Kybernetika

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Kybernetika, Vol. 53 (2017), No. 3, 530–544

Persistent URL: `http://dml.cz/dmlcz/146941`

KYBERNETIKA — VOLUME *53* (2017), NUMBER 3, PAGES 530–544

BOUNDED-INPUT-BOUNDED-STATE STABILIZATION OF SWITCHED PROCESSES AND PERIODIC ASYMPTOTIC CONTROLLABILITY

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The main result of this paper is a sufficient condition for the existence of periodic switching signals which render asymptotically stable at the origin a linear switched process defined by a pair of 2×2 real matrices. The interest of this result is motivated by the application to the problem of bounded-input-bounded-state (with respect to an external input) stabilization of linear switched processes.

Keywords: switched processes, asymptotic controllability, bounded-input-bounded-state stability

Classification: 93D20, 93B60

1. INTRODUCTION

In the recent, rapidly increased literature about switched systems, the fundamental notions of classical control theory, such as controllability and stability, have been revisited and extended ([8, 11, 14, 20]). In this paper, we deal with the extension to a linear switched process of the *bounded-input-bounded-state stability* property, and its relationship with asymptotic controllability of the associated unforced system.

Bounded-input-bounded-state stability is a natural requirement for systems whose behavior is affected by an external input. Roughly speaking, it means that the state variable (or, more generally, the output variable) remains bounded in the future for each initial condition, provided that the input variable is bounded (see for instance [4, 12, 19]). The nature of the external input depends on the application: it may be a reference signal, as in the servomechanisms, or a disturbance.

When bounded-input-bounded-state stability does not hold, it is natural to ask whether it can be achieved by exerting a suitable control action: this is the *bounded-input-bounded-state stabilization* problem. Note that in general, the external input and the control are injected into the system through separate input channels.

The class of switched processes considered in this paper is formally introduced in the next section. For the moment we limit ourselves to say that a switched process is formed by several components, and each component is represented by a finite-dimensional linear time-invariant system

DOI: 10.14736/kyb-2017-3-0530

$$\dot{x} = A_\nu x + B_\nu u \ , \quad \nu \in \mathcal{N} \tag{1}$$

where $\mathcal{N}$ is a set of indices. The switching policy is determined by a *switching signal*, that is a piecewise constant function $\sigma : [0, +\infty) \to \mathcal{N}$. The role of the external input is played by u, while the role of the control is played by the switching signal σ.

Our approach to the problem is inspired by the well known, classical result that, for a single linear, time-invariant system, bounded-input-bounded-state stability is implied by the asymptotic stability of the associated unforced system. In our case, the unforced system is obtained by setting $u = 0$ in (1).

We are so led to study the following problem: find sufficient conditions guaranteeing the existence of a switching signal $\sigma(t)$ such that, when implemented, the state $x(t)$ of the associated unforced switched system converges to the origin for each initial condition, when $t \to \infty$. This is the switched version of the classical *asymptotic controllability* problem. More precisely, in this paper we restrict the search to *periodic* switching signals.

In the literature, the stabilization problem for (unforced) switched systems is usually addressed by the (common or multiple) Lyapunov function method, which leads to closed-loop discontinuous (with respect to the state variable) feedback laws (see for instance [11, 18]). This implies in turn some mathematical problems about the existence of solutions ([5]). The asymptotic controllability problem, making use of open-loop controls, avoids this drawbacks, and the limitation to periodic signals seems to represent a reasonable compromise from the point of view of applications.

A general result about asymptotic controllability of switched systems can be found in [3], but the problem becomes more difficult when we restrict the search to periodic switching signals. To this respect, the most significant result available in the literature is a sufficient condition in [15] (however, as pointed out in [1], this condition is limited to switching laws of sufficiently small period). Apart from this remarkable exception, the problem is basically open: a summary with more technical details about this problem will be found in Section 4 (see also [2] and the references therein).

