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# CARISTI'S FIXED POINT THEOREM AND ITS EQUIVALENCES IN FUZZY METRIC SPACES

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In this article, we extend Caristi's fixed point theorem, Ekeland's variational principle and Takahashi's maximization theorem to fuzzy metric spaces in the sense of George and Veeramani [A. George , P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems. 64 (1994) 395-399]. Further, a direct simple proof of the equivalences among these theorems is provided.

*Keywords:* fuzzy metric space, Ekeland variational principle, Caristi's fixed point theorem, Takahashi's maximization theorem

*Classification:* 47H10, 58E30

## 1. INTRODUCTION

Since Ekeland [10] in 1972 (see also [11, 12]), the variational principle in the setting of complete metric spaces, now known as Ekeland's variational principle and its equivalent formulations emerged as one of the main subjects in many fields of nonlinear analysis with a wide applications in optimizations, optimal control theory, game theory, nonlinear equations, dynamical systems etc. Also it is well known that the primitive Ekeland variational principle is equivalent to the Caristi's fixed point theorem [7], and to the Takahashi's nonconvex minimization theorem [35, 36]. Let us state the following celebrated theorems.

**Theorem 1.1.** (Ekeland's $\epsilon$-variational principle [10]) Let $(X, d)$ be a complete metric space and let $f : X \to [0, +\infty]$ be a proper lower semi continuous (l.s.c.) function, where 'proper' means that there exists an $x \in X$ such that $f(x) \neq +\infty$. Given $\epsilon > 0$. For each $f(v) \neq +\infty$, there exists $u \in X$ such that

(i) $f(u) + \epsilon d(v, u) \leq f(v)$

(ii) for each $x \neq u, f(u) - \epsilon d(u, x) < f(x)$.

In 1976, Caristi proved the following fixed point theorem, which is a generalization of Banach contraction principle [4]. The Caristi's result has been studied by many authors (see e. g. [2,5–8,12]).

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**Theorem 1.2.** (Caristi [7], Caristi and Kirk [8]) Let $(X, d)$ be a complete metric space and let $T$ be a mapping from $X$ into itself. Suppose $f : X \to [0, +\infty]$ be a proper l.s.c. function and

$$d(x, Tx) \leq f(x) - f(Tx), \tag{1.1}$$

for all $x \in X$. Then $T$ has a fixed point in $X$.

Moreover Takahashi introduced the following theorem:

**Theorem 1.3.** (Takahashi's minimization theorem Takahaski [35]) Let $(X, d)$ be a complete metric space and $f : X \to [0, +\infty]$ be a proper l.s.c. function. Suppose that for each $w \in X$ with $\inf_{x \in X} f(x) < f(w)$, there exists $v \in X$ such that $v \neq w$ and $f(v) + d(w, v) \leq f(w)$. Then there exists $u \in X$ such that $f(u) = \inf_{x \in X} f(x)$.

Many generalizations of the above theorems have been investigated by a number of authors (see [2,3,22,23,26,32–37] and references therein). Moreover, Lee et al. [26], Zhu et al. [37], Jung et al. [23, 22] and Chang et al. [9], extended the above theorems to fuzzy metric spaces. In this paper, we establish the above theorems in fuzzy metric spaces in a different way. First, we present the Caristi's fixed point theorem in such spaces and then, we introduce the Ekeland's variational principle and Takahashi's maximization theorem. Moreover, we prove equivalencies among these three theorems. Also, motivated by the result of Aubin [2], we prove another fixed point theorem in the class of set-valued mappings. Finally, we mention that these theorems can be expressed in non-Archimedean fuzzy metric space and some examples are given to support the results.

## 2. PRELIMINARIES

Probabilistic metric spaces were introduced by Schweizer and Sklar [30] and Menger [27] who generalized the theory of metric spaces. These spaces have been widely studied, for instance [13,14,16–21,24,25,27,30,31]. George and Veeramani [13] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [25] as an extension of the concept of Menger space [27] to the fuzzy context with a view to obtain a Hausdorff topology on fuzzy metric spaces. In the following, we recall some well-known definitions, results and examples in the theory of fuzzy metric spaces which are needed. For more details we refer to [13–16,18–21].

Throughout the paper, we denote $\mathbb{N}$ the set of all natural numbers, $\mathbb{R}$ the set of real numbers and $\mathbb{R}^*$ the set of extended real numbers.

**Definition 2.1.** (see Schweizer and Sklar [30]) A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous triangular norm (t-norm) if $*$ satisfies the following conditions:

(a) $*$ be commutative and associative;

(b) $*$ be continuous;

(c) $a * 1 = a$ for all $a \in [0, 1]$;

(d) $a * b \leq c * d$ where $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0, 1]$.

