\colon U \ni \{x^i, y^\alpha\} \mapsto \{f^j(x^i, y^\alpha), f^\beta(x^i, y^\alpha)\} \in U'$ between open subsets U and U' of $\mathbb{R}^{n+m}$ is called a local automorphism of $\mathcal{F}_{n,m}$ if $\partial f^j/\partial y^\alpha = 0$. A codimension n foliation $\mathcal{F}$ on an $(n+m)$-dimensional smooth manifold M is given by an atlas $\mathcal{A}$ whose coordinate changes are local automorphisms of the model foliation $\mathcal{F}_{n,m}$ [4]. Charts from $\mathcal{A}$ are called *foliated charts*. A manifold M with given foliation $\mathcal{F}$ on it is called a *foliated manifold*. A foliated manifold is also denoted by $(M, \mathcal{F})$. A connected open subset U of a foliated manifold M is called *simple* if the induced foliation on U is generated by a submersion with connected leaves. A foliated chart (U, h) is called *simple* if U is a simple open subset of M. The *leaf* of a foliated manifold M passing through a point x is the maximal connected submanifold $L_x \ni x$ in M defined in terms of simple foliated charts by equations of the form $x^i = x^i_0 = const$. A smooth mapping $f\colon M \to M'$ between two foliated manifolds $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ is a foliated mapping (a morphism of foliations) if in terms of any foliated charts (U, h) on M and (U', h') on M' such that $f(U) \subset U'$ it has equations

$$x^{i'} = f^{i'}(x^i, y^\alpha), \quad y^{\alpha'} = f^{\alpha'}(x^i, y^\alpha), \quad \partial_\alpha f^{i'} = 0. \tag{1}$$

Here and in what follows we use the following notation for partial derivatives:

$$\partial_j f^{i'} = \partial f^{i'}/\partial x^j, \quad \partial_\alpha f^{i'} = \partial f^{i'}/\partial y^\alpha, \quad \partial^2_{jk} f^{i'} = \partial^2 f^{i'}/\partial x^j \partial x^k,$$
$$\partial^2_{j\beta} f^{\alpha'} = \partial^2 f^{\alpha'}/\partial x^j \partial y^\beta,$$

and so on.

A foliated mapping maps leaves of M into leaves of M'. If U is a simple open set, equations (1) take the form

$$x^{i'} = f^{i'}(x^i), \quad y^{\alpha'} = f^{\alpha'}(x^i, y^\alpha). \tag{2}$$

In what follows we will assume that equations of foliated mappings in question are written for simple open subsets of their domains.

A transverse 2-velocity on M at $x \in M$ is an equivalence class of germs of smooth curves on M with respect to the following equivalence relation: two germs $\gamma\colon (\mathbb{R}, 0) \to (M, x)$ and $\gamma'\colon (\mathbb{R}, 0) \to (M, x)$ are equivalent if and only if the 2-jets $j^2(p \circ h \circ \gamma)$ and $j^2(p \circ h \circ \gamma')$ coincide for any foliated chart (U, h), $x \in U$. The transverse 2-velocity defined by a germ γ is denoted by $j^2_{\mathrm{tr}}\gamma$ or $j^2_{\mathrm{tr}\,x}\gamma$. The numbers

$$\begin{aligned} x^i &= (h^i \circ \gamma)(0), & y^\alpha &= (h^\alpha \circ \gamma)(0), \\ \dot{x}^i &= d(h^i \circ \gamma)/dt|_0, & \ddot{x}^i &= \tfrac{1}{2} d^2(h^i \circ \gamma)/dt^2|_0 \end{aligned} \tag{3}$$

are the coordinates of the transverse 2-velocity $j^2_{\mathrm{tr}\,x}\gamma$ in terms of the chart (U, h). Let $T^2_{\mathrm{tr}\,x}M$ denote the set of all transverse 2-velocities at $x \in M$ and $T^2_{\mathrm{tr}}M = \cup_{x \in M} T^2_{\mathrm{tr}\,x}M$ the set of all transverse 2-velocities on M. $T^2_{\mathrm{tr}}M$ carries a structure of a smooth $(3n+m)$-dimensional manifold fibered over M. This structure is defined as follows. Let

$$\pi^2_0\colon T^2_{\mathrm{tr}}M \ni j^2_{\mathrm{tr}\,x}\gamma \mapsto x \in M$$

be the canonical projection assigning to a 2-velocity $j^2_{\mathrm{tr}\,x}\gamma \in T^2_{\mathrm{tr}\,x}M$ the point $x \in M$. A foliated chart (U, h) on M induces the chart

$$h^2\colon (\pi^2_0)^{-1}(U) \ni X = j^2_{\mathrm{tr}\,x}\gamma \mapsto \{x^i, y^\alpha, \dot{x}^i, \ddot{x}^i\} \in \mathbb{R}^{3n+m} \tag{4}$$

on $T^2_{tr}M$. If the change of coordinates on a simple open subset of the overlapping of the domains of two charts (U, h) and (U', h') on M is of the form (2), then the corresponding change of the induced coordinates on $T^2_{tr}M$ is of the form

$$\begin{gathered} x^{i'} = f^{i'}(x^i), \quad y^{\alpha'} = f^{\alpha'}(x^i, y^\alpha), \quad \dot{x}^{i'} = (\partial_j f^{i'})\dot{x}^j, \\ \ddot{x}^{i'} = (\partial_j f^{i'})\ddot{x}^j + \tfrac{1}{2}(\partial^2_{jk} f^{i'})\dot{x}^j \dot{x}^k. \end{gathered} \tag{5}$$

