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DEFECTS AND TRANSFORMATIONS OF QUASI-COPULAS

Michal Dibala, Susanne Saminger-Platz, Radko Mesiar, and Erich Peter Klement

Six different functions measuring the defect of a quasi-copula, i. e., how far away it is from a copula, are discussed. This is done by means of extremal non-positive volumes of specific rectangles (in a way that a zero defect characterizes copulas). Based on these defect functions, six transformations of quasi-copulas are investigated which give rise to six different partitions of the set of all quasi-copulas. For each of these partitions, each equivalence class contains exactly one copula being a fixed point of the transformation under consideration. Finally, an application to the construction of so-called imprecise copulas is given.

Keywords: copula, quasi-copula, transformation of quasi-copulas, imprecise copula

Classification: 26B25, 62E10, 26B35, 60E05, 62H10

1. INTRODUCTION

Copulas were introduced in [34] (see also [1, 14, 27, 33, 35]) in order to represent and construct joint distribution functions of random vectors by means of the related one-dimensional marginal distribution functions. As a more general concept, quasi-copulas were introduced in [2] and later characterized by means of their 1-Lipschitz property (with respect to the L_1-norm) in [16].

Quasi-copulas have interesting applications in several areas, such as fuzzy logic [18, 31], fuzzy preference modeling [9, 10] or similarity measures [8]. Other deep results concerning quasi-copulas can be found in [6, 19, 28].

While copulas are characterized by the non-negativity of the volume of each subrectangle of $[0,1]^2$ which is a Cartesian product of two subintervals of $[0,1]$, this is no more true for quasi-copulas. This defect of quasi-copulas can be described in several ways, indicating how far away they are from copulas. We introduce several such descriptions and apply them to transform the original quasi-copulas. Note that the sequence of iterative transformations always converges to a copula. This allows us to introduce an equivalence relation on the set of quasi-copulas by grouping quasi-copulas converging to the same copula into an equivalence class. An interesting application of our approach to the so-called imprecise copulas [25, 26, 30, 36] will also be given.

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The paper is organized as follows. In Section 2, some preliminary notions and examples concerning copulas and quasi-copulas are given. Six types of functions induced by quasi-copulas and characterizing their defects are introduced and discussed in Section 3. In Section 4, the corresponding transformations of quasi-copulas are studied. The concept of imprecise copulas is recalled, and their relations to the transformations in Section 4 are shown in Section 5.

2. COPULAS AND QUASI-COPULAS

In this paper, we restrict ourselves to the case of real functions defined on the unit square $[0,1]^2$. Therefore, there is no need to use adjectives like *binary*, 2-*dimensional* or *bivariate*, even if, e. g., for copulas, multivariate generalizations exist.

Definition 2.1. A function $C\colon [0,1]^2 \to [0,1]$ is called a *copula* if the following conditions are satisfied:

(i) C is *grounded*, i. e., we have $C(0,x) = C(x,0) = 0$ for all $x \in [0,1]$;

(ii) 1 is a *neutral element* of C, i. e., we have $C(x,1) = C(1,x) = x$ for all $x \in [0,1]$;

(iii) C is 2-*increasing*, i. e., for each rectangle $[a,b] \times [c,d] \subseteq [0,1]^2$ we have

$$V_C([a,b] \times [c,d]) = C(a,c) + C(b,d) - C(a,d) - C(b,c) \geq 0. \tag{2.1}$$

The set of all copulas will be denoted by $\mathcal{C}$. Note that for each copula C we have $W \leq C \leq M$, where the Fréchet–Hoeffding lower and upper bound W and M are given by $W(x,y) = \max(x+y-1,0)$ and $M(x,y) = x \wedge y$, respectively, and where the order on $\mathcal{C}$ is the pointwise partial order inherited from the linear order on $[0,1]$. This means that the partially ordered set $\mathcal{C}$ has M as top element and W as bottom element, but $\mathcal{C}$ is not a lattice since the supremum of two copulas is not necessarily a copula (see, e. g., $C_1 \vee C_2$ in Example 2.3 below). For more details about copulas and their applications see [14, 27].

The value $V_C([a,b] \times [c,d])$ given by (2.1) is called the C-*volume* of the rectangle $[a,b] \times [c,d]$. Observe that it formally can be defined for each function $F\colon G^2 \to \mathbb{R}$ and each rectangle $[a,b] \times [c,d] \subseteq G^2 \subseteq \mathbb{R}^2$.

Definition 2.2. A function $Q\colon [0,1]^2 \to [0,1]$ is called a *quasi-copula* if it satisfies conditions (i) and (ii) in Definition 2.1 and inequality (2.1) for all rectangles $[a,b]\times[c,d] \subseteq [0,1]^2$ such that $\{a,b,c,d\} \cap \{0,1\} \neq \emptyset$.

The set of all quasi-copulas will be denoted by $\mathcal{Q}$ and, evidently, we have $\mathcal{C} \subset \mathcal{Q}$. Quasi-copulas which are not copulas, i. e., elements of $\mathcal{Q} \setminus \mathcal{C}$, are called *proper quasi-copulas* (see, for instance, Example 2.4). From Definitions 2.1 and 2.2 it follows that they have a negative volume for some rectangle $[a,b] \times [c,d] \subset [0,1]^2$.

Observe that a function $Q\colon [0,1]^2 \to [0,1]$ is a quasi-copula if and only if it is monotone non-decreasing in each coordinate, grounded, has 1 as neutral element and is 1-Lipschitz, i. e., for all $(x_1,x_2),(y_1,y_2) \in [0,1]^2$ we have

$$|Q(x_1,x_2) - Q(y_1,y_2)| \leq |x_1 - y_1| + |x_2 - y_2|. \tag{2.2}$$

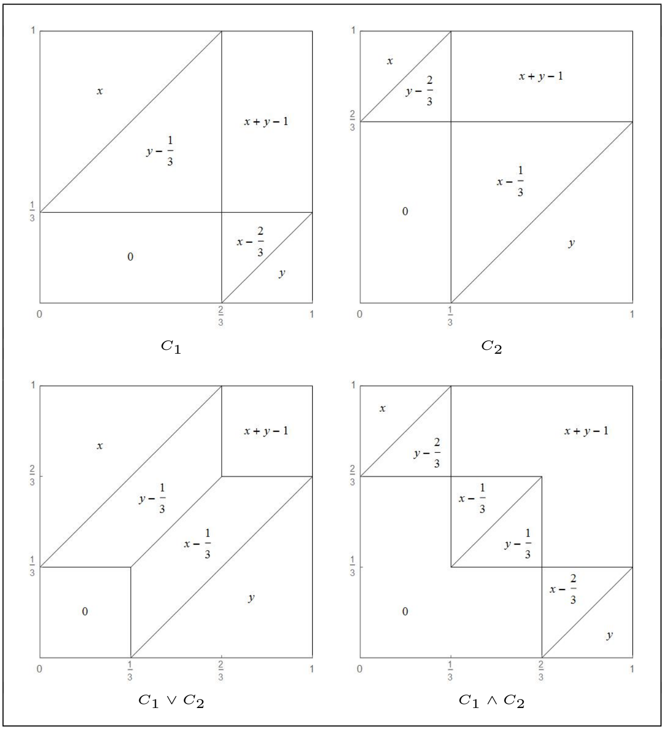

Fig. 1. Copulas C_1, C_2, $C_1 \wedge C_2$ and proper quasi-copula $C_1 \vee C_2$ from Example 2.3.

