Czechoslovak Mathematical Journal

Somayeh Bandari; Ali Soleyman Jahan
The cleanness of (symbolic) powers of Stanley-Reisner ideals

Czechoslovak Mathematical Journal, Vol. 67 (2017), No. 3, 767–778

Persistent URL: http://dml.cz/dmlcz/146858

THE CLEANNESS OF (SYMBOLIC) POWERS OF STANLEY-REISNER IDEALS

Somayeh Bandari, Buein Zahra, Tehran, Ali Soleyman Jahan, Sanandaj

Received April 8, 2016. First published August 8, 2017.

Abstract. Let Δ be a pure simplicial complex on the vertex set $[n]=\{1,\ldots,n\}$ and I_Δ its Stanley-Reisner ideal in the polynomial ring $S=K[x_1,\ldots,x_n]$. We show that Δ is a matroid (complete intersection) if and only if $S/I_\Delta^{(m)}$ (S/I_Δ^m) is clean for all $m\in\mathbb{N}$ and this is equivalent to saying that $S/I_\Delta^{(m)}$ (S/I_Δ^m, respectively) is Cohen-Macaulay for all $m\in\mathbb{N}$. By this result, we show that there exists a monomial ideal I with (pretty) cleanness property while S/I^m or $S/I^{(m)}$ is not (pretty) clean for all integer $m\geqslant 3$. If $\dim(\Delta)=1$, we also prove that $S/I_\Delta^{(2)}$ (S/I_Δ^2) is clean if and only if $S/I_\Delta^{(2)}$ (S/I_Δ^2, respectively) is Cohen-Macaulay.

Keywords: clean; Cohen-Macaulay simplicial complex; complete intersection; matroid; symbolic power

MSC 2010: 13F20, 05E40, 13F55

Introduction

Let Δ be a simplicial complex on the vertex set $[n]=\{1,\ldots,n\}$ and $S=k[x_1,\ldots,x_n]$ be the polynomial ring in n indeterminates over a field k. The Stanley-Reisner ideal of Δ, I_Δ, is defined by $I_\Delta := \Bigl(\prod_{i\in F} x_i\colon\ F\notin\Delta\Bigr)$.

There is a bijection between squarefree monomial ideals I and simplicial complexes. Cohen-Macaulayness (Buchsbaumness, cleanness, generalized Cohen-Macaulayness) of these ideals have been studied by several authors (see [4], [10], [8], [13], [15], [16], [18]). Minh and Trung in [13] and Varbaro in [17] independently proved that Δ is a matroid if and only if $S/I_\Delta^{(m)}$ is Cohen-Macaulay for all $m\in\mathbb{N}$, where $I_\Delta^{(m)}$

The first author was in part supported by a grant from Institute for Research in Fundamental Sciences (IPM) (No. 93130020).

DOI: 10.21136/CMJ.2017.0173-16

denotes the mth-symbolic power of I_Δ. Later on, Terai and Trung in [16] showed that Δ is a matroid if and only if $S/I_\Delta^{(m)}$ is Cohen-Macaulay for some integer $m \geqslant 3$. The similar characterizations of being Buchsbaum and generalized Cohen-Macaulay were also studied by them. Minh and Trung in [12] proved that for a simplicial complex Δ with $\dim(\Delta) = 1$, $I_\Delta^{(2)}$ is Cohen-Macaulay if and only if $\operatorname{diam}(\Delta) \leqslant 2$, where $\operatorname{diam}(\Delta)$ denotes the diameter of Δ. We pursue this line of research further.

This paper is organized as follows: in Section 1, we collect some preliminaries which will be needed later. In Section 2, we show that if Δ is a matroid, then $S/I_\Delta^{(m)}$ is clean for all $m \in \mathbb{N}$; see Theorem 2.1. Since I_Δ is unmixed, in particular, this shows that $S/I_\Delta^{(m)}$ is Cohen-Macaulay for all $m \in \mathbb{N}$. Therefore this result covers one direction of the result of Minh and Trung in [13] and Varbaro in [17]. Our proof is combinatorial and more elementary than that given in [13]. As our first corollary, by using [16], Theorem 3.6, we show that if Δ is pure and $I = I_\Delta \subset S$, then the following conditions are equivalent:

(a) Δ is a matroid.
(b) $S/I^{(m)}$ is clean for all integers $m > 0$.
(c) $S/I^{(m)}$ is clean for some integer $m \geqslant 3$.
(d) $S/I^{(m)}$ is Cohen-Macaulay for some integer $m \geqslant 3$.
(e) $S/I^{(m)}$ is Cohen-Macaulay for all integers $m > 0$.

Our second corollary asserts that a pure simplicial complex Δ is a complete intersection if and only if S/I_Δ^m is clean for all $m \in \mathbb{N}$ and if and only if S/I_Δ^m is clean for some integer $m \geqslant 3$.

Let $I \subset S$ be a monomial ideal such that S/I is (pretty) clean. It is natural to ask whether S/I^m or $S/I^{(m)}$ is again (pretty) clean for all $m \in \mathbb{N}$? Example 2.5 shows that the answer is negative in general.