The way the evolution of the state is affected by the input is more frequently studied in terms of the input-to-state stability property (a notion introduced in [13]). Bounded-input-bounded-state stability is indeed a weak version of input-to-state stability. Roughly speaking, input-to-state stability can be thought of as the extension of global asymptotic stability to systems with inputs, as long as bounded-input-bounded-state stability is an analogous extension of simple Lyapunov stability. In the context of switched systems theory, input-to-state stability has been extended and studied, essentially on the base of the Lyapunov method, in several papers ([10, 16, 17, 21, 23, 24]).

The paper is organized as follows. In Section 2 we introduce the class of systems under consideration. In Section 3 we show that if there exists a switching signal σ : $[0, +\infty) \to \mathcal{N}$ (independent of the initial state) such that all the switched solutions of the associated unforced system approach the origin for $t \to +\infty$, then the same switching signal stabilizes the given process in the bounded-input-bounded-state sense. This is not difficult to prove: it is actually a combination of some well know existing results (especially Theorem 5.1 of [6]). However, it provides a strong motivation for our main result presented in Section 4, where we give a new sufficient condition for the periodic asymptotic controllability of a pair of 2×2 matrices. Our approach is based on linear algebra ideas and controllability notions, and does not make use of Lyapunov functions. The proof of the main result is given in Section 5. Section 6 is devoted to some useful examples, while Section 7 resumes the conclusions.

2. LINEAR SWITCHED PROCESSES WITH INPUTS

Now we introduce formally the class of systems of interest in this paper. Let $\mathcal{N} = \{1, \ldots, N\}$ be a set of indices, endowed with the discrete topology. A *linear switched process* is a pair $(\mathcal{F}, \mathcal{G})$ where

$$\mathcal{F} = \{A_1, \ldots, A_N\} , \quad \mathcal{G} = \{B_1, \ldots, B_N\}$$

and, for each index $\nu \in \mathcal{N}$, A_ν is a real square $n \times n$ matrix and B_ν is a real $n \times m$ matrix. As already mentioned, the time-invariant linear systems

$$\dot{x} = A_\nu x + B_\nu u , \quad t \geq 0 , \quad x \in \mathbf{R}^n , \quad u \in \mathbf{R}^m, \quad \nu \in \mathcal{N} \tag{2}$$

constitute the *components* of the process. The switching policy is determined by a *switching signal*, that is a piecewise constant, right continuous function $\sigma : [0, +\infty) \to \mathcal{N}$.

For each fixed switched signal $\sigma : [0, +\infty) \to \mathcal{N}$, we associate to $(\mathcal{F}, \mathcal{G})$ the linear, time-varying system

$$\dot{x} = A(t)x + B(t)u \tag{3}$$

where $A(t) = A_{\sigma(t)}$ and $B(t) = B_{\sigma(t)}$. Note that the entries of $A(t)$ and $B(t)$ are piecewise constant functions, for $t \in [0, +\infty)$. Hence, for each initial state $x_0 \in \mathbf{R}^n$ and each measurable, locally bounded function $u : [0, +\infty) \to \mathbf{R}^m$, (3) admits a unique solution $x(t)$ defined for $t \in [0, +\infty)$, such that $x(0) = x_0$. It will be called the *solution* of $(\mathcal{F}, \mathcal{G})$ corresponding to x_0, $\sigma(t)$ and $u(t)$.

The *unforced switched system* associated to $(\mathcal{F}, \mathcal{G})$ is defined by the family of matrices $\mathcal{F} = \{A_1, \ldots, A_N\}$, and corresponds to the family of linear (unforced) systems

$$\dot{x} = A_\nu x , \quad \nu \in \mathcal{N} . \tag{4}$$

For an introduction to the general properties and the formalism about unforced switched systems we refer the reader to [3, 11, 14]; here, we limit ourselves to recall that a *switched solution* of $\mathcal{F}$ is a continuous, piecewise differentiable curve obtained by gluing together solutions of the linear systems (4). Hence, for a given switched signal $\sigma(t)$ and a given initial state x_0, there is a unique switched solution of $\mathcal{F}$, defined for $t \geq 0$. Moreover, such a switched solution can be represented on any compact interval $[0, \bar{t}]$ $(\bar{t} > 0)$ as a composition of exponentials