From $\mathcal{C} \subset \mathcal{Q}$ it follows that also each copula is non-decreasing in each coordinate and 1-Lipschitz.

From a lattice-theoretic point of view, $\mathcal{Q}$ is the smallest complete lattice containing $\mathcal{C}$, i.e., for each quasi-copula Q we have $Q = \inf\{C_i \mid i \in I\} = \sup\{C_j | j \in J\}$ for some families of copulas $(C_i)_{i\in I}$ and $(C_j)_{j\in J}$ (in fact, $\mathcal{Q}$ was shown in [29] to be order-isomorphic to the Dedekind-MacNeille completion of $\mathcal{C}$). In particular, for each family of copulas both its infimum and its supremum are quasi-copulas, and the Fréchet–Hoeffding bounds M and W are the top and the bottom element of $\mathcal{Q}$. For more details on quasi-copulas see [16] and [17].

Example 2.3. Consider the copulas C_1 and C_2 illustrated in Figure 1 (top). The unit mass of copula C_1 is uniformly distributed on the line segments connecting the points $(0, \frac{1}{3})$ and $(\frac{2}{3}, 1)$, and $(\frac{2}{3}, 0)$ and $(1, \frac{1}{3})$, respectively. The unit mass of copula C_2 is uniformly distributed on the line segments connecting the points $(0, \frac{2}{3})$ and $(\frac{1}{3}, 1)$, and $(\frac{1}{3}, 0)$ and $(1, \frac{2}{3})$, respectively. In Figure 1, also the functions $C_1 \vee C_2$ and $C_1 \wedge C_2$ are visualized. Observe that $C_1 \vee C_2$ is a proper quasi-copula since, e.g., $V_{C_1 \vee C_2}([\frac{1}{3}, \frac{2}{3}]^2) = -\frac{1}{3}$, while $C_1 \wedge C_2$ is a copula.

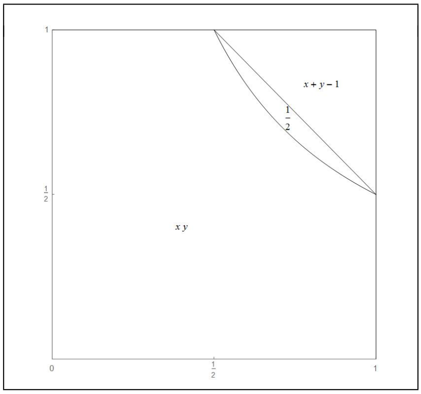

Fig. 2. The proper quasi-copula Q in Example 2.4.

Example 2.4. The function Q given by

$$Q(x,y) = \mathrm{med}(x \cdot y, \tfrac{1}{2}, x+y-1) = \begin{cases} x \cdot y & \text{if } x \cdot y \leq \frac{1}{2}, \\ \max(x+y-1, \frac{1}{2}) & \text{otherwise,} \end{cases} \tag{2.3}$$

where med is the shortcut for the median, is a proper quasi-copula, and it is visualized in Figure 2. Observe that the minimal value of the Q-volume of a rectangle in $[0,1]^2$ is attained for the square $\left[\frac{2}{3}, \frac{3}{4}\right]^2$ where $V_Q(\left[\frac{2}{3}, \frac{3}{4}\right]^2) = -\frac{1}{18}$.

3. DEFECTS OF QUASI-COPULAS

Consider an arbitrary point $(x_0, y_0) \in [0,1]^2$. Then it is clear that each rectangle $[a,b] \times [c,d] \subseteq [0,1]^2$ (i. e., its edges are parallel to the axes of the unit square) which has (x_0, y_0) as one of its vertices belongs (with the exception of line segments, i. e., when $a = b = x_0$ or $c = d = y_0$, or of the trivial rectangle consisting of the point (x_0, y_0) only, i. e., when $a = b = x_0$ and $c = d = y_0$) to exactly one of the following sets:

$$\begin{aligned}
\mathcal{R}_{\nearrow}(x_0,y_0) &= \{[x_0, x_0+\alpha] \times [y_0, y_0+\beta] \subseteq [0,1]^2 \mid \alpha, \beta \geq 0\}, \\
\mathcal{R}_{\searrow}(x_0,y_0) &= \{[x_0, x_0+\alpha] \times [y_0-\beta, y_0] \subseteq [0,1]^2 \mid \alpha, \beta \geq 0\}, \\
\mathcal{R}_{\swarrow}(x_0,y_0) &= \{[x_0-\alpha, x_0] \times [y_0-\beta, y_0] \subseteq [0,1]^2 \mid \alpha, \beta \geq 0\}, \\
\mathcal{R}_{\nwarrow}(x_0,y_0) &= \{[x_0-\alpha, x_0] \times [y_0, y_0+\beta] \subseteq [0,1]^2 \mid \alpha, \beta \geq 0\}.
\end{aligned}$$

Based on these sets of rectangles, we can introduce four different defect functions for a given quasi-copula:

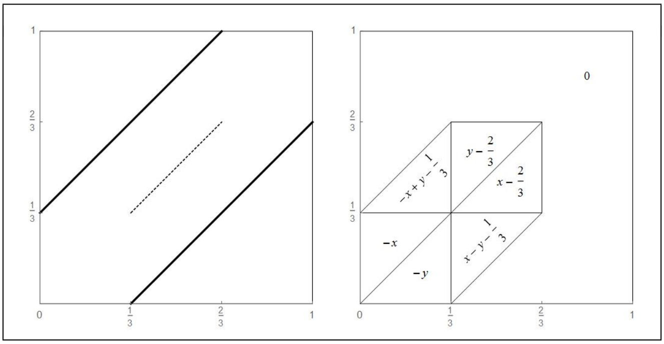

Fig. 3. Support of the proper quasi-copula Q (left), and the northeast-defect function $D^Q_{\nearrow}$ (Example 3.4).

Definition 3.1. Let $Q\colon [0,1]^2 \to [0,1]$ be a quasi-copula. Then we consider the following defect functions $D^Q_{\nearrow}, D^Q_{\searrow}, D^Q_{\swarrow}, D^Q_{\nwarrow} : [0,1]^2 \to \mathbb{R}$ given by

$$D^Q_{\nearrow}(x,y) = \inf\{V_Q(R) \mid R \in \mathcal{R}_{\nearrow}(x,y)\}, \qquad (\textit{northeast-defect of } Q) \tag{3.1}$$

$$D^Q_{\searrow}(x,y) = \inf\{V_Q(R) \mid R \in \mathcal{R}_{\searrow}(x,y)\}, \qquad (\textit{southeast-defect of } Q) \tag{3.2}$$

$$D^Q_{\swarrow}(x,y) = \inf\{V_Q(R) \mid R \in \mathcal{R}_{\swarrow}(x,y)\}, \qquad (\textit{southwest-defect of } Q) \tag{3.3}$$

$$D^Q_{\nwarrow}(x,y) = \inf\{V_Q(R) \mid R \in \mathcal{R}_{\nwarrow}(x,y)\}. \qquad (\textit{northwest-defect of } Q) \tag{3.4}$$

It is obvious that each of these defect functions is non-positive. As a consequence of the continuity of Q, each infimum in Definition 3.1 is actually attained (and, therefore, can be replaced by a minimum).