The following t-norms will be used in this paper.

**Example 2.2.** (see Klement et al. [24]) For every $x, y \in [0,1]$.

(i) The minimum t-norm is defined by $x * y = \wedge\{x, y\}$.

(ii) The product t-norm is defined by $x * y = x \cdot y$.

(iii) The Lukasiewicz t-norm is defined by $\mathfrak{L}(x, y) = \vee\{x + y - 1, 0\}$.

(iv) The family $(*_\lambda^{SW})_{\lambda \in (-1,+\infty)}$ of Sugeno–Weber t-norm is given by

$$*_\lambda^{SW}(x, y) = \vee\left\{\frac{x + y - 1 + \lambda xy}{1 + \lambda}, 0\right\}.$$

Regarding the pointwise ordering, the above t-norms satisfy $\wedge \geq \cdot \geq *_{\lambda_2}^{SW} \geq *_{\lambda_1}^{SW}$, for all $\lambda_2 \geq \lambda_1 > -1$.

The t-norm $*$ is called Archimedean if for each $a, b \in (0,1)$, there is $n \in \mathbb{N}$ such that

$$a^n = \overbrace{a * a * \cdots * a}^{n \text{ times}} < b.$$

This condition can be simplified to $\forall a, b \in (0,1], a * b \geq a \Rightarrow b = 1$. All the above t-norms except the minimum t-norm are Archimedean.

**Definition 2.3.** (George and Veeramani [13]) A fuzzy metric space (GV space, for short) is an ordered triple $(X, M, *)$ such that $X$ is a nonempty set, $*$ is a continuous t-norm and $M$ is a fuzzy set of $X \times X \times (0, +\infty)$ satisfy the followings

(GV1) $M(x, y, t) > 0$,

(GV2) $M(x, y, t) = 1$ if and only if $x = y$,

(GV3) $M(x, y, t) = M(y, x, t)$,

(GV4) $M(x, y, t + s) \geq M(x, z, t) * M(y, z, s)$,

(GV5) $M(x, y, \cdot) : (0, +\infty) \to [0, 1]$ is continuous,

where $x, y, z \in X, s, t > 0$.

If (GV4) is replaced by

(NA) $M(x, y, t) \geqslant M(x, z, t) * M(y, z, t)$ for all $x, y, z \in X, t > 0$,

then the triple $(X, M, *)$ is called a non-Archimedean fuzzy metric space (NA space, for short).

Recall that if $(X, M, *)$ is a GV space and $\diamond$ is a continuous t-norm such that $a * b \geq a \diamond b$, for each $a, b \in [0, 1]$, then $(X, M, \diamond)$ is a GV space but the converse, in general, is not true.

**Definition 2.4.** (see Gregori and Romaguera [20]) The GV space $(X, M, *)$ is said to be stationary (SGV space, for short) if $M$ is independent on $t$, i.e., if for each $x, y \in X$, $M_{x,y}(t) = M(x, y, t)$ is a constant function. In this case we say that $M$ is a fuzzy metric on $X$ and write $M(x, y)$ instead of $M(x, y, t)$.

**Lemma 2.5.** (see Grabiec [15]) Suppose that $(X, M, *)$ is a GV space. Then $M(x, y, .)$ is non-decreasing, for all $x, y \in X$.

George and Veeramani [13] proved that every fuzzy metric $M$ on $X$ generates a Hausdorff first countable topology $\tau_M$ which the family of the form $\{B_M(x, \epsilon, t) : x \in X, 0 < \epsilon < 1, t > 0\}$, where $B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ for all $x \in X, r \in (0, 1), t > 0$ are a base for the mentioned topology.

**Theorem 2.6.** (see George and Veeramani [13]) Suppose that $(X, M, *)$ is a GV space. A sequence $\{x_n\}$ in $X$ converges to $x$ if and only if $\lim_n M(x_n, x, t) = 1$, for all $t > 0$.

**Definition 2.7.** (see George and Veeramani [13])

(a) A sequence $\{x_n\}$ in a GV space $(X, M, *)$ is a Cauchy sequence if for each $\epsilon \in (0, 1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$.

(b) We say that a GV space $(X, M, *)$ is complete if any Cauchy sequence in $X$, is convergent.

**Theorem 2.8.** (see Rodríguez-López and Romaguera [29]) Let $(X, M, *)$ be a GV space. The function $M$ on $X \times X \times (0, +\infty)$ is continuous.

**Example 2.9.** The following are some examples of GV spaces:

(1) Let $f : X \to (0, +\infty)$ be a one-to-one function and define the fuzzy set $M$ on $X \times X$ by

$$M(x, y) = \frac{\wedge\{f(x), f(y)\}}{\vee\{f(x), f(y)\}}. \tag{2.1}$$

Then, $(X, M, \cdot)$ is a SGV space ([18, Example 3]).