$$e^{t_{Nk}A_N} \cdot \ldots \cdot e^{t_{1k}A_1} \cdot \ldots \ldots \cdot e^{t_{N2}A_N} \cdot \ldots \cdot e^{t_{12}A_1} \cdot e^{t_{N1}A_N} \cdot \ldots \cdot e^{t_{11}A_1} x_0$$

where $k \geq 1$ is any integer, $t_{ij} \geq 0$ $(i = 1, \ldots, N, \ j = 1, \ldots, k)$, and $\sum t_{ij} = \bar{t}$. The numbers t_{ij} are called *durations*; they correspond to the length of the time intervals separating the *switching times*, that are the contiguous discontinuity points of the switching signal σ. At a switching time, a switched solution of $\mathcal{F}$ could be not differentiable.

3. BIBS STABILITY

The following definition applies in general to linear time-varying systems of the form (3) (not necessarily associated to a switched process). It differs from the definition given in [6] for some details (basically, the addition of some more uniformity: see also [4, 19]). Recall that any square matrix $A(t)$ with measurable and locally bounded entries generates a *transition matrix* $G(t,\tau)$, defined for $t,\tau \geq 0$ (see [7]). The symbol $||\cdot||$ denotes any norm of (finite dimensional) vectors or matrices; the symbol $||u(\cdot)||_\infty$ denotes the infinity norm of the function $u(\cdot)$, that is $||u(\cdot)||_\infty = \operatorname{ess\,sup}_{t\geq 0}||u(t)||$.

Definition 3.1. The linear time-varying system (3) is (uniformly) bounded-input-bounded-state (in short, BIBS) stable if:

1. there exists $\gamma > 0$ such that
$$t \geq \tau \geq 0 \Longrightarrow ||G(t,\tau)|| < \gamma$$
2. there exists $k > 0$ such that
$$t \geq 0 \Longrightarrow \int_0^t ||G(t,s)B(s)||\,\mathrm{d}s < k.$$

The switched process $(\mathcal{F},\mathcal{G})$ is said to be BIBS stabilizable (with respect to the external input u) if there exists a switching signal $\sigma(t)$ such that the associated linear time-varying system (3) is BIBS stable.

The first requirement of Definition 3.1 is equivalent to uniform stability at the origin of the unforced time-varying system

$$\dot{x} = A(t)x \tag{5}$$

associated to (3). The first and the second requirement together imply that for each input $u(t)$, bounded on $[0,+\infty)$, and each $\tau \geq 0$, the solution $x(t)$ of (3) such that $x(\tau) = \chi \in \mathbf{R}^n$ satisfies the inequality

$$||x(t)|| \leq \gamma \cdot ||\chi|| + k \cdot ||u(\cdot)||_\infty$$

on $[0,+\infty)$ and it is so bounded.

Theorem 3.2. Assume that $B(t)$ is bounded for $t \geq 0$. Assume also that the unforced time-varying system (5) is exponentially asymptotically stable at the origin. Then, $(\mathcal{F},\mathcal{G})$ is BIBS stable.

This theorem is nothing else a restatement of Theorem 5.1 in [6]. By virtue of Theorem 3.2, the BIBS stabilization problem for the switched process $(\mathcal{F},\mathcal{G})$ is reduced to the problem of constructing, if possible, a switching signal (independent of the initial state) such that all the switched solutions of the associated switched unforced process $\mathcal{F}$ converge to the origin when $t \to +\infty$.