Moreover, we have the following result which immediately follows from the Definitions 2.1 and 2.2.

Proposition 3.2. Let $Q\colon [0,1]^2 \to [0,1]$ be a quasi-copula. Then Q is a copula if and only if one (and, subsequently, each) of the defect functions $D^Q_{\nearrow}$, $D^Q_{\searrow}$, $D^Q_{\swarrow}$, and $D^Q_{\nwarrow}$ given by (3.1)–(3.4) is identically zero.

Based on Definition 3.1, it is possible to introduce additional defect functions, two of which are given below:

Definition 3.3. Let $Q\colon [0,1]^2 \to [0,1]$ be a quasi-copula. Then the following defect functions $D^Q_{\mathrm{M}}, D^Q_{\mathrm{O}}\colon [0,1]^2 \to \mathbb{R}$ are given by

$$D^Q_{\mathrm{M}} = D^Q_{\nearrow} \wedge D^Q_{\swarrow}, \qquad (\textit{main-defect of } Q) \tag{3.5}$$

$$D^Q_{\mathrm{O}} = D^Q_{\nwarrow} \wedge D^Q_{\searrow}. \qquad (\textit{opposite-defect of } Q) \tag{3.6}$$

Observe that the main-defect $D^Q_{\mathrm{M}}(x,y)$ of Q is related to rectangles in $\mathcal{R}_{\nearrow}(x,y) \cup \mathcal{R}_{\swarrow}(x,y)$, i. e., having (x,y) as lower left or upper right vertex. Similarly, the opposite-

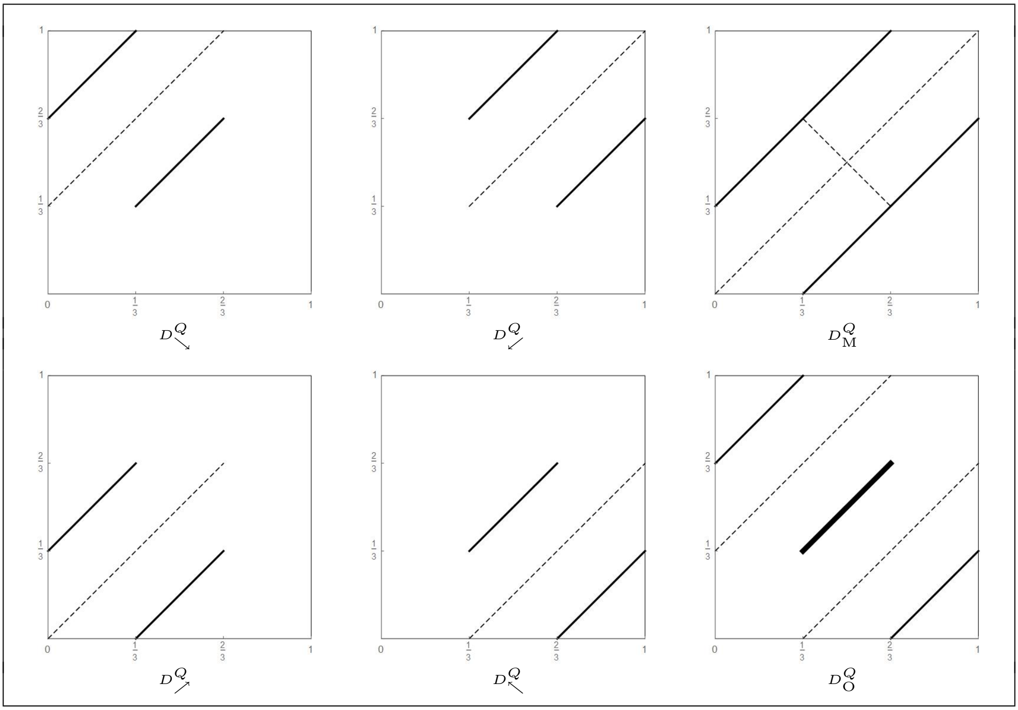

Fig. 4. Supports of the six defect functions of Q (Example 3.4).

defect $D^Q_{\mathrm{O}}(x,y)$ of Q is related to $\mathcal{R}_{\nwarrow}(x,y) \cup \mathcal{R}_{\searrow}(x,y)$, i.e., to rectangles having (x,y) as lower right or upper left vertex.

Obviously, all the six defect functions introduced in Definitions 3.1 and 3.3 vanish at the boundary points of the unit square $[0,1]^2$. On the other hand, for each proper quasi-copula each of these six defect functions is strictly negative on a subset of $[0,1]^2$ with positive Lebesgue measure.

Example 3.4. Put $Q = C_1 \vee C_2$, i.e., the proper quasi-copula considered in Example 2.3 and Figure 1 (bottom left). The support of Q is given in Figure 3 (left). Note that on the thick line segments the mass $\frac{4}{3}$ is uniformly distributed, while on the dashed line segment the mass $-\frac{1}{3}$ is uniformly distributed. The northeast-defect function $D^Q_{\nearrow}$ given by (3.1) is visualized in Figure 3 (right). Similarly the supports of all six defect functions of Q can be illustrated (see Figure 4 and notice that the extra thick line in the support of D^Q_{O} indicates that the line segment connecting the points $(\frac{1}{3},\frac{1}{3})$ and $(\frac{2}{3},\frac{2}{3})$ carries twice as much mass). Finally, in Figure 5, the main-defect function D^Q_{M} and the opposite-defect function D^Q_{O} of the quasi-copula Q are given.

All the defect functions discussed so far have their values in the interval $\left[-\frac{1}{3},0\right]$, and the extremal value can be attained only in vertices of the square $\left[\frac{1}{3},\frac{2}{3}\right]^2$ (compare Example 3.4). We even have the following stronger result:

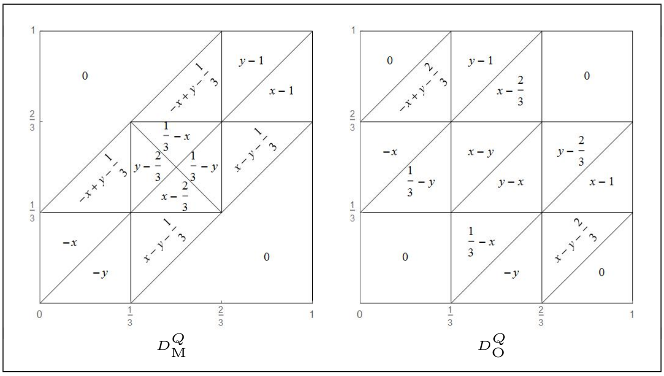

Fig. 5. Main- and opposite-defect functions (Example 3.4).