In Section 3, we show that if $I \subset S$ is the Stanley-Reisner ideal of a pure simplicial complex Δ with $\dim \Delta = 1$, then for an integer $m > 1$, $S/I^{(m)}$ (S/I^m) is clean if and only if $S/I^{(m)}$ (S/I^m, respectively) is Cohen-Macaulay.

1. Preliminary

A simplicial complex Δ on the vertex set $[n] = \{1, \ldots, n\}$ is a collection of subsets of $[n]$ with the property that if $F \subset G$ and $G \in \Delta$, then $F \in \Delta$. An element of Δ is called a *face*, and the maximal faces of Δ, under inclusion, are called *facets*. We denote by $\mathcal{F}(\Delta)$ the set of facets of Δ. When $\mathcal{F}(\Delta) = \{F_1, \ldots, F_t\}$, we write $\Delta = \langle F_1, \ldots, F_t \rangle$. For each $F \in \Delta$, we set $\dim F := |F| - 1$, and

$$\dim \Delta := \max\{\dim F\colon\ F \in \mathcal{F}(\Delta)\},$$

which is called the dimension of Δ. A simplicial complex Δ is called *pure* if all facets of Δ have the same dimension. According to Björner and Wachs in [3], a simplicial complex Δ is said to be *(non-pure) shellable* if there exists an order $F_1, \ldots, F_t$ of the facets of Δ such that for each $2 \leqslant i \leqslant t$, $\langle F_1, \ldots, F_{i-1}\rangle \cap \langle F_i\rangle$ is a pure $(\dim F_i - 1)$-dimensional simplicial complex. Such an ordering of facets is called a *shelling*.

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring in n indeterminates over a field K. The Stanley-Reisner ideal of Δ is denoted by I_Δ and defined as $I_\Delta := \Bigl(\prod_{i\in F} x_i\colon\ F \notin \Delta\Bigr)$. The facet ideal of Δ is defined as $I(\Delta) := \Bigl(\prod_{i\in F} x_i\colon\ F \in \mathcal{F}(\Delta)\Bigr)$.

The *Alexander dual* of Δ is given by $\Delta^\vee := \{F^c\colon\ F \notin \Delta\}$. Let I be a squarefree monomial ideal in S. We denote by $I^\vee$ the squarefree monomial ideal which is minimally generated by all monomials $x_{i_1} \ldots x_{i_k}$, where $(x_{i_1}, \ldots, x_{i_k})$ is a minimal prime ideal of I. It is easy to see that for any simplicial complex Δ, one has $I_{\Delta^\vee} = (I_\Delta)^\vee$. The complement of a face F is $[n] \setminus F$ and it is denoted by F^c. Also, the complement of a simplicial complex $\Delta = \langle F_1, \ldots, F_r\rangle$ is $\Delta^c := \langle F_1^c, \ldots, F_r^c\rangle$. It is known that for a simplicial complex Δ one has $I_{\Delta^\vee} = I(\Delta^c)$.

Definition 1.1. A *matroid* Δ is a simplicial complex with the property that for all faces F and G in Δ with $|F| < |G|$, there exists $i \in G \setminus F$ such that $F \cup \{i\} \in \Delta$.

The above definition implies that each matroid is pure. As a consequence of [7], Theorem 12.2.4, a matroid can be characterized by the following exchange property: a pure simplicial complex Δ is a matroid if and only if for any two facets F and G of Δ with $F \neq G$, and for any $i \in F \setminus G$, there exists $j \in G \setminus F$ such that $(F \setminus \{i\}) \cup \{j\} \in \Delta$. A squarefree monomial ideal I in S is called *matroidal* if $I = I(\Delta)$, where Δ is a matroid. On the other hand, by [14], Theorem 2.1.1, Δ is a matroid if and only if Δ^c is a matroid. Altogether, as $I(\Delta^c) = I_{\Delta^\vee}$, we have that Δ is a matroid if and only if $I_{\Delta^\vee}$ is matroidal.

A simplicial complex Δ is called a *complete intersection* if I_Δ is a complete intersection monomial ideal. It is well known that each complete intersection simplicial complex is a matroid.

If $F \subseteq [n]$, then we put $P_F := (x_i\colon\ i \in F)$. We have $I_\Delta = \bigcap_{F\in\mathcal{F}(\Delta^c)} P_F$, hence for each $m \in \mathbb{N}$, the mth-symbolic power of I_Δ is the ideal

$$I_\Delta^{(m)} = \bigcap_{F\in\mathcal{F}(\Delta^c)} P_F^m.$$

An ideal $I \subset S$ is called *normally torsionfree* if $\operatorname{Ass}(S/I^m) \subseteq \operatorname{Ass}(S/I)$ for all $m \in \mathbb{N}$. If I is a squarefree monomial ideal, then I is normally torsionfree if and only if $I^{(m)} = I^m$ for all m; see [7], Theorem 1.4.6.