4. THE UPAC PROPERTY

If the problem mentioned at the end of the previous section (existence of a switching signal independent of the initial state such that all the switched solutions of $\mathcal{F}$ converge to the origin for $t \to +\infty$) has a solution, we say that $\mathcal{F}$ has the *uniform asymptotic controllability* (in short, UAC) property. If the problem is solvable by means of a periodic switching signal, we say that $\mathcal{F}$ has the *uniform periodic asymptotic controllability* (in short, UPAC) property. It is well known that the UAC property and the UPAC property are actually equivalent ([2, 14]). Hence, the BIBS stabilization problem for $(\mathcal{F}, \mathcal{G})$ can be further reduced to the search for a periodic switching signal.

A complete characterization of the UPAC property in arbitrary dimension is still an open problem. In this section, we give a short description of the state of the art; then, we present a new result for the two-dimensional case. To simplify the notation, from now on we assume $N = 2$. We introduce the set $\mathbf{R}^k_+ = \{\mathbf{t} = (t_1, \dots, t_k) : t_j \geq 0\}$, and the vectors $\mathbf{t}_1 = (t_{11}, \dots, t_{1k}), \mathbf{t}_2 = (t_{21}, \dots, t_{2k}) \in \mathbf{R}^k_+$. Moreover, we write

$$E(\mathbf{t}_1, \mathbf{t}_2) = e^{t_{2k}A_2} \cdot e^{t_{1k}A_1} \cdot \ldots \cdot e^{t_{21}A_2} \cdot e^{t_{11}A_1} \ . \tag{6}$$

Recall that a matrix M is said to be *Hurwitz* when all its eigenvalues have negative real part; *Schur* when all its eigenvalues lie in the unit open disc. The proof of the following proposition can be found in [19] (see also [2]).

Proposition 4.1. The unforced switched process $\mathcal{F} = \{A_1, A_2\}$ has the UPAC property if and only if there exist $\mathbf{t}_1, \mathbf{t}_2 \in \mathbf{R}^k_+$ such that the matrix (6) is Schur.

A well known necessary condition for the UPAC property is that $\operatorname{tr} A_\nu < 0$, for at least one index $\nu \in \{1, 2\}$ ([2, 14]); however, this condition is not sufficient ([2]). It is also well known that if there exist $\alpha_1, \alpha_2 \geq 0$ such that[1] the matrix $\alpha_1 A_1 + \alpha_2 A_2$ is Hurwitz, then the unforced process $\mathcal{F}$ is UPAC ([11, 18]). In fact, under this assumption the matrix

$$e^{T\alpha_2 A_2} \cdot e^{T\alpha_1 A_1}$$

is Schur provided that the period T is sufficiently small. We refer the reader to [1] for a discussion about the converse statement; here, we limit ourselves to remark that in general the condition is not necessary (see the examples of Section 6). In particular, there exist pairs of 2×2 matrices for which asymptotic controllability can be achieved by periodic switching signals only if the period is sufficiently large. Next, we recall the notion of radial controllability ([3]).

Definition 4.2. The pair of matrices $\mathcal{F}$ is *radially controllable* (in short, RC) if for each pair $x_0, x_1 \in \mathbf{R}^2 \setminus \{0\}$ there exists $c > 0$ such that the point cx_1 is reachable from x_0 in finite time along a switched solution, that is

$$E(\mathbf{t}_1, \mathbf{t}_2)x_0 = cx_1$$

for some $\mathbf{t}_1, \mathbf{t}_2 \in \mathbf{R}^k_+$.

[1] Without loss of generality, one can also assume $\alpha_1 + \alpha_2 = 1$.

The role of the RC property in the asymptotic controllability problem is widely exploited in [1, 2, 3]. An obvious necessary condition for RC is the non-existence of positively invariant sectors common to the linear systems $\dot{x} = A_\nu x$, $(\nu \in \{1,2\})$. The main result of this note is the following sufficient condition, which applies also in certain cases where the UPAC property cannot be realized by switching signals of small period.

Theorem 4.3. Assume that the pair of matrices $\mathcal{F}$ fulfils the following conditions:

(i) there exists $\nu \in \{1,2\}$ such that $\mathrm{tr}\, A_\nu < 0$;

(ii) $\mathcal{F}$ is radially controllable.