Theorem 3.5. Define the functions $D_{\nearrow}, D_{\searrow}, D_{\swarrow}, D_{\nwarrow}, D_{\mathrm{M}}, D_{\mathrm{O}} \colon [0,1]^2 \to \mathbb{R}$ by

$$\begin{aligned} D_{\nearrow} &= \inf\{D^Q_{\nearrow} \mid Q \in \mathcal{Q}\}, & D_{\swarrow} &= \inf\{D^Q_{\swarrow} \mid Q \in \mathcal{Q}\}, \\ D_{\nwarrow} &= \inf\{D^Q_{\nwarrow} \mid Q \in \mathcal{Q}\}, & D_{\searrow} &= \inf\{D^Q_{\searrow} \mid Q \in \mathcal{Q}\}, \\ D_{\mathrm{M}} &= \inf\{D^Q_{\mathrm{M}} \mid Q \in \mathcal{Q}\}, & D_{\mathrm{O}} &= \inf\{D^Q_{\mathrm{O}} \mid Q \in \mathcal{Q}\}. \end{aligned}$$

Then we have $D_{\mathrm{M}} = D_{\nearrow} \wedge D_{\swarrow}$ and $D_{\mathrm{O}} = D_{\nwarrow} \wedge D_{\searrow}$ and, for all $(x,y) \in [0,1]^2$,

$$\begin{aligned} D_{\nearrow}(x,y) &= \max(-x, -y, \tfrac{x-1}{2}, \tfrac{y-1}{2}), \\ D_{\swarrow}(x,y) &= \max(x-1, y-1, -\tfrac{x}{2}, -\tfrac{y}{2}), \\ D_{\nwarrow}(x,y) &= \max(x-1, -y, -\tfrac{x}{2}, \tfrac{y-1}{2}), \\ D_{\searrow}(x,y) &= \max(-x, y-1, \tfrac{x-1}{2}, -\tfrac{y}{2}). \end{aligned}$$

Proof. The result for D_{M} and D_{O} follows directly from Definition 3.3.

Now fix a point $(x,y) \in [0,1]^2$. Then for each quasi-copula Q and each rectangle $R = [x, x_1] \times [y, y_1] \in \mathcal{R}_{\nearrow}(x,y)$ we have

$$\begin{aligned} V_Q(R) + x &= V_Q([x,x_1] \times [y,y_1]) + V_Q([0,x] \times [0,1]) \\ &= V_Q([0,x] \times [0,y]) + V_Q([0,x_1] \times [y,y_1]) + V_Q([0,x] \times [y_1,1]). \end{aligned}$$

Each of the three rectangles considered in the last line shares an edge with the boundary of $[0,1]^2$ and, therefore, has a nonnegative Q-volume. Therefore, $V_Q(R) \geq -x$ for each rectangle $R \in \mathcal{R}_{\nearrow}(x,y)$ and each $Q \in \mathcal{Q}$, implying $D_{\nearrow}(x,y) \geq -x$. In an analogous way $D_{\nearrow}(x,y) \geq -y$ is shown.

For each quasi-copula Q and each rectangle $R = [x, x_1] \times [y, y_1] \in \mathcal{R}_{\nearrow}(x,y)$ we get,

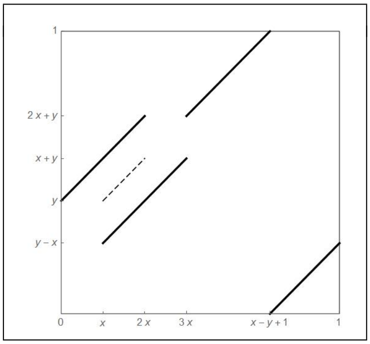

Fig. 6. Support of the quasi-copula $Q_{x,y}$.

using similar arguments, also

$$\begin{aligned}2V_Q(R)+1-x &= 2V_Q(R)+V_Q([x,1]\times[0,1])\\ &= V_Q([x,x_1]\times[y,1])+V_Q([x,x_1]\times[0,y_1])+V_Q([x,1]\times[y,y_1])\\ &\qquad +V_Q([x_1,1]\times[y_1,1])+V_Q([x_1,1]\times[0,y])\\ &\geq 0,\end{aligned}$$

implying $D_{\nearrow}(x,y)\geq\frac{x-1}{2}$. Of course, $D_{\nearrow}(x,y)\geq\frac{y-1}{2}$ is shown in complete analogy. Summarizing, we know now that $D_{\nearrow}(x,y)\geq\max(-x,-y,\frac{x-1}{2},\frac{y-1}{2})$.

Now fix a point (x,y) satisfying $x\leq\frac{1}{3}$ and $x\leq y\leq 1-2x$. Then $y-x\geq 0$, $y+2x\leq 1$ and $3x\leq 1$, implying $\max(-x,-y,\frac{x-1}{2},\frac{y-1}{2})=-x$. Denote now by $Q_{x,y}$ the quasi-copula whose support is visualized in Figure 6.

Observe that the restriction of $Q_{x,y}$ to the square $[0,3x]\times[y-x,y+2x]$ is a linear transformation of the proper quasi-copula $C_1\vee C_2$ considered in Examples 2.3 and 3.4. Because of $V_{Q_{x,y}}([x,2x]\times[y,y+x])=-x$ we have $D_{\nearrow}(x,y)=-x$.

In a similar way, for each point (x,y) the existence of a proper quasi-copula Q and of a rectangle $R\in\mathcal{R}_{\nearrow}(x,y)$ with $V_Q(R)=\max(-x,-y,\frac{x-1}{2},\frac{y-1}{2})$ is shown, proving the validity of $D_{\nearrow}(x,y)=\max(-x,-y,\frac{x-1}{2},\frac{y-1}{2})$.

Note that for each quasi-copula $Q\in\mathcal{Q}$ also the function $\widehat{Q}\colon[0,1]^2\to[0,1]$ given by

$$\widehat{Q}(x,y)=x+y-1+Q(1-x,1-y) \tag{3.7}$$

is a quasi-copula and, moreover, the mapping $Q\mapsto\widehat{Q}$ is an involution, i. e., we have

$$\widehat{(\widehat{Q})}=Q.$$

This implies $D_{\swarrow}(x,y)=D_{\nearrow}(1-x,1-y)$ for all $(x,y)\in[0,1]^2$ and, subsequently, $D_{\swarrow}(x,y)=\max(x-1,y-1,-\frac{x}{2},-\frac{y}{2})$.