(2) Let $k \in (0, 1)$. Define the fuzzy set $M$ on $X \times X$ by $M(x, y, t) = 1$ if $x = y$ and $M(x, y, t) = k$ if $x \neq y$, then $(X, M, \wedge)$ is a SGV space and $M$ is called the discrete fuzzy metric (see [18, Example 7]).

(3) Let $F : X \times X \to (0, 1/2)$ be a symmetric function (i. e. $F(x, y) = F(y, x)$ for all $x, y \in X$). Define the fuzzy set $M$ on $X \times X$ by $M(x, y) = 1$ if $x = y$ and $M(x, y) = F(x, y)$ if $x \neq y$ , then $(X, M, \mathfrak{L})$ is a SGV space (see [18, Example 8]). In this direction, the function $F$ can be extended as follows:
Suppose $F : X \times X \to (0, 1)$ is a symmetric function such that

$$F(x, y) + 1 \geq F(x, z) + F(y, z) \tag{2.2}$$

for all $x, y, z \in X$. Then $(X, M, \mathfrak{L})$ is a SGV space (see also Lemma 3.2 below). If $f : X \to (0, 1/2)$ is a function then $F(x, y) = f(x)+f(y)$ satisfies in (2.2). Moreover, one can consider $(0, 1/2]$ as the range of $f$ if $f$ attained to $1/2$ for at most one point of $X$.

## 3. MAIN RESULTS

In this section, we first prove an extension of Caristi's fixed point theorem in the complete GV spaces for a class of continuous and Archimedean t-norm. Then, we establish the Ekeland's variational principle and Takahashi's maximization theorem in such spaces. Also, The equivalence of them is proved. Finally, we mention that these theorems can be expressed for non-Archimedean fuzzy metric spaces.

**Theorem 3.1.** (Caristi's fixed point theorem) Let $(X, M, *)$ be a complete GV space and let $T$ be a mapping from $X$ into itself. Also, suppose that $*$ is a continuous and Archimedean t-norm and that $\phi$ is a non-trivial (i. e. $x \in X$ such that $\phi(x) \neq 0$) upper semi continuous (briefly u.s.c.) fuzzy set of $X$. Assume that

$$\phi(Tx) * M(x, Tx, t) \geq \phi(x), \tag{3.1}$$

for all $x \in X$ and $t > 0$. Then $T$ has a fixed point in $X$.

Proof. We divide our proof into three steps. First, for any $x \in X$ such that $\phi(x) \neq 0$ denote

$$P(x) = \{y \in X : \forall t > 0, \phi(y) * M(x, y, t) \geq \phi(x)\}$$

and

$$\alpha(x) = \sup\{\phi(y) : y \in P(x)\}.$$

Then we have, $x \in P(x)$ and $1 \geq \alpha(x) \geq \phi(x)$. Now let $x_1 = x$ and choose $x_{n+1} \in P(x_n)$ such that

$$\phi(x_{n+1}) \geq \alpha(x_n) - \frac{1}{n}. \tag{3.2}$$

Therefore, by the definition of $P$ and inequality (3.2) we have

$$\phi(x_{n+1}) * M(x_n, x_{n+1}, t) \geq \phi(x_n) \qquad \text{for all} \ \ n \in \mathbb{N}, t > 0. \tag{3.3}$$

So $\{\phi(x_n)\}$ is an increasing sequence and in a consequence it converges.
On the other hand

$$\alpha(x_n) \geq \phi(x_{n+1}) \geq \alpha(x_n) - \frac{1}{n} \qquad \text{for all} \ \ n \in \mathbb{N}.$$

So $\lim_{n\to+\infty}\phi(x_n) = \lim_{n\to+\infty}\alpha(x_n)$ exists. Setting

$$k = \lim_{n\to+\infty} \phi(x_n) = \lim_{n\to+\infty} \alpha(x_n). \tag{3.4}$$

(1) *For any $n \in \mathbb{N}$, by induction on $m$ we prove that the following inequality holds:*

$$\phi(x_m) * M(x_n, x_m, t) \geq \phi(x_n) \qquad \text{for all} \ \ t > 0, m > n. \tag{3.5}$$

By (3.3), we conclude that (3.5) holds for $m = n+1$. Suppose (3.5) holds for $m > n$, we must show it also true for $m + 1$. For all $t > 0$ note that

$$\begin{aligned} \phi(x_{m+1}) * M(x_n, x_{m+1}, t) &\geq \phi(x_{m+1}) * M(x_m, x_{m+1}, t/2) * M(x_n, x_m, t/2) \\ &\geq \phi(x_m) * M(x_n, x_m, t/2) \\ &\geq \phi(x_n). \end{aligned}$$

Therefore, (3.5) is true for $m + 1$.