Let $I \subset S$ be a monomial ideal. A chain of monomial ideals

$$\mathcal{F}\colon\ I = I_0 \subset I_1 \subset \ldots \subset I_r = S$$

is called a *prime filtration* of S/I if for each $i = 1, \ldots, r$ there exists a monomial prime ideal $\mathfrak{p}_i$ of S such that $I_i/I_{i-1} \cong S/\mathfrak{p}_i$. The set of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ which define the cyclic quotients of $\mathcal{F}$ will be denoted by $\operatorname{Supp}\mathcal{F}$. It is known (and easy to see) that

$$\operatorname{Ass} S/I \subseteq \operatorname{Supp}\mathcal{F} \subseteq \operatorname{Supp} S/I.$$

Let $\operatorname{Min} I$ denote the set of minimal prime ideals of $\operatorname{Supp} S/I$. Dress in [5] called a prime filtration $\mathcal{F}$ of S/I *clean* if $\operatorname{Supp}\mathcal{F} = \operatorname{Min} I$ and in [5], Theorem on page 53, proved that a simplicial complex Δ is (non-pure) shellable if and only if $K[\Delta]$ is a clean ring. Pretty clean filtrations were defined as a generalization of clean filtrations by Herzog and Popescu in [8]. A prime filtration $\mathcal{F}$ is called *pretty clean* if for all $i < j$ for which $\mathfrak{p}_i \subseteq \mathfrak{p}_j$, it follows that $\mathfrak{p}_i = \mathfrak{p}_j$. If $\mathcal{F}$ is a pretty clean filtration of S/I, then $\operatorname{Supp}\mathcal{F} = \operatorname{Ass} S/I$; see [8], Corollary 3.4. S/I is called *clean* (*pretty clean*) if it admits a clean (pretty clean) filtration. Obviously, cleanness implies pretty cleanness.

Let $I \subset S$ be a monomial ideal. Then S/I is *sequentially Cohen-Macaulay* if there exist a chain of monomial ideals

$$I = I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_r = S$$

such that each quotient I_i/I_{i-1} is Cohen-Macaulay and

$$\dim(I_1/I_0) < \dim(I_2/I_1) < \ldots < \dim(I_r/I_{r-1}).$$

Clearly, if S/I is Cohen-Macaulay, then it is sequentially Cohen-Macaulay. Also, if S/I is pretty clean, then by [8] it is sequentially Cohen-Macaulay.

The monomial ideal I has *linear quotients* if one can order the set of minimal monomial generators of I, $\mathrm{G}(I) = \{u_1, \ldots, u_m\}$, so that the colon ideal $(u_1, \ldots, u_{i-1}) : u_i$ is generated by a subset of the variables for all $i = 2, \ldots, m$. This means for each $j < i$ there exists a $k < i$ such that $u_k : u_i = x_t$ and $x_t \mid u_j : u_i$, where $t \in [n]$ and $u_k : u_i = u_k/\gcd(u_k, u_i)$. In the case I is squarefree, it is enough to show that for each $j < i$ there exists a $k < i$ such that $u_k : u_i = x_t$ and $x_t \mid u_j$ for some $t \in [n]$.

Let $u = \prod_{i=1}^{n} x_i^{a_i}$ be a monomial in S. Then

$$u^p := \prod_{i=1}^{n}\prod_{j=1}^{a_i} x_{i,j} \in K[x_{1,1}, \ldots, x_{1,a_1}, \ldots, x_{n,1}, \ldots, x_{n,a_n}]$$

is called the *polarization* of u. Let I be a monomial ideal of S with the unique set of minimal monomial generators $\mathrm{G}(I) = \{u_1, \ldots, u_m\}$. Then the ideal $I^p := (u_1^p, \ldots, u_m^p)$ of

$$T := K[x_{i,j} \colon\ i = 1, \ldots, n,\ j = 1, \ldots, a_i]$$

is called the *polarization* of I.

2. Matroids and complete intersection simplicial complexes

We will characterize matroids (complete intersection simplicial complexes) Δ in terms of the cleanness of the symbolic (ordinary) powers of I_Δ.

Theorem 2.1. *Let $I \subset S$ be the Stanley-Reisner ideal of a matroid Δ. Then $S/I^{(m)}$ is clean for all $m \in \mathbb{N}$.*

Proof. Let $I = I_\Delta = \bigcap_{i=1}^{t} P_{F_i}$ be the irredundant irreducible primary decomposition of I, where $\Delta^c = \langle F_1, \ldots, F_t\rangle$ and $r = |F_i|$ for all $i = 1, \ldots, t$. Then $I^{(m)} = \bigcap_{i=1}^{t} P_{F_i}^m$. By [11], Theorem 3.10, it is enough to show that $T/(I^{(m)})^p$ is clean.