The remaining equalities for $D_{\nwarrow}$ and $D_{\searrow}$ follow from results in [20]: for each quasi-copula $Q\in\mathcal{Q}$ also the functions $Q^-,Q_-\colon[0,1]^2\to[0,1]$ given by

$$Q^-(x,y)=x-Q(x,1-y),\qquad Q_-(x,y)=y-Q(1-x,y), \tag{3.8}$$

respectively, are quasi-copulas. Therefore $D_{\nwarrow}(x,y) = D_{\nearrow}(1-x,y)$ and $D_{\searrow}(x,y) = D_{\nearrow}(x,1-y)$ for each $(x,y) \in [0,1]^2$, completing the proof. $\square$

Remark 3.6. Note that there is no quasi-copula $Q \in \mathcal{Q}$ such that $D^Q_{\nearrow} = D_{\nearrow}$ (an analogous statement holds for each of the other defect functions). Assuming the contrary, i. e., $D^Q_{\nearrow} = D_{\nearrow}$ for some $Q \in \mathcal{Q}$, this means in particular

$$D^Q_{\nearrow}\Big(\frac{1}{3}, \frac{1}{3}\Big) = D_{\nearrow}\Big(\frac{1}{3}, \frac{1}{3}\Big) = -\frac{1}{3}.$$

Then, as a consequence of [14, Theorem 7.4.4], we necessarily get $Q(\frac{1}{3}, \frac{1}{3}) = 0$ and

$$Q\Big(\frac{1}{3}, \frac{2}{3}\Big) = Q\Big(\frac{2}{3}, \frac{1}{3}\Big) = Q\Big(\frac{2}{3}, \frac{2}{3}\Big) = \frac{1}{3}.$$

From the latter equality it follows easily that Q coincides with the Fréchet–Hoeffding lower bound W on the square $\big[\frac{2}{3}, 1\big]^2$, implying that $D^Q_{\nearrow}(\frac{2}{3}, \frac{2}{3}) = D^W_{\nearrow}(\frac{2}{3}, \frac{2}{3}) = 0$, i. e.,

$$D_{\nearrow}\Big(\frac{2}{3}, \frac{2}{3}\Big) = -\frac{1}{6} < 0 = D^Q_{\nearrow}\Big(\frac{2}{3}, \frac{2}{3}\Big).$$

Because of some symmetries of the defects introduced in (3.1)–(3.6) we may restrict our considerations to, say, northeast-defects of quasi-copulas only.

Remark 3.7. Using similar arguments as in the proof of Theorem 3.5, it is possible to show that for each quasi-copula $Q \in \mathcal{Q}$ and for the quasi-copulas $\widehat{Q}$, Q^- and Q_- considered in (3.7) and (3.8) the following equalities hold for all $(x,y) \in [0,1]^2$:

$$\begin{aligned}
D^{\widehat{Q}}_{\nearrow}(x,y) &= D^Q_{\swarrow}(1-x,1-y), & D^{\widehat{Q}}_{\searrow}(x,y) &= D^Q_{\nwarrow}(1-x,1-y),\\
D^{Q^-}_{\nearrow}(x,y) &= D^Q_{\searrow}(x,1-y), & D^{Q^-}_{\swarrow}(x,y) &= D^Q_{\nwarrow}(x,1-y),\\
D^{Q_-}_{\nearrow}(x,y) &= D^Q_{\nwarrow}(1-x,y), & D^{Q_-}_{\swarrow}(x,y) &= D^Q_{\searrow}(1-x,y).
\end{aligned}$$

As a consequence, the main- and the opposite-defect of a quasi-copula Q can be expressed by northeast-defects of the quasi-copulas Q, $\widehat{Q}$, Q^-, and Q_-, i. e., for all $(x,y) \in [0,1]^2$ we have

$$\begin{aligned}
D^Q_{\mathrm{M}}(x,y) &= D^Q_{\nearrow}(x,y) \wedge D^{\widehat{Q}}_{\nearrow}(1-x,1-y),\\
D^Q_{\mathrm{O}}(x,y) &= D^{Q^-}_{\nearrow}(1-x,y) \wedge D^{Q_-}_{\nearrow}(x,1-y).
\end{aligned}$$

An important tool for the construction of new quasi-copulas from given ones is the so-called ordinal sum. Based on earlier results in the context of partially ordered sets [4] and of abstract semigroups [5], the concept of an ordinal sum of triangular norms was introduced in [24, 32] (compare also [1, 21, 33]), and it can be carried over to the case of (quasi-)copulas in a straightforward way.

The following example shows that, for an ordinal sum of quasi-copulas, also the corresponding defect functions given in Definitions 3.1 and 3.3 have an ordinal sum structure:

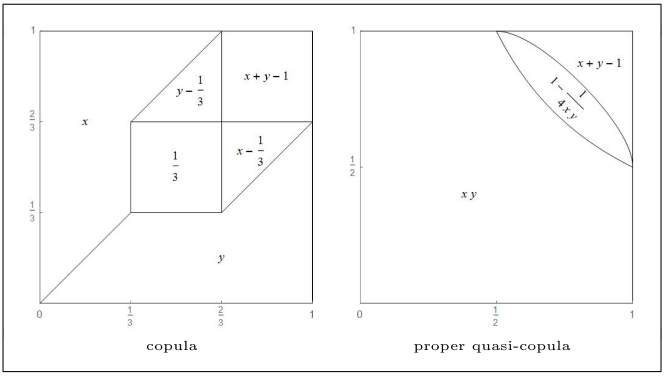

Fig. 7. Examples 4.1 (left) and 4.2.

Example 3.8. Let $(]a_i, b_i[)_{i\in I}$ be a pairwise disjoint family of non-empty open subintervals of $[0,1]$, $(Q_i)_{i\in I}$ be a family of quasi-copulas, and assume that the quasi-copula $Q = (\langle a_i, b_i, Q_i\rangle)_{i\in I}$ is the ordinal sum of the summands $(]a_i, b_i[\,, Q_i)_{i\in I}$ given by

$$Q(x,y) = \begin{cases} a_i + (b_i - a_i)\cdot Q_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) & \text{if } (x,y)\in [a_i,b_i]^2\,, \\ M(x,y) & \text{otherwise.} \end{cases} \tag{3.9}$$

Then also the structure of the defect functions $D^Q_{\nearrow}, D^Q_{\searrow}, D^Q_{\swarrow}, D^Q_{\nwarrow}, D^Q_{\mathrm{M}}$, and D^Q_{O} given in (3.1)–(3.6) is that of an ordinal sum. We give here the exact formula for $D^Q_{\nearrow}$ only (remember that $D^M_{\nearrow}(x,y) = 0$ for all $(x,y)\in[0,1]^2$):

$$D^Q_{\nearrow}(x,y) = \begin{cases} (b_i - a_i)\cdot D^{Q_i}_{\nearrow}\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) & \text{if } (x,y)\in [a_i,b_i]^2\,, \\ D^M_{\nearrow}(x,y) & \text{otherwise.} \end{cases}$$

4. DEFECT-BASED TRANSFORMATIONS OF QUASI-COPULAS

Several constructions and transformations of copulas and quasi-copulas have been considered so far (see, e. g., [3, 7, 11, 13, 15, 20, 22, 23]). Here we use the defect functions given in Definitions 3.1 and 3.3 to introduce new types of transformations of quasi-copulas.