(2) $\{x_n\}$ *is a Cauchy sequence.*
Suppose that $\{x_n\}$ is not a Cauchy sequence, so there exist $0 < \epsilon < 1$ and $t > 0$ such that for each $n \in \mathbb{N}$ there exists $m \geq n$ such that $M(x_n, x_m, t) \leq (1-\epsilon)$. By (3.4) for each $0 < \epsilon' < 1$ there is $N \in \mathbb{N}$ such that $k \geq \phi(x_n) \geq k(1-\epsilon')$, for all $n \geq N$. From (3.5) one can conclude that $k * (1-\epsilon) \geq k(1-\epsilon')$, for all $0 < \epsilon' < 1$. But this is contradicted by Archimedean condition, so $\{x_n\}$ is a Cauchy sequence.

(3) $\{x_n\}$ *is convergent to a fixed point of* $T$.
Since $X$ is complete GV space, $\{x_n\}$ converges to $u \in X$. One can show that $k = \phi(u) = \phi(Tu)$. Since $\phi$ is u.s.c. (3.4) shows that $k = \limsup_{n\to+\infty} \phi(x_n) \leq \phi(u)$. On the other hand, by taking limit from both sides of (3.5) we have

$$\phi(x_n) \leq \limsup_{m\to+\infty}(\phi(x_m) * M(x_n, x_m, t)) \leq \phi(u) * M(x_n, u, t) \qquad \text{for all} \;\; t > 0,$$

thus,

$$u \in P(x_n) \qquad \text{for all} \;\; n \in \mathbb{N}. \tag{3.6}$$

Therefore, $\phi(u) \leq \alpha(x_n)$, for all $n \in \mathbb{N}$. So by (3.4), we obtain $\phi(u) \leq k$ and so $\phi(u) = k$. Since $u \in P(x_n)$, for all $n \in \mathbb{N}$ and (3.1) holds, $Tu \in P(u)$. Note that

$$\begin{aligned} \phi(Tu) * M(x_n, Tu, t) &\geq \phi(Tu) * M(x_n, u, t/2) * M(u, Tu, t/2) \\ &\geq \phi(u) * M(x_n, u, t/2) \\ &\geq \phi(x_n). \end{aligned}$$

for all $t > 0$, hence $Tu \in P(x_n)$ for all $n \in \mathbb{N}$. This implies that

$$\phi(Tu) \leq \alpha(x_n) \qquad \text{for all} \;\; n \in \mathbb{N}.$$

Hence (3.4) shows that $\phi(Tu) \leq k$. Since (3.1) holds and $\phi(u) = k$, we can obtain

$$\phi(u) = k \geq \phi(Tu) \geq \phi(u).$$

Thus, $k = \phi(Tu) = \phi(u)$. Also (3.1) shows that $k * M(u, Tu, t) \geq k$, for all $t > 0$. It means that $M(u, Tu, t) = 1$, for all $t > 0$ and so $Tu = u$.

□

**Lemma 3.2.** Consider the GV space $(X, M, (*_\lambda^{SW})_{\lambda\in(-1,+\infty)})$. The triangular inequality (GV4) is equivalent to the following inequality

$$(1+\lambda)M(x, y, t+s) + 1 \geq M(x, z, s) + M(y, z, t) + \lambda M(x, z, s)M(y, z, t),$$

for all $x, y, z \in X$ and $s, t > 0$.

Proof. By the definition of Sugeno–Weber t-norms we can obtain the desired result easily. □

The Archimedean assumption on the t-norm in the preceding theorem is not needed in the two following corollaries.

**Corollary 3.3.** Let $(X, M, *)$ be a complete GV space and let $T$ be a mapping from $X$ into itself. Also assume that $*$ is a continuous t-norm such that $a * b \geq \mathfrak{L}(a, b)$ for all $a, b \in [0, 1]$ and $\phi$ be an u.s.c. upper bounded function from $X$ into $\mathbb{R}^*$. Additively, suppose that $\phi(x) \not\equiv -\infty$ and

$$M(x, Tx, t) \geq 1 + \phi(x) - \phi(Tx), \tag{3.7}$$

holds for all $x \in X$ and $t > 0$. Then $T$ has a fixed point in $X$.