Thus, the collection $\mathcal{A}^2_{\mathrm{tr}}$ of charts of the form (4), where h runs through the atlas $\mathcal{A}$, is an atlas defining a structure of a smooth manifold on $T^2_{\mathrm{tr}}M$.

As it follows from (5), the bundle $T^2_{\mathrm{tr}}M$ carries a foliation $\mathcal{F}^2_{\mathrm{tr}}$ with basic coordinates $x^i, \dot{x}^i, \ddot{x}^i$. We will call $\mathcal{F}^2_{\mathrm{tr}}$ the lifted foliation [4] and consider $T^2_{\mathrm{tr}}M$ as a foliated manifold with foliation $\mathcal{F}^2_{\mathrm{tr}}$. The projection π^2_0 is a morphism of foliations $(T^2_{\mathrm{tr}}M, \mathcal{F}^2_{\mathrm{tr}})$ and $(M, \mathcal{F})$.

The second order transverse bundle $T^2_{\mathrm{tr}}M$ can be viewed as the bundle $T^{\mathbb{D}^2}_{\mathrm{tr}}M$ of transverse $\mathbb{D}^2$-velocities on M [7, 8], where $\mathbb{D}^2$ is the algebra of truncated polynomials of degree less or equal to 2 in one variable, i.e. the three-dimensional commutative associative algebra whose elements are of the form $a + b\varepsilon + c\varepsilon^2$,

$a, b, c \in \mathbb{R}$, with multiplication defined by the relation $\varepsilon^3 = 0$, and so $T^2_{\mathrm{tr}}M$ carries a natural structure of a smooth manifold over $\mathbb{D}^2$. This structure can be described as follows.

On the manifold $T^2_{\mathrm{tr}}\mathbb{R}^{n+m}$, there arises a structure of a $\mathbb{D}^2$-module naturally isomorphic to the $\mathbb{D}^2$-module $(\mathbb{D}^2)^n \oplus \mathbb{R}^m$ with the action of $\mathbb{D}^2$ on $(\mathbb{D}^2)^n \oplus \mathbb{R}^m$ defined by the relation

$$\sigma(u \oplus v) = \sigma u \oplus 0$$

for $\sigma = b\varepsilon + c\varepsilon^2$. Coordinate chart (4) defines the mapping

$$T^2_{\mathrm{tr}}h\colon \pi^{-1}U \ni X = j^2_{\mathrm{tr}}\gamma \mapsto \{X^i = x^i + \varepsilon\dot{x}^i + \varepsilon^2\ddot{x}^i, y^\alpha\} \in T^2_{\mathrm{tr}}\mathbb{R}^{n+m} = (\mathbb{D}^2)^n \oplus \mathbb{R}^m.$$

Let U be a simple open subset of $(\mathbb{D}^2)^n \oplus \mathbb{R}^m$. An arbitrary $\mathbb{D}^2$-smooth mapping $F\colon U \to (\mathbb{D}^2)^n \oplus \mathbb{R}^m$ is of the form [8]

$$\begin{aligned} X^{i'} &= f^{i'}(x^i) + \varepsilon\big(\dot{x}^j\partial_j f^{i'} + g^{i'}(x^i)\big) \\ &+ \varepsilon^2\big(\ddot{x}^j\partial_j f^{i'} + \tfrac{1}{2}\dot{x}^j\dot{x}^k\partial^2_{jk}f^{i'} + \dot{x}^j\partial_j g^{i'} + h^{i'}(x^i, y^\alpha)\big),\ y^{\alpha'} = f^{\alpha'}(x^i, y^\alpha). \end{aligned} \tag{6}$$

Therefore, coordinate changes (5) are $\mathbb{D}^2$-smooth diffeomorphisms between open subsets of the module $(\mathbb{D}^2)^n \oplus \mathbb{R}^m$, and $T^2_{\mathrm{tr}}M$ carries a structure of a smooth manifold over the algebra $\mathbb{D}^2$ modelled by the module $(\mathbb{D}^2)^n \oplus \mathbb{R}^m$.

Let T^2_{tr} denote the functor which assigns to a foliated manifold its second order transverse bundle and to a foliated mapping $f\colon M \to M'$ the mapping $T^2_{\mathrm{tr}}f\colon T^2_{\mathrm{tr}}M \to T^2_{\mathrm{tr}}M'$ defined by the composition of jets: $T^2_{\mathrm{tr}}f\colon j^2_{\mathrm{tr}}\gamma \mapsto j^2_{\mathrm{tr}}(f \circ \gamma)$. In terms of local coordinates, $T^2_{\mathrm{tr}}f$ is of the form (5). In what follows we assume that the functor T^2_{tr} assigns to a foliated manifold M the bundle $T^2_{\mathrm{tr}}M$ endoved with the above described structure of a $\mathbb{D}^2$-smooth manifold.