For a quasi-copula Q, consider the functions $Q_{\nearrow}, Q_{\swarrow}, Q_{\nwarrow}, Q_{\searrow}, Q_{\mathrm{M}}, Q_{\mathrm{O}}\colon [0,1]^2 \to [0,1]$ defined by, respectively,

$$Q_{\nearrow} = Q - D^Q_{\nearrow}, \qquad Q_{\swarrow} = Q - D^Q_{\swarrow}, \tag{4.1}$$

$$Q_{\nwarrow} = Q + D^Q_{\nwarrow}, \qquad Q_{\searrow} = Q + D^Q_{\searrow}, \tag{4.2}$$

$$Q_{\mathrm{M}} = Q - D^Q_{\mathrm{M}}, \qquad Q_{\mathrm{O}} = Q + D^Q_{\mathrm{O}}. \tag{4.3}$$

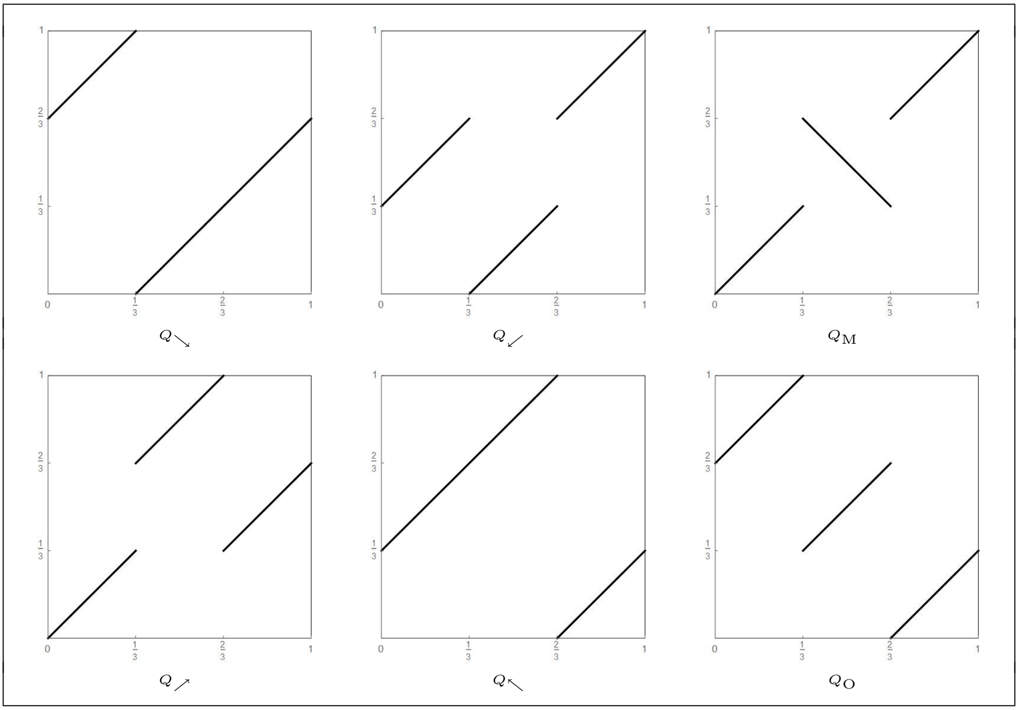

Fig. 8. Supports of the six transformations of Q (Example 4.1).

A natural question arises: given an arbitrary quasi-copula Q, is each of the functions $Q_{\nearrow}$, $Q_{\swarrow}$, $Q_{\nwarrow}$, $Q_{\searrow}$, Q_{M}, and Q_{O} given above a (quasi-)copula?

Example 4.1. Consider again the quasi-copula $Q = C_1 \vee C_2$ introduced in Example 2.3 and discussed in Example 3.4. Observe that the function $Q_{\nearrow}$ visualized in Figure 7 (left) is a copula (actually, it turns out to be a shuffle of the Fréchet–Hoeffding upper bound M). For more details about shuffles of M see [12, 14, 27]. In analogy, the functions $Q_{\swarrow}, Q_{\nwarrow}, Q_{\searrow}, Q_{\mathrm{M}}$, and Q_{O} are shuffles of M and, therefore, copulas (for a visualization of their supports see Figure 8).

In general, however, we don't obtain copulas using these transformations.

Example 4.2. Consider the proper quasi-copula Q given by (2.3) in Example 2.4 (see also Figure 2). Then the function $Q_{\swarrow}$ (see Figure 7 right) is a proper quasi-copula (observe that, e.g., $V_{Q_{\swarrow}}\big(\big[\frac{3}{4}, \frac{4}{5}\big]^2\big) = -\frac{1}{80}$). Also the other functions $Q_{\nearrow}$, $Q_{\nwarrow}$, $Q_{\searrow}$, Q_{M}, and Q_{O} are proper quasi-copulas.

Theorem 4.3. Let $Q \in \mathcal{Q}$ be a quasi-copula. Then each of the six functions $Q_{\nearrow}$, $Q_{\swarrow}$, $Q_{\nwarrow}$, $Q_{\searrow}$, Q_{M}, and Q_{O} given in (4.1)–(4.3) is a quasi-copula.

P r o o f. Fix an arbitrary quasi-copula $Q \in \mathcal{Q}$. Then each of the six functions $Q_{\nearrow}$, $Q_{\swarrow}$,

$Q_{\nwarrow}$, $Q_{\searrow}$, Q_{M}, and Q_{O} satisfies the boundary conditions, i. e., it is grounded and has 1 as neutral element.

Now fix an arbitrary point $(x,y) \in \,]0,1[^2$ and $\varepsilon \in \,]0, 1-x[$. Then we have $Q(x,y) \leq Q_{\nearrow}(x,y)$ and $Q(x+\varepsilon,y) \leq Q_{\nearrow}(x+\varepsilon,y)$.

If $D^Q_{\nearrow}(x,y) = 0$ then $Q_{\nearrow}(x,y) = Q(x,y) \leq Q(x+\varepsilon,y) \leq Q_{\nearrow}(x+\varepsilon,y)$. If $D^Q_{\nearrow}(x,y) < 0$ then the continuity of Q implies the existence of a rectangle $R = [x,x_1] \times [y,y_1] \in \mathcal{R}_{\nearrow}(x,y)$ such that $D^Q_{\nearrow}(x,y) = V_Q(R)$.

Suppose first that $x_1 \leq x+\varepsilon$. Then we get

$$Q_{\nearrow}(x+\varepsilon,y) - Q_{\nearrow}(x,y) \geq Q(x+\varepsilon,y) - Q(x_1,y) + Q(x_1,y_1) - Q(x,y_1) \geq 0,$$

i. e., $Q_{\nearrow}(x,y) \leq Q_{\nearrow}(x+\varepsilon,y)$.

On the other hand, if $x_1 > x+\varepsilon$ then

$$\begin{aligned} Q_{\nearrow}(x,y) &= Q(x,y_1) + Q(x_1,y) - Q(x_1,y_1) \\ &= Q(x+\varepsilon,y) - (Q(x_1,y_1) - Q(x+\varepsilon,y_1) - Q(x_1,y) + Q(x+\varepsilon,y)) \\ &\quad - (Q(x+\varepsilon,y_1) - Q(x,y_1)) \\ &\leq Q(x+\varepsilon,y) - V_Q([x+\varepsilon,x_1] \times [y,y_1]) \\ &\leq Q_{\nearrow}(x+\varepsilon,y). \end{aligned}$$

Therefore, $Q_{\nearrow}$ is monotone non-decreasing in its first coordinate. The monotonicity in the second coordinate is shown analogously. Using similar arguments, the monotonicity of the functions $Q_{\swarrow}$, $Q_{\nwarrow}$, and $Q_{\searrow}$ is verified.