P r o o f. Consider the GV space $X$ endowed with t-norm $\mathfrak{L}$. Define $M'(x, y, t) = e^{M(x,y,t)-1}$ on $X \times X \times (0, +\infty)$. We claim that $(X, M', \cdot)$ is a complete GV space. It is sufficient to show that the triangular inequality holds. Since $(X, M, \mathfrak{L})$ be a GV space, Lemma 3.2 shows that (for $\lambda = 0$)

$$M'(x, z, s)M'(y, z, t) = e^{M(x,z,s)+M(y,z,t)-2} \leq e^{M(x,y,t+s)-1} = M'(x, y, t + s),$$

for all $x, y, z \in X$ and $s, t > 0$. It is easy to see that if $\{x_n\}$ is a Cauchy sequence in $(X, M, \mathfrak{L})$ then it is a Cauchy in $(X, M', \cdot)$, so $(X, M', \cdot)$ is a complete GV space. Setting $\phi'(x) = e^{\phi(x)}$, for all $x \in X$. One can conclude that $\phi'$ is a nontrivial u.s.c. upper bounded function from $X$ into $[0, +\infty)$. Note that (3.7) implies $\phi'(Tx)M'(x, Tx, t) \geq \phi'(x)$, for all $x \in X$. Without loss of generality we can assume that $K \leq 1$. Since otherwise we can consider $\phi'/K$, where $K = \sup\{\phi'(x) : x \in X\}$. Therefore Theorem 3.1 concludes that $T$ has a fixed point in $X$. □

**Corollary 3.4.** Let $(X, M, *)$ be a complete GV space and let $T$ be a mapping from $X$ into itself. Also assume that $*$ is a continuous t-norm such that $a * b \geq *^{SW}_\lambda(a, b)$, for all $a, b \in [0, 1]$ and for some $\lambda > -1$. Let $\phi$ be a nontrivial u.s.c. upper bounded function from $X$ into $[0, +\infty)$. Moreover suppose that

$$\phi(Tx)M(x, Tx, t) \geq \phi(x), \tag{3.8}$$

for all $x \in X$ and $t > 0$. Then $T$ has a fixed point in $X$.

P r o o f. Suppose that $\lambda > -1$ and consider $X$ endowed with $*^{SW}_\lambda$. Setting $a = \lambda + 1$, $M'(x, y, t) = \log_{a+1}(aM(x, y, t) + 1)$ and let $\phi'(x) = \log_{a+1}\phi(x)$ (define $\log_{a+1} 0 = -\infty$), for all $x, y \in X$ and $t > 0$. One can easily to verify that $\phi'$ is a nontrivial u.s.c. and upper bounded function from $X$ into $\mathbb{R}^*$ and $\phi(x) \not\equiv -\infty$. We claim that $(X, M', \mathfrak{L})$ is a complete GV space. To show that $(X, M', \mathfrak{L})$ is a GV space we only prove the triangle inequality. By Lemma 3.2 we have

$$\begin{aligned} M'(x, z, s) + M'(y, z, t) &= \log_{a+1}(aM(x, z, s) + 1) + \log_{a+1}(aM(y, z, t) + 1) \\ &= \log_{a+1}(a(aM(x, z, s)M(y, z, t) \\ &\qquad\qquad + M(x, z, s) + M(y, z, t)) + 1) \\ &\leq \log_{a+1}(a((a + 1)M(x, y, s + t) + 1) + 1) \\ &= \log_{a+1}(aM(x, y, s + t) + 1) + 1 = M'(x, y, s + t) + 1. \end{aligned}$$

for all $x, y, z \in X$ and $s, t > 0$. So Lemma 3.2 shows that $M'$ satisfies in (GV4). It is easy to see that if $\{x_n\}$ is a Cauchy sequence in $(X, M, *^{SW}_\lambda)$ then it is a Cauchy sequence in $(X, M', \mathfrak{L})$. So $(X, M', \mathfrak{L})$ is a complete GV space. Moreover, (3.8) shows that

$$\begin{aligned} M'(x,Tx,t) &= \log_{a+1}(aM(x,Tx,t)+1) \\ &\geq \log_{a+1}((a+1)M(x,Tx,t)) \\ &\geq 1+\log_{a+1}\varphi(x)-\log_{a+1}\varphi(Tx) \\ &= 1+\varphi'(x)-\varphi'(Tx). \end{aligned}$$

Finally, Corollary 3.3 concludes that $T$ has a fixed point in $X$. □

The following metric fixed point theorem is concluded by Corollary 3.4.

**Corollary 3.5.** Let $(X, d)$ be a complete metric space such that $d < 1$ and let $T$ be a mapping from $X$ into itself. Let $\phi$ be a u.s.c. upper bounded function from $X$ into $(0, +\infty)$. Suppose that

$$d(x,Tx) \leq 1-\frac{\phi(x)}{\phi(Tx)}, \tag{3.9}$$

for all $x \in X$. Then $T$ has a fixed point in $X$.