Let $i_0\colon M \to T^2_{\mathrm{tr}}M$ denote the zero section which assigns to a point $x \in M$ the jet $j^2_{\mathrm{tr}}\gamma$ of the constant curve $\gamma(t) = x$. We will identify the image of the zero section $i_0(M) \subset T^2_{\mathrm{tr}}M$ with M. From (6) it follows that an arbitrary $\mathbb{D}^2$-smooth mapping $F\colon T^2_{\mathrm{tr}}M \to T^2_{\mathrm{tr}}M'$ is defined by its restriction $f = F|M \to T^2_{\mathrm{tr}}M'$ to M. It also follows from (6) that a $\mathbb{D}^2$-smooth mapping $F\colon T^2_{\mathrm{tr}}M \to T^2_{\mathrm{tr}}M'$ is a morphism of foliations (the functions $h^{i'}$ in (6) do not depend on y^α) if and only if $f = F|M$ is a morphism of foliations. This being the case, we call F a foliated $\mathbb{D}^2$-smooth mapping. If a $\mathbb{D}^2$-smooth mapping $F\colon T^2_{\mathrm{tr}}M \to T^2_{\mathrm{tr}}M'$ is defined by a morphism of foliations $f\colon M \to T^2_{\mathrm{tr}}M'$, we denote it by $f^{\mathbb{D}^2}$ and say that it is the $\mathbb{D}^2$-prolongation of f. In the case when the image of f belongs to the zero section of $T^2_{\mathrm{tr}}M'$, the $\mathbb{D}^2$-prolongation of f coincides with the mapping $T^2_{\mathrm{tr}}f\colon T^2_{\mathrm{tr}}M \to T^2_{\mathrm{tr}}M'$. Let in addition $\bar{f} = \pi^2_0 \circ f$. The above mentioned mappings form the commutative diagram

$$\begin{array}{ccc} T^2_{\mathrm{tr}}M & \xrightarrow{F=f^{\mathbb{D}^2}} & T^2_{\mathrm{tr}}M' \\ {\scriptstyle \pi^2_0}\big\downarrow & \overset{f}{\nearrow} & \big\downarrow{\scriptstyle \pi^2_0} \\ M & \xrightarrow{\ \bar{f}\ } & M'. \end{array} \tag{7}$$

3 Foliated linear connections and their lifts to the second order transverse bundles

With a foliated manifold $(M, \mathcal{F})$ one can associate the following fiber bundles.

1. The bundle $P^2_{fol}M$ of second order foliated frames on M whose elements are 2-jets of germs of morphisms of foliations

$$f\colon (\mathbb{R}^{n+m}, 0) \ni \{u^a, v^\rho\} \to M, \quad a = 1, \ldots, n, \ \rho = n+1, \ldots n+m. \tag{8}$$

A local foliated chart (x^i, y^α) on M induces the chart

$$(x^i, y^\alpha; x^i_a, x^i_{ab}; y^\alpha_a, y^\alpha_\rho, y^\alpha_{ab}, y^\alpha_{a\rho}, y^\alpha_{\rho\sigma}), \tag{9}$$

where $x^i_a = \partial_a x^i = \partial x^i/\partial u^a$, $x^i_{ab} = \partial^2_{ab} x^i$, $y^\alpha_a = \partial_a y^\alpha = \partial y^\alpha/\partial u^a$, $y^\alpha_\rho = \partial_\rho y^\alpha = \partial y^\alpha/\partial v^\rho$, $y^\alpha_{ab} = \partial^2_{ab} y^\alpha$, $y^\alpha_{a\rho} = \partial^2_{a\rho} y^\alpha$, $y^\alpha_{\rho\sigma} = \partial^2_{\rho\sigma} y^\alpha$. We will consider $P^2_{fol}M$ as a foliated manifold with basic coordinates (x^i, x^i_a, x^i_{ab}). $P^2_{fol}M$ is a principal fiber bundle over M with structure group $G^2_{n,m}$ consisting of 2-jets of germs at zero of automorphisms of the model foliation

$$g\colon (\mathbb{R}^{n+m}, 0) \ni \{u^a, v^\rho\} \mapsto \{u^{a'}, v^{\rho'}\} \in (\mathbb{R}^{n+m}, 0),$$

where $a = 1, \ldots, n$, $\rho = n+1, \ldots n+m$, $a' = 1', \ldots, n'$, $\rho' = (n+1)', \ldots (n+m)'$. The action $P^2_{fol}M \times G^2_{n,m} \to P^2_{fol}M$ is defined by the rule of composition of jets: $j^2_x f \circ j^2 g = j^2_x(f \circ g)$.

2. The principal bundle $P^1_{fol}M$ of first order foliated frames on M whose elements are 1-jets of germs of morphisms of foliations (8).