Recall that for the quasi-copula Q^- given by $Q^-(x,y) = x - Q(x,1-y)$ we have $D^Q_{\nearrow}(x,y) = D^{Q^-}_{\searrow}(x,y)$ for all $(x,y) \in [0,1]^2$. As a consequence, for each $\varepsilon \in \,]0,1-x[$ we obtain

$$\begin{aligned} 0 &\leq Q_{\nearrow}(x+\varepsilon,y) - Q_{\nearrow}(x,y) \\ &= Q(x+\varepsilon,y) - D^Q_{\nearrow}(x+\varepsilon,y) - Q(x,y) + D^Q_{\nearrow}(x,y) \\ &= x+\varepsilon - Q^-(x+\varepsilon,1-y) - D^{Q^-}_{\searrow}(x+\varepsilon,1-y) - x + Q^-(x,1-y) + D^{Q^-}_{\searrow}(x,1-y) \\ &= \varepsilon + Q^-_{\searrow}(x,1-y) - Q^-_{\searrow}(x+\varepsilon,1-y) \\ &\leq \varepsilon, \end{aligned}$$

where the latter inequality follows from the monotonicity of $Q^-_{\searrow}$, thus proving the 1-Lipschitz property of $Q^-_{\nearrow}$ in the first coordinate. In a similar way, the 1-Lipschitz property of $Q_{\nearrow}$ in the second coordinate can be shown, completing the proof that $Q_{\nearrow}$ is a quasi-copula.

In the same way, one verifies that $Q_{\swarrow}$, $Q_{\nwarrow}$, and $Q_{\searrow}$ are quasi-copulas. Since $\mathcal{Q}$ is a lattice, also $Q_{\mathrm{M}} = Q_{\nearrow} \vee Q_{\swarrow}$ and $Q_{\mathrm{O}} = Q_{\nwarrow} \wedge Q_{\searrow}$ are quasi-copulas. $\square$

Observe that for each quasi-copula $Q \in \mathcal{Q}$ we have the inequalities

$$\begin{aligned} Q_{\mathrm{O}} \leq Q_{\nwarrow} \leq Q \leq Q_{\nearrow} \leq Q_{\mathrm{M}}, \\ Q_{\mathrm{O}} \leq Q_{\searrow} \leq Q \leq Q_{\swarrow} \leq Q_{\mathrm{M}}. \end{aligned}$$

If $Q \in \mathcal{Q}$ is a proper quasi-copula then we obtain the strict inequalities

$$Q_{\nwarrow} < Q, \qquad Q_{\searrow} < Q, \qquad Q < Q_{\nearrow}, \qquad Q < Q_{\swarrow}.$$

Example 4.4. Let $(]a_i, b_i[)_{i\in I}$ be a pairwise disjoint family of non-empty open subintervals of $[0,1]$, $(Q_i)_{i\in I}$ be a family of quasi-copulas, and assume that the quasi-copula $Q = (\langle a_i, b_i, Q_i\rangle)_{i\in I}$ is the ordinal sum of the summands $(]a_i, b_i[\,, Q_i)_{i\in I}$ given by (3.9).

Then each of the six quasi-copulas $Q_{\nearrow}$, $Q_{\swarrow}$, $Q_{\nwarrow}$, $Q_{\searrow}$, Q_{M}, and Q_{O} given in (4.1)–(4.3) is an ordinal sum of quasi-copulas, i. e., $Q_{\nearrow} = (\langle a_i, b_i, (Q_i)_{\nearrow}\rangle)_{i\in I}$, and analogously for $Q_{\swarrow}$, $Q_{\nwarrow}$, $Q_{\searrow}$, Q_{M}, and Q_{O}.

Based on the results of Theorem 4.3, it is possible to define the following six transformations $\nearrow, \swarrow, \searrow, \nwarrow, \mathrm{M}, \mathrm{O}\colon \mathcal{Q} \to \mathcal{Q}$ by

$$\nearrow(Q) = Q_{\nearrow}, \qquad \swarrow(Q) = Q_{\swarrow}, \qquad \searrow(Q) = Q_{\searrow}, \qquad \nwarrow(Q) = Q_{\nwarrow},$$

$$\mathrm{M}(Q) = Q_{\mathrm{M}}, \qquad \mathrm{O}(Q) = Q_{\mathrm{O}}.$$

Of course, any composition of these transformations is again a transformation on $\mathcal{Q}$.

In particular, if we write $\swarrow^1 = \swarrow$, $\swarrow^2 = \swarrow \circ \swarrow$, ..., $\swarrow^{n+1} = \swarrow \circ \swarrow^n$, we obtain, for each quasi-copula $Q \in \mathcal{Q}$, the sequence $(\swarrow^n(Q))_{n\in\mathbb{N}}$ which is monotone non-decreasing, implying that its supremum $\swarrow^*(Q)$ coincides with its pointwise limit. Obviously, we have $\swarrow \circ \swarrow^* = \swarrow^*$ and, subsequently, $D_{\swarrow}^{\swarrow^*(Q)} = 0$, implying that $\swarrow^*(Q)$ is a copula. In a similar way, we can construct the copulas $\nearrow^*(Q)$ and $\mathrm{M}^*(Q)$.

For the transformations $\searrow$, $\nwarrow$, and O, the sequences $(\searrow^n(Q))_{n\in\mathbb{N}}$, $(\nwarrow^n(Q))_{n\in\mathbb{N}}$, and $(\mathrm{O}^n(Q))_{n\in\mathbb{N}}$ are monotone non-increasing, and their respective limits (i. e., infima) $\searrow_*(Q)$, $\nwarrow_*(Q)$, and $\mathrm{O}_*(Q)$ are copulas, too.

This allows us to construct six different partitions of the set of all quasi-copulas, considering the six equivalence relations $\sim_{\nearrow}$, $\sim_{\swarrow}$, $\sim_{\searrow}$, $\sim_{\nwarrow}$, $\sim_{\mathrm{M}}$, and $\sim_{\mathrm{O}}$ on $\mathcal{Q}$ which are defined by

$$\begin{aligned}
Q_1 \sim_{\nearrow} Q_2 &\iff \nearrow^*(Q_1) = \nearrow^*(Q_2);\\
Q_1 \sim_{\swarrow} Q_2 &\iff \swarrow^*(Q_1) = \swarrow^*(Q_2);\\
Q_1 \sim_{\mathrm{M}} Q_2 &\iff \mathrm{M}^*(Q_1) = \mathrm{M}^*(Q_2);\\
Q_1 \sim_{\searrow} Q_2 &\iff \searrow_*(Q_1) = \searrow_*(Q_2);\\
Q_1 \sim_{\nwarrow} Q_2 &\iff \nwarrow_*(Q_1) = \nwarrow_*(Q_2);\\
Q_1 \sim_{\mathrm{O}} Q_2 &\iff \mathrm{O}^*(Q_1) = \mathrm{O}^*(Q_2).
\end{aligned}$$

Obviously, each of the respective equivalence classes contains exactly one element which is a copula.