Then, $\mathcal{F}$ has the UPAC property.

5. PROOF OF THEOREM 4.3

Without loss of generality, assume that *(i)* holds for $\nu = 1$, and let

$$\alpha_0 = \max\left\{\frac{\mathrm{tr}\, A_2}{-\mathrm{tr}\, A_1}, 0\right\} .$$

First we establish the following claim.

Claim 1. There exist a vector $x_0 \neq 0$, a real number $c > 0$, an integer $k \geq 1$ and vectors $\mathbf{t}_1, \mathbf{t}_2 \in \mathbf{R}^k_+ \setminus \{0\}$ such that

$$E(\mathbf{t}_1, \mathbf{t}_2)x_0 = -cx_0 \tag{7}$$

and moreover

$$\frac{t_{11} + \ldots + t_{1k}}{t_{21} + \ldots + t_{2k}} > \alpha_0 \ . \tag{8}$$

Proof. By virtue of *(i)*, the eigenvalues of A_1 cannot be both on the imaginary axis. Thus, the eigenvalues of A_1 can be either complex conjugate with nonzero imaginary part, or real. We give different proofs for these two cases.

Case 1. The eigenvalues of A_1 are complex conjugate (with nonzero imaginary part). In this case the nontrivial orbits of system $\dot{x} = A_1 x$ rotate around the origin. Fix an arbitrary $x_0 \neq 0$. Clearly, there exists $t_{12} > 0$ and $c > 0$ such that

$$e^{t_{12}A_1}x_0 = -cx_0 \ .$$

For any $t_{21} > 0$, we so have

$$e^{t_{12}A_1} \cdot e^{t_{21}A_2} \cdot e^{-t_{21}A_2}x_0 = -cx_0 \ . \tag{9}$$

Let $v_0 = v_0(t_{21}) = e^{-t_{21}A_2}x_0$. Obviously, $v_0(t_{21})$ is a continuous function of t_{21}, and

$$\lim_{t_{21}\to 0^+} v_0(t_{21}) = x_0 \ .$$

Consider the function

$$f(b,\varepsilon,v) = be^{\varepsilon A_1}v : \mathbf{R}\times\mathbf{R}\times\mathbf{R}^2 \to \mathbf{R}^2 \ .$$

Since $f(1,0,x_0) = x_0$, by the Implicit function Theorem there exist functions $\varepsilon = \varepsilon(v), b = b(v)$ defined in a small neighborhood of $v = x_0$ such that

$$b(v)e^{\varepsilon(v)A_1}v = f(b(v),\varepsilon(v),v) = x_0 \ , \qquad \varepsilon(x_0) = 0 \ , \quad b(x_0) = 1 \ .$$

Replacing $v = v_0(t_{21})$, for small t_{21} we get

$$b(v_0(t_{21}))e^{\varepsilon(v_0(t_{21}))A_1}v_0(t_{21}) = x_0 \tag{10}$$

for small t_{21}, and

$$\lim_{t_{21}\to 0^+} \varepsilon(v_0(t_{21})) = 0 \ , \qquad \lim_{t_{21}\to 0^+} b(v_0(t_{21})) = 1 \tag{11}$$