3. The principal bundles $P^1_{\mathrm{tr}}M$ and $P^2_{\mathrm{tr}}M$ of first and second order transverse frames on M defined as bundles whose elements are equivalence classes of germs $f\colon (\mathbb{R}^n, 0) \ni \{u^a\} \to M$ such that $p \circ h \circ f$ is a germ of diffeomorphism for any foliated chart (U, h) with respect to the following equivalence relation: two germs f and f' are equivalent if and only if the jets, respectively, of the first and the second order of $p \circ h \circ f$ and $p \circ h \circ f'$ coincide. A local foliated chart (x^i, y^α) on M induces the charts $(x^i, y^\alpha; x^i_a)$ and $(x^i, y^\alpha; x^i_a, x^i_{ab})$ on $P^1_{\mathrm{tr}}M$ and $P^2_{\mathrm{tr}}M$ respectively. There are natural projections $p^2_{\mathrm{tr}}\colon P^2_{fol}M \to P^2_{\mathrm{tr}}M$ and $p^1_{\mathrm{tr}}\colon P^1_{fol}M \to P^1_{\mathrm{tr}}M$.

4. The transverse bundle (or the first order transverse bundle) $T_{\mathrm{tr}}M$ is defined as the quotient bundle of the tangent bundle TM by the distribution of tangent spaces to leaves or, equivalently, as the bundle of transverse 1-velocities on M, i.e. the fiber bundle over M whose elements are equivalence classes $j^1_{tr\,x}\gamma$ of germs of smooth curves on M with respect to the equivalence relation: two germs $\gamma\colon (\mathbb{R}, 0) \to (M, x)$ and $\gamma'\colon (\mathbb{R}, 0) \to (M, x)$ are equivalent if and only if the 1-jets $j^1(p \circ h \circ \gamma)$ and $j^1(p \circ h \circ \gamma')$ coincide. A foliated chart (U, h) on M induces the chart $h^1\colon (\pi^1_0)^{-1}(U) \ni X = j^1_{tr\,x}\gamma \mapsto \{x^i, y^\alpha, \dot{x}^i\} \in \mathbb{R}^{2n+m}$ on $T_{\mathrm{tr}}M$, where $\pi^1_0\colon T_{\mathrm{tr}}M \ni j^1_{tr\,x}\gamma \mapsto x \in M$ and the numbers $\dot{x}^i$ are the same as in (3). The bundle $T_{\mathrm{tr}}M$ can also be obtained as the base of the projection $\pi^2_1\colon T^2_{\mathrm{tr}}M \to T_{\mathrm{tr}}M$ induced by the algebra epimorphism $\pi^2_1\colon \mathbb{D}^2 \to \mathbb{D}$, where $\mathbb{D}$

is the algebra of Study dual numbers. $T_{\mathrm{tr}}M$ carries a natural structure of a smooth manifold over the algebra $\mathbb{D}$ modeled by the $\mathbb{D}$-module $\mathbb{D}^n \oplus \mathbb{R}^m$.

A linear connection on M is a right invariant horizontal distribution on the first order frame bundle P^1M [1, 3] and can be viewed as a field $\Gamma\colon P^2M \to \mathbb{R}^{(n+m)^3}$ of second order geometric objects on M corresponding to the representation $G^2_{n+m} \times \mathbb{R}^{(n+m)^3} \to \mathbb{R}^{(n+m)^3}$ of the second order differential group G^2_{n+m} [1, 3] on the space $\mathbb{R}^{(n+m)^3}$ defined as follows:

$$\Gamma^A_{BC} = z^A_{A'} z^{A'}_{BC} + \Gamma^{A'}_{B'C'} z^{B'}_B z^{C'}_C z^A_{A'},$$
$$A, B, C = 1, \dots n+m, \quad A', B', C' = 1', \dots (n+m)',$$

where Γ^A_{BC} and $\Gamma^{A'}_{B'C'}$ are the coordinates of elements of $\mathbb{R}^{(n+m)^3}$ and $z^{A'}_A = \partial_A z^{A'}$, $z^{A'}_{AB} = \partial^2_{AB} z^{A'}$ are the coordinates of an element from G^2_{n+m} defined by a germ of diffeomorphism at zero given by equations $z^{A'} = z^{A'}(z^A)$.

The subgroup $G^2_{n,m} \subset G^2_{n+m}$ of 2-jets of germs of automorphisms of the model foliation leaves invariant the submanifold $F \subset \mathbb{R}^{(n+m)^3}$ defined by the equations

$$\Gamma^a_{b\rho} = \Gamma^a_{\rho b} = \Gamma^a_{\rho\tau} = 0$$

and acts on F as follows:

$$\begin{aligned}
\Gamma^a_{bc} &= u^a_{a'} u^{a'}_{bc} + \Gamma^{a'}_{b'c'} u^{b'}_b u^{c'}_c u^a_{a'},\\
\Gamma^\rho_{bc} &= v^\rho_{\rho'} v^{\rho'}_{bc} + \Gamma^{a'}_{b'c'} u^{b'}_b u^{c'}_c v^\rho_{a'}\\
&\quad + v^\rho_{\rho'} (\Gamma^{\rho'}_{b'c'} u^{b'}_b u^{c'}_c + \Gamma^{\rho'}_{b'\sigma'} u^{b'}_b v^{\sigma'}_c + \Gamma^{\rho'}_{\sigma'c'} v^{\sigma'}_b u^{c'}_c + \Gamma^{\rho'}_{\sigma'\tau'} v^{\sigma'}_b y^{\tau'}_c),\\
\Gamma^\rho_{\sigma c} &= v^\rho_{\rho'} v^{\rho'}_{\sigma c} + v^\rho_{\rho'} (\Gamma^{\rho'}_{\sigma'c'} v^{\sigma'}_\sigma u^{c'}_c + \Gamma^{\rho'}_{\sigma'\tau'} v^{\sigma'}_\sigma v^{\tau'}_c),\\
\Gamma^\rho_{b\tau} &= v^\rho_{\rho'} v^{\rho'}_{b\tau} + v^\rho_{\rho'} (\Gamma^{\rho'}_{b'\tau'} u^{b'}_b v^{\tau'}_\tau + \Gamma^{\rho'}_{\sigma'\tau'} v^{\sigma'}_b v^{\tau'}_\tau),\\
\Gamma^\rho_{\sigma\tau} &= v^\rho_{\rho'} v^{\rho'}_{\sigma\tau} + \Gamma^{\rho'}_{\sigma'\tau'} v^\rho_{\rho'} v^{\sigma'}_\sigma y^\tau_{\tau'}.
\end{aligned} \tag{10}$$

The manifold F is fibered over $\mathbb{R}^{n^3}$ with coordinates Γ^a_{bc}, and action (10) defines the action of the differential group G^2_n on $\mathbb{R}^{n^3}$ given by the first relation of (10).

Denote by $E(M)$ the bundle associated to $P^2_{fol}M$ corresponding to action (10). A local foliated chart (x^i, y^α) on M induces the chart $(x^i, y^\alpha, \Gamma^i_{jk}, \Gamma^\alpha_{\beta k}, \Gamma^\alpha_{j\gamma}, \Gamma^\alpha_{\beta\gamma}, \Gamma^\alpha_{jk})$ on $E(M)$. By a foliated linear connection on M we will mean a foliated section

$$\nabla\colon M \to E(M) \tag{11}$$

with respect to the foliation on $E(M)$ with basic coordinates x^i, Γ^i_{jk}. In terms of a simple foliated chart, such a section is given by equations

$$\Gamma^i_{jk} = \Gamma^i_{jk}(x^\ell), \tag{12}$$

$$\Gamma^\alpha_{\beta k} = \Gamma^\alpha_{\beta k}(x^\ell, y^\delta), \quad \Gamma^\alpha_{j\gamma} = \Gamma^\alpha_{j\gamma}(x^\ell, y^\delta), \quad \Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma}(x^\ell, y^\delta), \quad \Gamma^\alpha_{jk} = \Gamma^\alpha_{jk}(x^\ell, y^\delta). \tag{13}$$

A foliated connection ∇ defines a projectable connection in the transverse frame bundle $P^1_{\mathrm{tr}}M$ with coefficients (12) in terms of simple foliated charts. A projectable connection in $P^1_{\mathrm{tr}}M$ exists if and only if the Atiah class $a(M)$ of M is zero [4]. Therefore, vanishing of the Atiah class $a(M)$ is necessary condition for existence of a foliated linear connection on M. This condition is also sufficient. In fact, let $\mathfrak{g}^1_n$ be the Lie algebra of the Lie group $G^1_n \cong GL(n, \mathbb{R})$, $\mathfrak{g}^1_{n,m}$ the Lie algebra of the Lie group $G^1_{n,m}$, and let ω_{tr} be the $\mathfrak{g}^1_n$-valued connection form of a projectable connection in $P^1_{\mathrm{tr}}M$. A local trivilization of the bundle $P^1_{fol}M$ over a domain of a foliated chart $U \subset M$ defines a local trivialization of $P^1_{\mathrm{tr}}M$ over U. Along a section of $P^1_{fol}M$ over U one can choose a $\mathfrak{g}^1_{n,m}$-valued connection form ω_U which projects into ω_{tr} and then extend it by right translations on $P^1_{fol}M$ over U. Then, using a partition of zero for M over a covering $\{U_\lambda\}$ consisting of domains of foliated charts over which $P^1_{fol}M$ is trivial, one can glue such local connection forms and obtain a connection form which defines a foliated linear connection on M. In what follows we assume that the Atiah classes of foliated manifolds under consideration are zero. This takes place, e.g., for foliations defined by submersions.