One can see by [6], Proposition 2.3 (3), that $((I^{(m)})^p)^\vee = \sum_{i=1}^{r} ((P_{F_i}^m)^p)^\vee$. If $F_i = \{s_1, \ldots, s_r\}$, then by [6], Proposition 2.5 (2), $(P_{F_i}^m)^p$ has the irredundant irreducible primary decomposition

$$(P_{F_i}^m)^p = \bigcap_{\substack{1 \leqslant t_j \leqslant m \\ \sum t_j \leqslant m+r-1}} (x_{s_1,t_1}, \ldots, x_{s_r,t_r}).$$

It follows that the ideal $J := ((I^{(m)})^p)^\vee$ is generated by the monomials

$$x_{i_1,a_1} x_{i_2,a_2} \ldots x_{i_r,a_r} \quad \text{with } \{i_1, \ldots, i_r\} \in \mathcal{F}(\Delta^c),$$

where a_j are positive integers satisfying $1 \leqslant a_j \leqslant m$ and $\sum_{j=1}^{r} a_j \leqslant m + r - 1$. For showing that $T/(I^{(m)})^p$ is clean, it is enough to show that J has linear quotients; see for example [2], Lemma 2.1.

Now, we order the variables in T as follows:

$x_{i,a} > x_{j,b} \Leftrightarrow (i, a) < (j, b)$, and $(i, a) < (j, b)$ if $a < b$, or $a = b$ and $i < j$. Then we show that J has linear quotients with respect to the reverse lexicographical order of its generators induced from the above ordering. Indeed, let $u = x_{i_1,a_1} x_{i_2,a_2} \ldots x_{i_r,a_r}$ and $v = x_{j_1,b_1} x_{j_2,b_2} \ldots x_{j_r,b_r}$ be two monomials in $G(J)$ with $u > v$. We have to show that there exists $w \in G(J)$ with $w > v$ such that $w : v = x_{i_l,a_l}$ and $x_{i_l,a_l} \mid u$.

Since $u > v$, there exists an integer t such that $x_{i_t,a_t} > x_{j_t,b_t}$ and $x_{i_k,a_k} = x_{j_k,b_k}$ for all $k > t$. In particular, we have $a_t < b_t$, or $a_t = b_t$ and $i_t < j_t$. We first claim that there exists $1 \leqslant l \leqslant t$ such that

$$x_{j_1} \dots x_{j_{t-1}} x_{i_l} x_{j_{t+1}} \dots x_{j_r} \in G(I_{\Delta^\vee}) = G(I(\Delta^c)).$$

This is obvious, if $x_{j_t} \mid x_{i_1} x_{i_2} \dots x_{i_t}$, and if $x_{j_t} \nmid x_{i_1} x_{i_2} \dots x_{i_t}$, then, as $I^\vee$ is matroidal, it follows that there exists $1 \leqslant l \leqslant t$ such that $x_{j_1} \dots x_{j_{t-1}} x_{i_l} x_{j_{t+1}} \dots x_{j_r} \in G(I^\vee)$. Here, we used the fact that $i_k = j_k$ for $k = t+1, \dots, r$. Then

$$w := x_{j_1,b_1} x_{j_2,b_2} \dots x_{j_{t-1},b_{t-1}} x_{i_l,a_l} x_{j_{t+1},b_{t+1}} \dots x_{j_r,b_r} \in G(J),$$

because $a_l \leqslant b_t$. Moreover, we have $w : v = x_{i_l,a_l}$ and $x_{i_l,a_l} \mid u$.

Next, we will show that $w > v$. If $x_{i_l,a_l} > x_{j_{t-1},b_{t-1}}$, then $w > v$ because $x_{j_{t-1},b_{t-1}} > x_{j_t,b_t}$. Otherwise, one has $x_{i_l,a_l} < x_{j_{t-1},b_{t-1}}$. We know that $a_t < b_t$, or $a_t = b_t$ and $i_t < j_t$. Since $a_l \leqslant a_t$, if $a_t < b_t$, then $w > v$. Now, assume that $a_t = b_t$ and $i_t < j_t$. Since $a_l < a_t$ or $a_l = a_t$, and $i_l < i_t < j_t$, one has $x_{i_l,a_l} > x_{j_t,b_t}$ and $w > v$. $\square$

We shall use the following lemma.

Lemma 2.2. *Let $I \subset S$ be a monomial ideal. Then S/I is Cohen-Macaulay if and only if S/I is sequentially Cohen-Macaulay and I is unmixed.*

P r o o f. If S/I is Cohen-Macaulay, then it is obvious that S/I is sequentially Cohen-Macaulay and I is unmixed. Conversely, assume that S/I is sequentially Cohen-Macaulay and I is unmixed. Then there exists a chain of monomial ideals

$$I = I_0 \subset I_1 \subset I_2 \subset \dots \subset I_r = S$$

such that each quotient I_i/I_{i-1} is Cohen-Macaulay and

$$\dim(I_1/I_0) < \dim(I_2/I_1) < \dots < \dim(I_r/I_{r-1}).$$

By [9], Lemma 1.2, $\operatorname{depth}(S/I) = \dim(I_1/I_0)$. On the other hand, by [8], Proposition 2.5, $\operatorname{Ass}(S/I) = \bigcup_{i=1}^{r} \operatorname{Ass}(I_i/I_{i-1})$. Since I is unmixed, it follows that $\dim(S/I) = \dim(I_i/I_{i-1})$ for all i. Hence $\operatorname{depth}(S/I) = \dim(I_1/I_0) = \dim(S/I)$, and so S/I is Cohen-Macaulay. $\square$

If we combine our results with [16], Theorem 3.6, we get the following characterization of matroids.