(see the picture in Figure 1). Taking into account of (10), (9) becomes

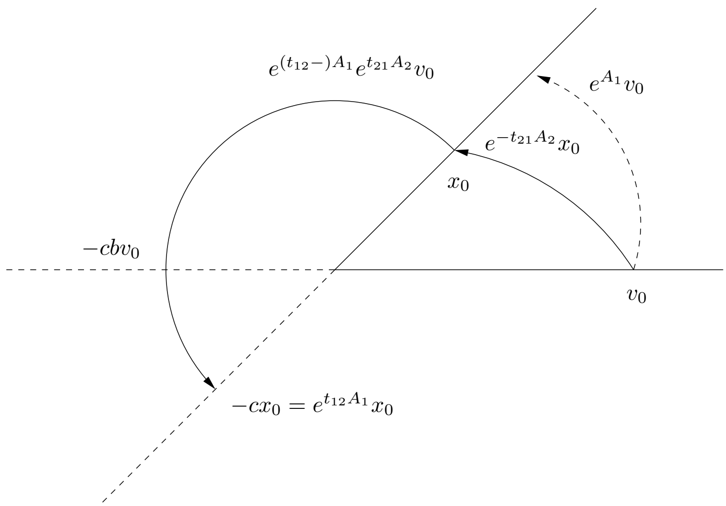

Fig. 1. The picture illustrates formulae (9) and (10) for the case where the vectors A_1x_0 and A_2x_0 point in the same directions; the other cases are similar.

$$e^{t_{12}A_1}\cdot e^{t_{21}A_2}v_0(t_{21}) = -cb(v_0(t_{21}))e^{\varepsilon(v_0(t_{21}))A_1}v_0(t_{21})$$

that is

$$e^{(t_{12}-\varepsilon(v_0(t_{21})))A_1}\cdot e^{t_{21}A_2}v_0(t_{21}) = -cb(v_0(t_{21}))v_0(t_{21})$$

which is nothing else than (7), with $k = 2$, c, x_0 and t_{12} respectively replaced by $cb(v_0(t_{21}))$, $v_0(t_{21})$ and $t_{12} - \varepsilon(v_0(t_{21}))$, $t_{11} = t_{22} = 0$, and t_{21} any sufficiently small positive number.

Notice that by construction, $\mathbf{t}_1$ and $\mathbf{t}_2$ are both nonzero, as required. As far as condition (8) is concerned, in our case it reduces to

$$\frac{t_{12} - \varepsilon(v_0(t_{21}))}{t_{21}} > \alpha_0$$

which is fulfilled for sufficiently small t_{21}, because of the first limit in (11).

Case 2. The eigenvalues of A_1 are real. Because of *(i)* at least one of them, say λ, must be negative. Let x_0 be a real eigenvector of A_1 corresponding to λ. The curve $e^{\lambda t}x_0$ is a solution of system $\dot{x} = A_1 x$, lying on the positive half-line generated by x_0. Moreover, because of *(ii)*, there exist $\mathbf{t}_1, \mathbf{t}_2 \in \mathbf{R}_+^k$ such that

$$E(\mathbf{t}_1, \mathbf{t}_2)x_0 = -cx_0 \ . \tag{12}$$

Let $t_{10} > 0$. We may write

$$E(\mathbf{t}_1, \mathbf{t}_2) \cdot e^{0A_2} \cdot e^{t_{10}A_1}x_0 = E(\mathbf{t}_1, \mathbf{t}_2) \cdot e^{t_{10}\lambda}x_0 = -ce^{t_{10}\lambda}x_0 \ .$$

Thus, condition (7) is fulfilled for any $t_{10} > 0$, with c replaced by $ce^{t_{10}\lambda}$, and

$$\tilde{\mathbf{t}}_1 = (t_{10}, t_{11}, \ldots, t_{1k}) \in \mathbf{R}_+^{k+1} \ , \qquad \tilde{\mathbf{t}}_2 = (t_{20}, t_{21}, \ldots, t_{2k}) \in \mathbf{R}_+^{k+1} \ .$$

Note that necessarily $\tilde{\mathbf{t}}_1 \neq 0$, since $t_{10} > 0$. But also $\tilde{\mathbf{t}}_2 \neq 0$; indeed, in the opposite case

$$E(\mathbf{t}_1, \mathbf{t}_2)x_0 = e^{tA_1}x_0 = e^{t\lambda}x_0$$

for some $t > 0$, a contradiction with (12).

To complete the proof, we remark that also condition (8) is fulfilled, for t_{10} sufficiently large.

□

Next claim is another important step of our proof.