Applying the functor T^2_{tr} to the bundle $P^2_{fol}M$ with structure group $G^2_{n,m}$, we arrive at the $\mathbb{D}^2$-smooth principal bundle $T^2_{\mathrm{tr}}P^2_{fol}M$ over $T^2_{\mathrm{tr}}M$ with structure group $T^2_{\mathrm{tr}}G^2_{n,m}$. A local chart (9) induces the chart

$$(X^i, y^\alpha; X^i_a, X^i_{ab}; y^\alpha_a, y^\alpha_\rho, y^\alpha_{ab}, y^\alpha_{a\rho}, y^\alpha_{\rho\sigma}) \tag{14}$$

on $T^2_{\mathrm{tr}}P^2_{fol}M$, where the coordinates X^i, X^i_a, X^i_{ab} take values in $\mathbb{D}^2$. The application of the functor T^2_{tr} to relations (10) gives the expressions for the action of $T^2_{\mathrm{tr}}G^2_{n,m}$ on $T^2_{\mathrm{tr}}F$. To write down these expressions, one should replace the first relation in (10) by

$$\widetilde{\Gamma}^a_{bc} = U^a_{a'}U^{a'}_{bc} + \widetilde{\Gamma}^{a'}_{b'c'}U^{b'}_b U^{c'}_c U^a_{a'}, \tag{15}$$

where all components in (15) belong to $\mathbb{D}^2$. This action leads in turn to the associated bundle $T^2_{\mathrm{tr}}E(T^2_{\mathrm{tr}}M)$. A local foliated chart (x^i, y^α) on M induces the chart $(X^i, y^\alpha, \widetilde{\Gamma}^i_{jk}, \Gamma^\alpha_{\beta k}, \Gamma^\alpha_{j\gamma}, \Gamma^\alpha_{\beta\gamma}, \Gamma^\alpha_{jk})$ on $T^2_{\mathrm{tr}}E(T^2_{\mathrm{tr}}M)$ with $\widetilde{\Gamma}^i_{jk} \in \mathbb{D}^2$. Finally, the application of T^2_{tr} to (11) defines a $\mathbb{D}^2$-smooth $\mathbb{D}^2$-linear connection $T^2_{\mathrm{tr}}\nabla$ on $T^2_{\mathrm{tr}}M$, which will be called the *lift* of a foliated connection (11), or a *lifted connection*. If a foliated connection ∇ on M is given in terms of a simple foliated chart by equations (12) and (13), then to get the equation of its lift in terms of the unduced chart on $T^2_{\mathrm{tr}}E(T^2_{\mathrm{tr}}M)$, one should take all equations (13) and replace equations (12) by the equations

$$\widetilde{\Gamma}^i_{jk}(X^\ell) = \Gamma^i_{jk}(x^\ell) + \varepsilon\dot{x}^\ell\partial_\ell\Gamma^i_{jk} + \varepsilon^2\left(\ddot{x}^j\partial_\ell\Gamma^i_{jk} + \tfrac{1}{2}\dot{x}^\ell\dot{x}^p\partial^2_{\ell p}\Gamma^i_{jk}\right). \tag{16}$$

Let now M and M' be two isomorphic foliated manifolds and $F\colon T^2_{\mathrm{tr}}M \to T^2_{\mathrm{tr}}M'$ a foliated $\mathbb{D}^2$-smooth diffeomorphism. Our aim is to find conditions under which a foliated $\mathbb{D}^2$-smooth diffeomorphism F maps the lift of a given foliated connection on $T^2_{\mathrm{tr}}M$ into a lifted connection on $T^2_{\mathrm{tr}}M'$.

Consider diagram (7) for a foliated $\mathbb{D}^2$-smooth diffeomorphism F. It is obvious that the prolongation $T^2_{\mathrm{tr}}\bar{f}$ of an isomorphism of foliations $\bar{f}\colon M \to M'$

maps the lift $T^2_{\mathrm{tr}}\nabla$ of any foliated connection ∇ into the lift of the image of ∇ under $\bar{f}$. Hence F maps the lift $T^2_{\mathrm{tr}}\nabla$ of a foliated connection ∇ into a lifted connection $T^2_{\mathrm{tr}}\nabla'$ if and only if the composition $T^2_{\mathrm{tr}}(\bar{f}^{-1})\circ F$ maps $T^2_{\mathrm{tr}}\nabla$ into itself. This composition is the $\mathbb{D}^2$-prolongation of the section $\varphi = T^2_{\mathrm{tr}}(\bar{f}^{-1})\circ F|M\colon M \to T^2_{\mathrm{tr}}M$. In terms of local charts, the section φ and the $\mathbb{D}^2$-diffeomorphism $T^2_{\mathrm{tr}}(\bar{f}^{-1})\circ F = \varphi^{\mathbb{D}^2}$ are given, respectively, by equations of the form $X'^i = x^i + \varepsilon g^i(x^k) + \varepsilon^2 h^i(x^k)$, $y'^\alpha = y^\alpha$ and

$$X'^i = x^i + \varepsilon\left(\dot{x}^i + g^i(x^k)\right) + \varepsilon^2\left(\ddot{x}^i + \dot{x}^j\partial_j g^i + h^i(x^k)\right), \quad y'^\alpha = y^\alpha. \qquad (17)$$

Note 1 As was mentioned above, the first order transverse bundle $T_{\mathrm{tr}}M$ is the base of the projection $\pi^2_1\colon T^2_{\mathrm{tr}}M \to T_{\mathrm{tr}}M$ corresponding to the algebra epimorphism $\pi^2_1\colon \mathbb{D}^2 \to \mathbb{D}$, where the algebra of dual numbers is viewed as the quotient algebra of $\mathbb{D}^2$ by the ideal $\varepsilon^2\mathbb{D}^2$. Applying this epimorphism to the relations in the above discussion, we obtain the respective formulas for the bundle $T_{\mathrm{tr}}M$. To write down these formulas, one should reject in formulas for $T^2_{\mathrm{tr}}M$ the terms containing ε^2.