Corollary 2.3. *Let Δ be a pure simplicial complex and $I = I_\Delta \subset S$. Then the following conditions are equivalent:*

(a) *Δ is a matroid.*
(b) *$S/I^{(m)}$ is clean for all integers $m > 0$.*
(c) *$S/I^{(m)}$ is clean for some integer $m \geqslant 3$.*
(d) *$S/I^{(m)}$ is Cohen-Macaulay for some integer $m \geqslant 3$.*
(e) *$S/I^{(m)}$ is Cohen-Macaulay for all integers $m > 0$.*

Proof. In view of Theorem 2.1, (a) $\Rightarrow$ (b) holds. The implications (a) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) follow from [16], Theorem 3.6. The implication (b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (d) Suppose that for an integer $m \geqslant 3$, $S/I^{(m)}$ is clean. Then by [8], Corollary 4.3, $S/I^{(m)}$ is sequentially Cohen-Macaulay. On the other hand, $I^{(m)}$ is an unmixed monomial ideal for all m, because I is unmixed and $\mathrm{Ass}(S/I^{(m)}) = \mathrm{Ass}(S/I)$. Hence by Lemma 2.2, $S/I^{(m)}$ is Cohen-Macaulay. □

It is known [1] that a simplicial complex Δ is a complete intersection if and only if S/I_Δ^m is Cohen-Macaulay for all $m \in \mathbb{N}$. Since for a complete intersection monomial ideal I_Δ the symbolic powers coincide with its ordinary powers, we have:

Corollary 2.4. *Let Δ be a pure simplicial complex and $I = I_\Delta \subset S$. Then the following conditions are equivalent:*

(a) *Δ is a complete intersection.*
(b) *S/I^m is clean for all integers $m > 0$.*
(c) *S/I^m is clean for some integer $m \geqslant 3$.*
(d) *S/I^m is Cohen-Macaulay for some integer $m \geqslant 3$.*
(e) *S/I^m is Cohen-Macaulay for all integers $m > 0$.*

Proof. The equivalences (a) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) follow from [16], Theorem 4.3. The implication (b) $\Rightarrow$ (c) is obvious. The proof of (c) $\Rightarrow$ (d) is similar to that of the same case in Corollary 2.3. Note that, as S/I^m is clean for some integer $m \geqslant 3$, it follows that

$$\mathrm{Ass}(S/I^m) = \mathrm{Min}(I^m) = \mathrm{Min}(I) = \mathrm{Ass}(S/I).$$

It remains to show (a) $\Rightarrow$ (b). Since I is complete intersection, for any $m > 0$, one has $\mathrm{Ass}(S/I^m) = \mathrm{Min}(I^m) = \mathrm{Min}(I)$. Hence by the definition of symbolic powers (see [18], Definition 3.3.22), $I^m = I^{(m)}$ for all $m > 0$. Since any complete intersection complex is a matroid, therefore by Theorem 2.1, S/I^m is clean for all $m > 0$. □

Example 2.5. Let $I := (x_1x_2, x_2x_3, x_3x_4)$. Obviously, I is an unmixed squarefree monomial ideal. Since $|\operatorname{G}(I)| \leqslant 3$, it follows by [2], Corollary 2.6, that S/I is clean. On the other hand, $I^\vee = (x_1x_3, x_2x_3, x_2x_4)$ is not matroidal. Hence, I is not the Stanley-Reisner ideal of a matroid. So by Corollary 2.3, $S/I^{(m)}$ is not clean for all integers $m \geqslant 3$. Also, S/I is not complete intersection, so by Corollary 2.4 S/I^m is not clean for all integers $m \geqslant 3$. Now, consider the ideal I as the edge ideal of a graph G. Obviously, G is a bipartite graph, so by [7], Corollary 10.3.17, I is normally torsionfree. Therefore for any m,

$$\operatorname{Ass}(S/I^m) = \operatorname{Ass}(S/I) = \operatorname{Min}(I) = \operatorname{Min}(I^m).$$

It follows by [8], Corollary 3.5, that S/I^m is not pretty clean for all integers $m \geqslant 3$.

We note that the above example shows that, if $I \subset S$ is a pretty clean monomial ideal, then necessarily $S/I^{(m)}$ cannot be pretty clean for all integers $m > 0$.

3. Second symbolic power and cleanness

Let Δ be a 1-dimensional simplicial complex and $I = I_\Delta \subset S$. Minh and Trung in [12] studied under which conditions $S/I^{(2)}$ and S/I^2 are Cohen-Macaulay. In this section we will give a characterization for the Cohen-Macaulayness of $S/I^{(2)}$ and S/I^2 in terms of the cleanness property.

Let $G = (V, E)$ be a simple graph. In graph theory, the distance between two vertices u and v of G is the minimal length of paths from u to v and is denoted by $d(u, v)$. This length is infinite if there is no path connecting them. The diameter of G, $\operatorname{diam}(G)$, is defined by $\operatorname{diam}(G) := \max\{d(u, v) \colon u, v \in V\}$.