Proof. Set $M(x, y) = 1 - d(x, y)$, for all $x, y \in X$. Then $(X, M, \mathfrak{L})$ is a complete SGV space (see [17]), and the inequality (3.9) is equivalent to inequality (3.8), so $T$ has a fixed point by Corollary 3.4. □

**Remark 3.6.** Note that we can prove Corollary 3.4 like the same argument used in the proof of Theorem 3.1, when $* \geq \cdot$.

To show that $\{x_n\}$ is a Cauchy sequence we can conclude that $\phi(x_{n+1})M(x_n, x_{n+1}, t) \geq \phi(x_n)$, for all $t > 0, n \in \mathbb{N}$, instead of (3.3). For all $m > n + 1$ we have

$$\begin{aligned} M(x_n,x_m,t) &\geq M(x_n,x_{n+1},t/2)*M(x_{n+1},x_m,t/2) \\ &\geq \cdots \\ &\geq M(x_n,x_{n+1},t/2)*\cdots*M(x_{m-1},x_m,t/2^{m-n-1}) \\ &\geq \frac{\phi(x_n)}{\phi(x_{n+1})}*\cdots*\frac{\phi(x_{m-1})}{\phi(x_m)} \\ &\geq \frac{\phi(x_n)}{\phi(x_m)} \\ &\geq \frac{\phi(x_n)}{k}, \end{aligned}$$

for all $t > 0$. It means that $\{x_n\}$ is Cauchy sequence. The rest of the proof is very similar to the proof of Theorem 3.1.

**Example 3.7.** Let $X = [1/2, 1]$ and $(X, M, \cdot)$ be the same space in Example 2.9 − (1), when $f : X \to (0, +\infty)$ is given as the identity function. $(X, M, \cdot)$ is complete (see [16, 28]). Suppose $T : X \to X$ defined by $T(x) = (x^2 + 1)/2$, for all $x \in X$. Consider the function $\phi : X \to [0, 1]$ defined by $\phi(x) = x$, for all $x \in X$. Then we have

$$\phi(Tx)M(x,Tx) = ((x^2+1)/2)\frac{\wedge\{x,(x^2+1)/2\}}{\vee\{x,(x^2+1)/2\}} = x = \phi(x),$$

for all $x \in X$. All conditions of Theorem 3.1 (or Corollary 3.4) hold, thus $T$ has a fixed point in $X$. Also it is easy to check that condition (3.9) when $d$ is the usual Euclidean metric on $X$ and $\phi : X \to (0, \infty)$ is defined by $\phi(x) = 1/x$. So the same result is obtained by Corollary 3.5.

The following shows that Theorem 3.1 is not true when $*$ is the minimum t-norm.

**Example 3.8.** Let $X = \mathbb{R}$ and let $(X, M, \wedge)$ be the same space in Example 2.9−(2). Since the induced topology by $M$ is discrete, $X$ is complete. Take the u.s.c. function $\phi : X \to [0, 1]$ in Theorem 3.1 defined by $\phi(x) = k$, for all $x \in X$ and let $T : X \to X$ defined by $T(x) = x + 1$, for all $x \in X$. Then, all of the conditions in Theorem 3.1 except the t-norm hold and $T$ has no fixed point in $X$.

**Theorem 3.9.** (Ekeland's variational principle) Let $(X, M, *)$ be a complete GV space endowed with a continuous and Archimedean t-norm $*$ and let $\phi$ be a nontrivial u.s.c. fuzzy set of $X$. Consider $v \in X$ such that $\phi(v) \neq 0$. Then there exists $u \in X$ such that for every $t > 0$,

(i) $\phi(u) * M(v, u, t) \geq \phi(v)$,

(ii) $\phi(x) * M(u, x, t) < \phi(u) \qquad$ for all $x \in X, x \neq u$.

Proof. Define the sequence $\{x_n\}$ starting on $x_1 = v$. We shall show that the point $u$ found in the Proof of Theorem 3.1 is the requested point. By (3.6), we have $u \in p(x_1)$ so condition (i) holds. Finally, we need to show that $x \notin P(u)$, for all $x \neq u$. Suppose that there exists $x \neq u$ such that $x \in P(u)$. Since $\phi(x) * M(u, x, t) \geq \phi(u)$, for all $t > 0$, we have $\phi(x) > \phi(u) = k$. For all $t > 0$, we have

$$\phi(x) * M(x_n, x, t) \geq \phi(x) * M(x_n, u, t/2) * M(u, x, t/2) \geq \phi(u) * M(x_n, u, t/2) \geq \phi(x_n),$$

and this concludes that $x \in P(x_n)$ and so we have

$$\alpha(x_n) \geq \phi(x) \qquad \text{for all } n \in \mathbb{N}. \tag{3.10}$$

Taking limit from both sides of (3.10) we obtain $\phi(x) \leq k$ and this is a contradiction with $\phi(x) > k = \phi(u)$. So condition (ii) holds too. □