Example 4.5. Consider again the proper quasi-copula Q given by (2.3) in Example 2.4 (see also Example 4.2 and Figures 2 and 7). After some computation we get for each $n \in \mathbb{N}$

$$\swarrow^n(Q)(x,y) = \begin{cases} x \cdot y & \text{if } x \cdot y \le \frac{1}{2},\\ \max(x+y-1, K_n(x,y)) & \text{otherwise,}\end{cases}$$

where

$$K_n(x,y) = \frac{2^{n-2}}{\sqrt[2^{n-1}]{2xy}} + \frac{1+2^{n-1}}{2}.$$

Since, for $x \cdot y \geq \frac{1}{2}$, we have $\lim\limits_{n\to\infty} K_n(x,y) = \dfrac{1+\log(2xy)}{2}$, the copula $\swarrow^*(Q)$ is given by

$$\swarrow^*(Q)(x,y) = \begin{cases} x \cdot y & \text{if } x \cdot y \leq \frac{1}{2}, \\ \max\left(x+y-1, \frac{1+\log(2xy)}{2}\right) & \text{otherwise.} \end{cases}$$

5. AN APPLICATION TO IMPRECISE COPULAS

Imprecise copulas were studied in [30] and [26] (see also [25, 36]) in order to construct two-dimensional *probability boxes* (briefly *p-boxes*), which are represented by ordered pairs of comparable distribution functions, from two given one-dimensional p-boxes.

Definition 5.1. A pair (A, B) of functions $A, B\colon [0,1]^2 \to [0,1]$ is called an *imprecise copula* if A and B are grounded and have 1 as neutral element, and if for each rectangle $[a,b] \times [c,d] \subseteq [0,1]^2$ we have

$$A(b,d) + B(a,c) - A(a,d) - A(b,c) \geq 0; \tag{IC1}$$

$$B(b,d) + A(a,c) - A(a,d) - A(b,c) \geq 0; \tag{IC2}$$

$$B(b,d) + B(a,c) - B(a,d) - A(b,c) \geq 0; \tag{IC3}$$

$$B(b,d) + B(a,c) - A(a,d) - B(b,c) \geq 0. \tag{IC4}$$

It is not difficult to check that, for each imprecise copula (A, B), we have $A, B \in \mathcal{Q}$, i. e., both A and B are quasi-copulas, and $A \leq B$.

The properties (IC1)–(IC4) in Definition 5.1 can be equivalently expressed in the following form:

$$B \geq A_{\nearrow}; \tag{IC1*}$$

$$B \geq A_{\swarrow}; \tag{IC2*}$$

$$A \leq B_{\nwarrow}; \tag{IC3*}$$

$$A \leq B_{\searrow}. \tag{IC4*}$$

Obviously, (IC1*) and (IC2*) are simultaneously satisfied if and only if we have $B \geq A_{\nearrow} \vee A_{\swarrow} = A_{\mathrm{M}}$. Similarly, $A \leq B_{\nwarrow} \wedge B_{\searrow} = B_{\mathrm{O}}$ is equivalent to the joint validity of (IC3*) and (IC4*).

Summarizing these observations, the following result is immediate.

Theorem 5.2. A pair (A, B) of quasi-copulas $A, B\colon [0,1]^2 \to [0,1]$ is an imprecise copula if and only if $B \geq A_{\mathrm{M}}$ and $A \leq B_{\mathrm{O}}$.

Evidently, each pair (C_1, C_2) of copulas satisfying $C_1 \leq C_2$ is an imprecise copula, in particular, each pair (C, C) with $C \in \mathcal{C}$. However, for a proper quasi-copula Q, the pair (Q, Q) is never an imprecise copula because of $Q_{\mathrm{O}} < Q < Q_{\mathrm{M}}$.

The partially ordered set of imprecise copulas forms an upper semi-lattice with top element (W, M):

Theorem 5.3. Let $((A_i, B_i))_{i\in I}$ be a family of imprecise copulas. Then also

$$\bigvee_{i\in I}(A_i, B_i) = \Big(\bigwedge_{i\in I} A_i, \bigvee_{i\in I} B_i\Big)$$

is an imprecise copula.

Proof. Evidently, both $A = \bigwedge_{i\in I} A_i$ and $B = \bigvee_{i\in I} B_i$ are quasi-copulas. Note first that for each $i \in I$ and for each rectangle $[a, b] \times [c, d] \subseteq [0, 1]^2$ we have $A_i(b, d) + B_i(a, c) - A_i(a, d) - A_i(b, c) \geq 0$ and, therefore, also

$$A_i(b, d) + B_i(a, c) \geq A_i(a, d) + A_i(b, c) \geq A(a, d) + A(b, c).$$

As a consequence, $A(b, d) + B(a, c) \geq A(a, d) + A(b, c)$, i.e., (IC1) holds for the pair (A, B). The validity of the other inequalities (IC2)–(IC4) for the pair (A, B) is shown in a similar way. □

As an immediate corollary of Theorem 5.3 we have the following result already mentioned in [26, 30]:

Corollary 5.4. For each family $(C_i)_{i\in I}$ of copulas the pair $(\underline{C}, \overline{C})$, where the two functions $\underline{C}, \overline{C}\colon [0, 1]^2 \to [0, 1]$ are given by

$$\underline{C} = \bigwedge_{i\in I} C_i \quad \text{and} \quad \overline{C} = \bigvee_{i\in I} C_i,$$

is an imprecise copula.

The problem whether all imprecise copulas can be obtained in this way (already posed in [26, 30]) is still open, namely, whether for each each imprecise copula (A, B) there is a family $(C_i)_{i\in I}$ of copulas such that $A = \underline{C}$ and $B = \overline{C}$.

Example 5.5. For each quasi-copula $Q \in \mathcal{Q}$ and for each $n \in \mathbb{N}$ the two pairs $(\mathrm{M}^n(Q), \mathrm{M}^*(Q))$ and $(\mathrm{O}^*(Q), \mathrm{O}^n(Q))$ are imprecise copulas.

CONCLUSION

We have introduced different functions measuring the defect of a quasi-copula expressed by means of extremal non-positive volumes of specific rectangles (such that defect zero characterizes copulas). These defect functions were applied to define new transformations of quasi-copulas, each of them leading to a new quasi-copula comparable with the original one, and each of them having copulas as the only fixed points. Therefore, starting with any quasi-copula Q, the iterative application of each of these transformations has a limit which necessarily is a copula, and which can be seen as an attractor of Q under the respective transformation. Then, for each transformation, the domains of attraction of copulas form a partition of the set of all quasi-copulas. Finally, an application to the construction of so-called imprecise copulas was added.

Several problems are still open and will be the subject of future research.

First of all, starting with a quasi-copula Q and considering Examples 4.1, 4.2 and Theorem 4.3, what are (sufficient) conditions for the six functions $Q_{\nearrow}$, $Q_{\swarrow}$, $Q_{\nwarrow}$, $Q_{\searrow}$, Q_{M}, and Q_{O} to be copulas?

Second, are there (for some of the transformations under consideration) any singleton members of the corresponding partition of quasi-copulas, i.e., is there a copula which does not admit any proper quasi-copula in its domain of attraction (up to the trivial cases of the Fréchet–Hoeffding upper bound M with respect to the transformations $\nwarrow$, $\searrow$ and O, and of the Fréchet–Hoeffding lower bound W with respect to the transformations $\nearrow$, $\swarrow$ and M)?

A third problem is to think about similar defect functions and transformations of quasi-copulas in the case of dimensions higher than two. A positive solution of this problem could help to solve the still open problem of imprecise copulas of higher dimensions.

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