Theorem 3.1. *Let Δ be a pure simplicial complex on $[n]$ with $\dim \Delta = 1$ and $I = I_\Delta \subset S$. Then the following conditions are equivalent:*

(a) *$S/I^{(2)}$ is clean.*

(b) *$S/I^{(2)}$ is Cohen-Macaulay.*

(c) $\operatorname{diam} \Delta \leqslant 2$.

Proof. (a) $\Rightarrow$ (b) Since $S/I^{(2)}$ is sequentially Cohen-Macaulay and $I^{(2)}$ is unmixed, the desired conclusion follows from Lemma 2.2.

(b) $\Rightarrow$ (c) follows from [12], Theorem 2.3.

(c) $\Rightarrow$ (a) By [11], Theorem 3.10, it is enough to show that $S/(I^{(2)})^p$ is clean. Let $I = I_\Delta = \bigcap_{i=1}^{t} P_{F_i}$ be a primary decomposition of I. Then $\Delta^c = \langle F_1, \ldots, F_t \rangle$ with

$|F_i| = n-2$ for all $i = 1, \ldots, t$. We know that

$$(I^{(2)})^p = \bigcap_{i=1}^{t} (P_{F_i}^2)^p.$$

If $F \subset [n]$, then by [6], Proposition 2.5 (2),

$$(P_F^2)^p = \bigcap_{1 \leqslant j \leqslant n-2} P_{(F,2_j)} \cap P_{(F,1)},$$

where if $F = \{r_1, \ldots, r_{n-2}\}$ with $r_1 < r_2 < \ldots < r_{n-2}$, then we set $(F,1) := \{(r_i,1)\colon r_i \in F\}$ and $(F,2_j) := \{(r_j,2)\} \cup \{(r_i,1)\colon\ 1 \leqslant i \leqslant n-2,\ i \neq j\}$. Note that $(I^{(2)})^p$ is a monomial ideal in a polynomial ring $T = K[x_{(1,1)}, \ldots, x_{(n,1)}, x_{(1,2)}, \ldots, x_{(n,2)}]$. Since $(I^{(2)})^p$ is the Stanley-Reisner ideal of the simplicial complex

$$\Gamma = \langle (F_i,1)^c, (F_i,2_j)^c\colon\ 1 \leqslant i \leqslant t,\ 1 \leqslant j \leqslant n-2 \rangle,$$

by a result of Dress in [5] it is enough to prove that Γ is shellable.

We set $A_0 := \emptyset$ and $A_i := \left\{F_j^c \in \mathcal{F}(\Delta)\colon\ i \in F_j^c \text{ and } F_j^c \notin \bigcup_{s=1}^{i-1} A_s\right\}$ for all $i = 1, \ldots, n$. Note that $\mathcal{F}(\Delta) = \bigcup_{i=1}^{n} A_i$. We order the facets of Γ by the following process and show that the given order is a shelling order. For the convenience we can assume that $A_1 = \{F_1^c, \ldots, F_{s_1}^c\}$ for some $1 \leqslant s_1 \leqslant t$. Let the initial part of our order be

$$(*) \quad (F_1,1)^c, (F_1,2_1)^c, \ldots, (F_1,2_{n-2})^c, (F_2,1)^c, (F_2,2_1)^c, \ldots, (F_2,2_{n-2})^c, \ldots,$$
$$(F_{s_1},1)^c, (F_{s_1},2_1)^c, \ldots, (F_{s_1},2_{n-2})^c.$$

Then the following inequalities hold:

$$\begin{aligned} n &= |(F_1,1)^c \cap (F_1,2_j)^c| - 1 = \dim(\langle (F_1,1)^c \rangle \cap \langle (F_1,2_j)^c \rangle) \\ &\leqslant \dim(\langle (F_1,1)^c, (F_1,2_1)^c, \ldots, (F_1,2_{j-1})^c \rangle \cap \langle (F_1,2_j)^c \rangle) \\ &\leqslant \dim\langle (F_1,2_j)^c \rangle - 1 = |(F_1,2_j)^c| - 2 = n. \end{aligned}$$

Now, let $2 \leqslant d \leqslant s_1$. Then

$$\begin{aligned} n &= \dim(\langle (F_1,1)^c \rangle \cap \langle (F_d,1)^c \rangle) \\ &\leqslant \dim(\langle (F_1,1)^c, (F_1,2_1)^c, \ldots, (F_{d-1},2_{n-2})^c \rangle \cap \langle (F_d,1)^c \rangle) \\ &\leqslant \dim\langle (F_d,1)^c \rangle - 1 = n. \end{aligned}$$

Also, for any $1 \leqslant j \leqslant n-2$, we have

$$\begin{aligned} n &= \dim(\langle (F_d,1)^c\rangle \cap \langle (F_d,2_j)^c\rangle) \\ &\leqslant \dim(\langle (F_1,1)^c,(F_1,2_1)^c,\dots,(F_d,1)^c,\dots,(F_d,2_{j-1})^c\rangle \cap \langle (F_d,2_j)^c\rangle) \\ &\leqslant \dim\langle (F_d,2_j)^c\rangle - 1 = n. \end{aligned}$$

Suppose that Γ_1 is a simplicial complex whose facets are all of the sets belonging to $(*)$. If we rename the facets of Γ_1 in the same order as above by $G_1,\dots,G_{s_1(n-1)}$, then it is easy to see that $\langle G_1,\dots,G_{i-1}\rangle \cap \langle G_i\rangle$ is a pure simplicial complex for all $i = 1,\dots,s_1(n-1)$. Therefore, Γ_1 is shellable.