To introduce the Ekeland's $\epsilon$-variational principle's theorem in GV spaces, we need the following preliminaries:

**Proposition 3.10.** Let $\epsilon > 0$ and $(X, M, *)$ is a GV space such that, for all $a, b \in [0, 1]$ the t-norm $*$ satisfies the following inequality,

$$(a * b)^\epsilon \geq a^\epsilon * b^\epsilon. \tag{3.11}$$

Then $(X, M^\epsilon, *)$ is a GV space where the fuzzy set $M^\epsilon$ has been defined by $M^\epsilon(x, y, t) = M(x, y, t)^\epsilon$, for all $x, y \in X$ and $t > 0$.

**Remark 3.11.** Note that the t-norms $\wedge$ and $\cdot$ are satisfy $(a * b)^\epsilon = a^\epsilon * b^\epsilon$, for every $\epsilon > 0$ and $a, b \in [0, 1]$. For Lukasiewicz t-norm we have two cases:

(1) Given $0 < \epsilon \le 1$ and $a, b \in [0,1]$. If $a + b - 1 \le 0$ then obviously we have $(a * b)^\epsilon \le a^\epsilon * b^\epsilon$ and if $a+b-1 \ge 0$ then by Minkowski inequality we have $a^\epsilon + b^\epsilon - 1 \ge (a+b-1)^\epsilon$, so $(a * b)^\epsilon = (a + b - 1)^\epsilon \le a^\epsilon + b^\epsilon - 1 = a^\epsilon * b^\epsilon$, thus we have $(a * b)^\epsilon \le a^\epsilon * b^\epsilon$, for all $a, b \in [0, 1]$.

(2) Given $1 \le \epsilon$ and $a, b \in [0,1]$. If $a^\epsilon + b^\epsilon - 1 \le 0$ then obviously we have $a^\epsilon * b^\epsilon \ge (a*b)^\epsilon$ and if $a^\epsilon + b^\epsilon - 1 \ge 0$ then by Minkowski inequality we have $a^\epsilon + b^\epsilon - 1 \le (a+b-1)^\epsilon$, so $a^\epsilon * b^\epsilon = a^\epsilon + b^\epsilon - 1 \le (a + b - 1)^\epsilon = (a * b)^\epsilon$, thus we have $(a * b)^\epsilon \ge a^\epsilon * b^\epsilon$ for all $a, b \in [0, 1]$.

Also note that the equality does not hold when $\epsilon \ne 1$, in the both cases.

**Theorem 3.12.** (Ekeland's $\epsilon$-variational principle) Let $(X, M, *)$ be a complete GV space endowed with a continuous and Archimedean t-norm $*$ and suppose that condition (3.11) holds for some $\epsilon > 0$ and that $\phi$ is a nontrivial u.s.c. fuzzy set of $X$. Consider $v \in X$ such that $\phi(v) \ne 0$.Then there exists $u \in X$ such that for every $t > 0$,

(i) $\phi(u) * M(v, u, t)^\epsilon \ge \phi(v)$,

(ii) $\phi(x) * M(u, x, t)^\epsilon < \phi(u) \qquad$ for all $x \in X, x \ne u$.

P r o o f. Since by Proposition 3.10, $(X, M^\epsilon, *)$ is a GV space. Therefore, the desired result is obtained by Ekeland's variational principle in a GV space (Theorem 3.9). $\square$

**Theorem 3.13.** (Takahashi's maximization theorem) Let $(X, M, *)$ be a complete GV space endowed with a continuous and Archimedean t-norm $*$ and let $\phi$ be a nontrivial u.s.c. fuzzy set of $X$. Suppose that for each $w \in X$ with $\sup_{x\in X} \phi(x) > \phi(w)$, there exists $v \in X$ such that $v \ne w$ and $\phi(v) * M(w, v, t) \ge \phi(w)$, for all $t > 0$. Then there exists $u \in X$ such that $\phi(u) = \sup_{x\in X} \phi(x)$.

P r o o f. Consider the element $u \in X$ obtained in Theorem 3.9. If $\phi(u) < \sup_{x\in X} \phi(x)$ then there exists $v \in X$ such that $v \ne u$ and $\phi(v) * M(u, v, t) \ge \phi(u)$, for all $t > 0$, which is impossible by the conditions of Theorem 3.9. $\square$