In accordance with Note 1 made above, we apply first the $\mathbb{D}$-prolongation $g^{\mathbb{D}}\colon T_{\mathrm{tr}}M \to T_{\mathrm{tr}}M$ of the section $g = \pi^2_1\circ\varphi\colon M \to T_{\mathrm{tr}}M$ to the connection object

$$\widetilde{\Gamma}^{1}{}^i_{jk}(X^\ell) = \Gamma^i_{jk}(x^\ell) + \varepsilon\dot{x}^\ell\partial_\ell\Gamma^i_{jk}.$$

Using formulas similar to (15) in which $U^{a'}_a$ are replaced by $\partial X'^i/\partial X^k = \partial X'^i/\partial x^k = \delta^i_k + \varepsilon\partial_k g^i$ and $U^{a'}_{bc}$ by $\partial^2 X'^i/\partial X^k\partial X^j = \varepsilon\partial^2_{jk}g^i$, we obtain the following formulas for this image:

$$\Gamma^i_{jk}(x^\ell) + \varepsilon\left(\dot{x}^\ell\partial_\ell\Gamma^i_{jk} + \partial^2_{jk}g^i + g^\ell\partial_\ell\Gamma^i_{jk} - \Gamma^\ell_{jk}\partial_\ell g^i + \Gamma^i_{\ell k}\partial_j g^\ell + \Gamma^i_{j\ell}\partial_k g^\ell\right). \quad (18)$$

The formulas

$$\partial^2_{jk}g^i + g^\ell\partial_\ell\Gamma^i_{jk} - \Gamma^\ell_{jk}\partial_\ell g^i + \Gamma^i_{\ell k}\partial_j g^\ell + \Gamma^i_{j\ell}\partial_k g^\ell \qquad (19)$$

are the coordinate expression for a projectable section of the tensor bundle $T^1_{2\,\mathrm{tr}}M$ of type $(1,2)$ associated to the vector bundle $T_{\mathrm{tr}}M$. We will denote it by $\mathcal{L}_g\Gamma$ and call the *Lie derivative* of the connection object of the foliated connection ∇ on $T_{\mathrm{tr}}M$ with respect to a projectable section g of $T_{\mathrm{tr}}M$. The Lie derivative (19) can be defined pointwise as the inverse image of the Lie derivative with respect to the vector field $g^i(x^\ell)$ of the connection object $\Gamma^i_{jk}(x^\ell)$ of the linear connection given on a local quotient manifold of M [4] relative to the foliation (within a simple foliated domain). Thus, vanishing of the Lie derivative $\mathcal{L}_g\Gamma$ is a necessary condition for the image of $T^2_{\mathrm{tr}}\nabla$ to be a lifted connection.

It is a matter of direct verification that a projectable section $\varphi\colon M \to T^2_{\mathrm{tr}}M$ given locally by equations (17) defines in addition a projectable section $u\colon M \to T_{\mathrm{tr}}M$ given locally by the equations $\dot{x}^i = h^i - \frac{1}{2}g^k\partial_k g^i$. We will call the two sections g and u of $T_{\mathrm{tr}}M$ the sections *associated* to the diffeomorphism $F\colon T^2_{\mathrm{tr}}M \to T^2_{\mathrm{tr}}M'$ in question.

Theorem 1 *Let M and M' be two isomorphic foliated manifolds and ∇ a foliated linear connection on M with connection object Γ (12), (13). A foliated $\mathbb{D}^2$-smooth diffeomorphism $F\colon T^2_{\rm tr}M \to T^2_{\rm tr}M'$ maps the lift $T^2_{\rm tr}\nabla$ of ∇ to $T^2_{\rm tr}M$ into a lifted connection on $T^2_{\rm tr}M'$ if and only if*

$$\mathcal{L}_g\Gamma = \mathcal{L}_u\Gamma = 0,$$

where g and u are the two projectable sections of $T_{\rm tr}M$ associated to F.