Assume that $A_i = \{F^c_{s_{i-1}+1},\dots,F^c_{s_i}\}$ for $1 \leqslant i \leqslant h-1 < n$, where $s_0 = 0$ and $s_{i-1} < s_i$. Then we may assume by induction process that the following order is a shelling order for the simplicial complex with the set of facets

$$\begin{gathered} (F_1,1)^c,(F_1,2_1)^c,\dots,(F_1,2_{n-2})^c,\dots,(F_j,1)^c,(F_j,2_1)^c,\dots,(F_j,2_{n-2})^c, \\ (F_{j+1},1)^c,(F_{j+1},2_1)^c,\dots,(F_{j+1},2_{n-2})^c,\dots, \\ (F_{s_{h-1}},1)^c,(F_{s_{h-1}},2_1)^c,\dots,(F_{s_{h-1}},2_{n-2})^c, \end{gathered}$$

where $1 < j < s_{h-1}$.

Now, let $1 < h \leqslant n$. If there exists $F^c \in \bigcup_{i=1}^{h-1} A_i$ such that $h \in F^c$, then we take an arbitrary element G of A_h and set $F^c_{s_{h-1}+1} := G$. In this case, we have

$$\begin{aligned} n &= \dim(\langle (F,1)^c\rangle \cap \langle (F_{s_{h-1}+1},1)^c\rangle) \\ &\leqslant \dim(\langle (F_1,1)^c,(F_1,2_1)^c,\dots,(F_{s_{h-1}},2_{n-2})^c\rangle \cap \langle (F_{s_{h-1}+1},1)^c\rangle) \\ &\leqslant \dim\langle (F_{s_{h-1}+1},1)^c)\rangle - 1 = n. \end{aligned}$$

Otherwise, for any $F^c \in \bigcup_{i=1}^{h-1} A_i$, $h \notin F^c$. Hence $\{1,h\} \notin \mathcal{F}(\Delta)$. Since $\operatorname{diam}(\Delta) \leqslant 2$, it follows that there exists $m \in [n]$ such that $m \neq 1$, $m \neq h$ and $\{m,h\} \in A_h$, and $F^c := \{1,m\} \in A_1$. In this case we set $F^c_{s_{h-1}+1} := \{m,h\}$.

Now, the following inequalities hold:

$$\begin{aligned} n &= \dim(\langle (F,1)^c\rangle \cap \langle (F_{s_{h-1}+1},1)^c\rangle) \\ &\leqslant \dim(\langle (F_1,1)^c,(F_1,2_1)^c,\dots,(F_{s_{h-1}},2_{n-2})^c\rangle \cap \langle (F_{s_{h-1}+1},1)^c\rangle) \\ &\leqslant \dim\langle (F_{s_{h-1}+1},1)^c\rangle - 1 = n. \end{aligned}$$

We order all the other facets of Γ which correspond to A_h as

$$(F_{s_{h-1}+1},2_1)^c,\dots,(F_{s_{h-1}+1},2_{n-2})^c,\dots,(F_{s_h},1)^c,(F_{s_h},2_1)^c,\dots,(F_{s_h},2_{n-2})^c,$$

where $s_{h-1} < s_h$.

In the same way as previously, we can easily check that the given order is a shelling order. □

A 1-dimensional simplicial complex Δ on the vertex set $[n]$ is called a cycle of length n if the facets of Δ are $\{1, n\}$ and $\{i, i+1\}$ for all $i = 1, \dots, n-1$.

Corollary 3.2. *Let Δ be a pure simplicial complex on $[n]$ with $\dim \Delta = 1$ and $I = I_\Delta \subset S$. Then the following conditions are equivalent:*

(a) *S/I^2 is clean.*

(b) *S/I^2 is Cohen-Macaulay.*

(c) *Δ is a path of length $1, 2$ or a cycle of length $3, 4, 5$.*

Proof. (a) $\Rightarrow$ (b) Since S/I^2 is sequentially Cohen-Macaulay and I^2 is unmixed, the desired conclusion follows from Lemma 2.2.

(b) $\Rightarrow$ (c) If $n = 2$, then Δ is a path of length 1. If $n = 3$, then Δ is either a path of length 2 or a triangle (a cycle of length 3). Finally, if $n \geqslant 4$, then by [12], Corollary 3.4, Δ is a cycle of length 4 or 5.