**Remark 3.14.** Note that, Theorems 1.1, 1.2 and 1.3 are concluded by Theorems 3.12, 3.1 and 3.13, respectively.

**Theorem 3.15.** (Equivalencies) Theorems 3.1, 3.9 and 3.13 are equivalent.

P r o o f. Comparing the proofs of Theorem 3.1 and Theorem 3.9 show that, we only need to prove that Theorem 3.13 is deduced from Theorem 3.1. Consider the element $u \in X$ obtained in Theorem 3.13. Then we have $\phi(Tu) \le \phi(u)$. By (3.1) we have that $\phi(Tu) = \phi(u)$ and (3.1) shows that $M(u, Tu, t) = 1$, for all $t > 0$. It means that $u = T(u)$. $\square$

**Theorem 3.16.** Let $(X, M, *)$ be a complete GV space endowed with a continuous and Archimedean t-norm $*$, let $T$ be a set-valued mapping from $X$ into $2^X$. Suppose that $\phi$ is a nontrivial u.s.c. fuzzy set of $X$ and let

$$\forall x \in X, \exists y \in T(x) \qquad \text{such that} \qquad \phi(y) * M(x, y, t) \geq \phi(x), \tag{3.12}$$

for all $t > 0$. Then there exists some $u \in X$ such that $T(u) = \{u\}$.

Proof. By Theorem 3.9 there exists $u \in X$ such that $\phi(x) * M(u, x, t) < \phi(u)$, for all $x \in X, x \neq u, t > 0$. Also, there exists a $y \in T(u)$ such that $\phi(y) * M(u, y, t) \geq \phi(u)$, for all $t > 0$. So we have $y = u$. □

Now we give another fixed point theorem in a GV space without the upper semi-continuity of $\phi$.

**Theorem 3.17.** Let $(X, M, *)$ be a complete GV space endowed with continuous and Archimedean t-norm $*$, $T$ be a set-valued mapping from $X$ into $2^X$ such that graph$T :=$ $\{(x, y) \in X \times X : y \in T(x)\}$ is closed. Assume that $\phi$ is a nontrivial u.s.c. fuzzy set of $X$ and (3.12) holds. Then there exist $u \in X$ such that $u \in T(u)$.

Proof. Let $x_1 \in X$ such that $\phi(x_1) \neq 0$. By (3.12) choose $x_{n+1} \in G(x_n)$ such that $\phi(x_{n+1}) * M(x_n, x_{n+1}, t) \geq \phi(x_n)$, for all $t > 0$. This implies that $\{\phi(x_n)\}$ is an increasing sequence. Thus, $\lim_{n\to+\infty}\phi(x_n)$ exists. Setting $k = \lim_{n\to+\infty}\phi(x_n)$. By the same argument in Theorem 3.1, we can conclude that $\{x_n\}$ is a Cauchy sequence. Therefore, there exists $u \in X$ such that $x_n \to u$. Since $(x_n, x_{n+1}) \in$ graph$T$, for all $n \in \mathbb{N}$ and graph$T$ is closed, we have $(u, u) \in$ graph$T$. It means that, $u \in T(u)$. □

Next we are ready to prove Caristi's fixed point theorem in a non-Archimedean fuzzy metric spaces.

**Theorem 3.18.** Let $(X, M, *)$ be a complete NA space endowed with continuous and Archimedean t-norm $*$. Let $T$ be a mapping from $X$ into itself and $\phi$ be a fuzzy set of $X \times (0, +\infty)$. Suppose that $t_0 > 0$ and $\phi(\cdot, t_0)$ is a nontrivial and u.s.c. upper bounded function on $X$. Assume that

$$\phi(Tx, t) * M(x, Tx, t) \geq \phi(x, t) \qquad \text{for all } \ x \in X, t > 0. \tag{3.13}$$

Then $T$ has a fixed point in $X$.

Proof. Since $(X, M, *)$ is a NA space then $(X, M', *)$ is a SGV space with the fuzzy set defined by $M'(x, y) = M(x, y, t_0)$, for all $x, y \in X$. Define $\phi'$ on $X$ by $\phi'(x) = \phi(x, t_0)$, for all $x \in X$. Then $\phi'$ is a nontrivial and u.s.c. upper bounded fuzzy set of $X$. If we take $t = t_0$ then (3.13) concludes (3.1). So the desired result is obtained from Theorem 3.1. □

The following is similar to [1, Corollary 2.11] without the continuity of $T$.

**Corollary 3.19.** Let $(X, M, *)$ be a complete NA space, $* \geq \mathfrak{L}$, $T$ be a mapping from $X$ into itself and $\phi$ be a fuzzy set of $X \times (0, +\infty)$. Suppose that $t_0 > 0$ and that $\phi(\cdot, t_0)$ is an u.s.c. upper bounded function on $X$. Additively, suppose that $\phi(x, t_0) \not\equiv -\infty$ and

$$M(x, Tx, t) \geq 1 + \phi(x, t) - \phi(Tx, t) \qquad \text{for all} \qquad x \in X, t > 0.$$

Then $T$ has a fixed point in $X$.

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