Proof A direct verification shows that a projectable section $g\colon M \to T_{\rm tr}M$ with local coordinate expression $\dot{x}^i = g^i(x^k)$ defines a projectable section $\widetilde{g}\colon M \to T^2_{\rm tr}M$ with local coordinate expression $\dot{x}^i = g^i(x^k)$, $\ddot{x}^i = \frac{1}{2}g^k\partial_k g^i$. We show next that if $\mathcal{L}_g\Gamma = 0$, then the prolongation $\widetilde{g}^{\mathbb{D}^2}\colon T^2_{\rm tr}M \to T^2_{\rm tr}M$ defined by diagram (7) maps the connection $T^2_{\rm tr}\nabla$ into itself. Using a partition of zero for M over a covering $\{U_\lambda\}$ of M consisting of domains of simple foliated charts, one can glue vector fields $\widehat{g}_\lambda$ which are defined on U_λ and are projected under the mapping $\pi\colon TM \to T_{\rm tr}M$ into the restrictions $g|U_\lambda$ of the section $g\colon M \to T_{\rm tr}M$ to U_λ and obtain, as a result, a vector field $\widehat{g}$ on M which is projected by π into the section g. In terms of a local foliated chart, the vector field $\widehat{g}$ is given by equations $\{g^i(x^k), g^\alpha(x^k, y^\beta)\}$. Applying the functor $T_{\rm tr}$ to the vector field $\widehat{g}$ and the section g, one obtains a $\mathbb{D}^2$-smooth vector field $\widehat{G} = T_{\rm tr}\widehat{g}$ on $T^2_{\rm tr}M$ and a projectable section G of the transverse bundle of $T^2_{\rm tr}M$ with respect to the lifted foliation. The functor $T_{\rm tr}$ applied to the relation $\mathcal{L}_g\Gamma = 0$ gives $\mathcal{L}_G T_{\rm tr}\Gamma = 0$, and the vector field $\widehat{G}$ generates a local $\mathbb{D}^2$-smooth one-parameter group $\Psi = \{\Psi_T(X)\}$, $T = t + \dot{t}\varepsilon + \ddot{t}\varepsilon^2$, $X \in T^2_{\rm tr}M$ of transformations of $T^2_{\rm tr}M$ which transforms the connection $T^2_{\rm tr}\nabla$ into lifted connections. We also have $\Psi = T^2_{\rm tr}\psi$, where $\psi = \{\psi_t(x)\}$ is the local one-parameter group of transformations of M generated by the vector field $\widehat{g}$. If, in terms of a simple foliated chart, ψ is given by equations $\psi^i(x^k, t)$, $\psi^\alpha(x^k, y^\beta, t)$, then Ψ has equations

$$\begin{aligned}&\Psi^i(X^k, T) = \psi^i(x^k, t) + \varepsilon\left(\dot{x}^k\partial_k\psi^i + \dot{t}\partial_t\psi^i\right)\\ &+ \varepsilon^2\left(\ddot{x}^k\partial_k\psi^i + \ddot{t}\partial_t\psi^i + \tfrac{1}{2}\dot{x}^k\dot{x}^j\partial^2_{kj}\psi^i + \tfrac{1}{2}(\dot{t})^2\partial^2_{tt}\psi^i + \dot{x}^k\dot{t}\partial^2_{kt}\psi^i\right),\ \psi^\alpha(x^k, y^\beta, t).\end{aligned} \tag{20}$$

The $\mathbb{D}^2$-valued parameter T is equivalent to the three independent $\mathbb{R}$-valued parameters $t, \dot{t}, \ddot{t}$. If a transformation $\psi_{t_0}(x)$ is defined for some t_0 and $x \in M$, then the transformation $\Psi_T(X)$ is defined for all $T = t_0 + \dot{t}\varepsilon + \ddot{t}\varepsilon^2$ and $X \in (\pi^2_0)^{-1}(x)$. Letting $t = \ddot{t} = 0$, $\dot{t} = 1$ in (20), we obtain the transformation $\widetilde{g}^{\mathbb{D}^2}\colon T^2_{\rm tr}M \to T^2_{\rm tr}M$.

Let $i^2_1\colon T_{\rm tr}M \to T^2_{\rm tr}M$ denote the canonical embedding given in terms of foliated charts by equations $\{x^i, y^\alpha, \dot{x}^i\} \mapsto \{x^i, y^\alpha, 0, \dot{x}^i\}$. The composition $i^2_1 \circ u$ is a section of $T^2_{\rm tr}M$, and the $\mathbb{D}^2$-diffeomorphism $\varphi^{\mathbb{D}^2}$ can be represented as the composition $\varphi^{\mathbb{D}^2} = (i^2_1 \circ u)^{\mathbb{D}^2} \circ \widetilde{g}^{\mathbb{D}^2}$. It remains to apply $(i^2_1 \circ u)^{\mathbb{D}^2}$ to the connection object (16). Using again formulas similar to (15) where $U^{a'}_a$ are replaced by $\partial X'^i/\partial X^k = \partial X'^i/\partial x^k = \delta^i_k + \varepsilon^2\partial_k u^i$ and $U^{a'}_{bc}$ by $\partial^2 X'^i/\partial X^k\partial X^j = \varepsilon^2\partial^2_{jk}u^i$, we obtain the following formulas for the image:

$$\widetilde{\Gamma}^i_{jk}(X^\ell) + \varepsilon^2 \left(\partial^2_{jk}u^i + u^\ell \partial_\ell \Gamma^i_{jk} - \Gamma^\ell_{jk}\partial_\ell u^i + \Gamma^i_{\ell k}\partial_j u^\ell + \Gamma^i_{j\ell}\partial_k u^\ell\right),$$

which proves the theorem. □

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