(c) $\Rightarrow$ (a) It is easy to see that in each case, we have $\operatorname{diam} \Delta \leqslant 2$ and $I^{(2)} = I^2$. Hence the desired conclusion follows by Theorem 3.1. □

It is known that if I is a monomial ideal and S/I is clean, then S/I is sequentially Cohen-Macaulay. In particular when I is unmixed, then S/I is Cohen-Macaulay. But the converse is not true in general. In some special cases, like edge ideals of unmixed bipartite graphs, it is known that Cohen-Macaulayness and cleanness are equivalent. As another corollary of our results we get the following:

Corollary 3.3. *Let $m > 1$ be an integer, Δ a pure simplicial complex with $\dim \Delta = 1$, and $I = I_\Delta \subset S$. Then $S/I^{(m)}$ (S/I^m) is clean if and only if $S/I^{(m)}$ (S/I^m, respectively) is Cohen-Macaulay.*

Acknowledgment. We would like to thank Jürgen Herzog for raising the question of whether all (symbolic) powers of a matroid are clean, and for reading an earlier version of this manuscript.

References

[1] *R. Achilles, W. Vogel*: Über vollständige Durchschnitte in lokalen Ringen. Math. Nachr. *89* (1979), 285–298. (In German.) zbl MR doi

[2] *S. Bandari, K. Divaani-Aazar, A. S. Jahan*: Almost complete intersections and Stanley's conjecture. Kodai Math. J. *37* (2014), 396–404. zbl MR doi

[3] *A. Björner, M. L. Wachs*: Shellable nonpure complexes and posets. I. Trans. Am. Math. Soc. *348* (1996), 1299–1327. zbl MR doi

[4] *W. Bruns, J. Herzog*: Cohen-Macaulay Rings. Cambridge Studies in Advanced Mathematics 39, Cambridge University Press, Cambridge, 1993. zbl MR
[5] *A. Dress*: A new algebraic criterion for shellability. Beitr. Algebra Geom. *34* (1993), 45–55. zbl MR
[6] *S. Faridi*: Monomial ideals via square-free monomial ideals. Commutative Algebra: Geometric, Homological, Combinatorial and Computational Aspects (A. Corso et al., eds.). Proc. Conf., Sevilla, 2003. Lecture Notes in Pure and Applied Mathematics 244, Chapman & Hall/CRC, Boca Raton, 2006, pp. 85–114. zbl MR doi
[7] *J. Herzog, T. Hibi*: Monomial Ideals. Graduate Texts in Mathematics 260, Springer, London, 2011. zbl MR doi
[8] *J. Herzog, D. Popescu*: Finite filtrations of modules and shellable multicomplexes. Manuscr. Math. *121* (2006), 385–410. zbl MR doi
[9] *J. Herzog, D. Popescu, M. Vladoiu*: On the Ext-modules of ideals of Borel type. Commutative Algebra: Interactions with Algebraic Geometry (L. L. Avramov et al., eds.). Proc. Conf., Grenoble, 2001, Contemp. Math. 331, AMS, Providence, 2003, pp. 171–186. zbl MR doi
[10] *D. T. Hoang, N. C. Minh, T. N. Trung*: Combinatorial characterizations of the Cohen-Macaulayness of the second power of edge ideals. J. Comb. Theory, Ser. A *120* (2013), 1073–1086. zbl MR doi
[11] *A. S. Jahan*: Prime filtrations of monomial ideals and polarizations. J. Algebra *312* (2007), 1011–1032. zbl MR doi
[12] *N. C. Minh, N. V. Trung*: Cohen-Macaulayness of powers of two-dimensional squarefree monomial ideals. J. Algebra *322* (2009), 4219–4227. zbl MR doi
[13] *N. C. Minh, N. V. Trung*: Cohen-Macaulayness of monomial ideals and symbolic powers of Stanley-Reisner ideals. Adv. Math. *226* (2011), 1285–1306; corrigendum ibid. *228* (2011), 2982–2983. zbl MR doi
[14] *J. G. Oxley*: Matroid Theory. Oxford Graduate Texts in Mathematics 3, Oxford Science Publications, Oxford University Press, Oxford, 1992. zbl MR
[15] *R. P. Stanley*: Combinatorics and Commutative Algebra. Progress in Mathematics 41, Birkhäuser, Basel, 1996. zbl MR doi
[16] *N. Terai, N. V. Trung*: Cohen-Macaulayness of large powers of Stanley-Reisner ideals. Adv. Math. *229* (2012), 711–730. zbl MR doi
[17] *M. Varbaro*: Symbolic powers and matroids. Proc. Am. Math. Soc. *139* (2011), 2357–2366. zbl MR doi
[18] *R. H. Villarreal*: Monomial Algebras. Pure and Applied Mathematics 238, Marcel Dekker, New York, 2001. zbl MR

Authors' addresses: Somayeh Bandari, Department of Engineering Sciences and Physics, Buein Zahra Technical University, 3451745346, Buein Zahra, Iran, and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P. O. Box 19395-5746, Tehran, Iran, e-mail: `somayeh.bandari@yahoo.com`; Ali Soleyman Jahan, Department of Mathematics, University of Kurdistan, 66177-15175, Sanandaj, Iran, e-mail: `solymanjahan@gmail.com`.