Claim 2. Let k, $\mathbf{t}_1, \mathbf{t}_2 \in \mathbf{R}_+^k$, x_0 and c be such that (8) holds. Let $T > 0$, and let us denote by $\mu_1(T), \mu_2(T)$ the eigenvalues (not necessarily distinct) of $E(T\mathbf{t}_1, T\mathbf{t}_2)$. Then,

$$0 < \mu_1 \cdot \mu_2 < 1 \ .$$

Proof. We have

$$\begin{aligned} \mu_1 \cdot \mu_2 &= \det E(T\mathbf{t}_1, T\mathbf{t}_2) = \det e^{Tt_{2k}A_2} \cdot \det e^{Tt_{1k}A_1} \cdots \det e^{Tt_{21}A_2} \cdot \det e^{Tt_{11}A_1} \\ &= e^{Tt_{2k} \operatorname{tr} A_2} \cdot e^{Tt_{1k} \operatorname{tr} A_1} \cdots e^{Tt_{21} \operatorname{tr} A_2} \cdot e^{Tt_{11} \operatorname{tr} A_1} \\ &= e^{T((t_{2k}+\ldots+t_{21}) \operatorname{tr} A_2 + (t_{1k}+\ldots+t_{11}) \operatorname{tr} A_1)} \ . \end{aligned}$$

This immediately shows that $\mu_1 \cdot \mu_2 > 0$. Moreover, $\mu_1 \cdot \mu_2 < 1$ for each $T > 0$ by virtue of (8). □

Remark 5.1. Claim 2 implies that if μ_1, μ_2 are real for some $T > 0$, then they must have the same sign.

Remark 5.2. Actually, by virtue of (7), for at least one index $i = 1, 2$ (and, hence, for both) μ_i is real when $T = 1$. Moreover, again by (7), at least one (and, hence, both) must be negative.

Remark 5.3. Formula (8) states that the time spent along the solutions of the component defined by A_1 should be large enough with respect to the time spent along the solutions of the component defined by A_2.

We are finally in a position to complete the proof. Let k, $\mathbf{t}_1, \mathbf{t}_2 \in \mathbf{R}_+^k$, x_0 and c be such that (7), (8) hold. Let $T > 0$. As before, let us denote by $\mu_1(T), \mu_2(T)$ the eigenvalues (not necessarily distinct) of $E(T\mathbf{t}_1, T\mathbf{t}_2)$. Since $\mu_1 \cdot \mu_2 < 1$ (Claim 2) for at least one index $i = 1, 2$ we necessarily have $|\mu_i| < 1$ for each $T > 0$. Without loss of generality, we assume that $|\mu_i(T)| < 1$ when $i = 1$.

If for some $T > 0$ the eigenvalues μ_1, μ_2 are complex conjugate or real coincident, the Theorem is proved, since in these cases $|\mu_1| = |\mu_2|$. Thus, it remains to discuss the case where for each $T > 0$, μ_1, μ_2 are real and distinct.

Notice that the matrix $E(T\mathbf{t}_1, T\mathbf{t}_2)$ approaches the identity for $T \to 0^+$, so that $\mu_1(0) = \mu_2(0) = 1$. Notice also that $\mu_1(T), \mu_2(T)$ are continuous functions of T. Assume that for each $T > 0$, we have $\mu_2(T) \geq 1$. Since μ_1, μ_2 have the same sign (Remark 5.1), we actually have $0 < \mu_1(T) < 1 \leq \mu_2(T)$. This is a contradiction, since according to Remark 5.2, μ_1, μ_2 must be negative for $T = 1$. In conclusion, for some $T > 0$, we have $|\mu_i(T)| < 1$ for both $i = 1, 2$. The proof is finished.

Remark 5.4. As a matter of fact, under the assumptions of Theorem 3.2, the eigenvalues of matrix $E(T\mathbf{t}_1, T\mathbf{t}_2)$ cannot be real for all $T > 0$. Indeed, as noticed in the proof of the Theorem, μ_1, μ_2 are surely real and positive for T small. In order to take negative values (while remaining real) when T increases, they should cross zero, which is impossible because of Claim 2.

6. EXAMPLES

Example 6.1. Consider the pair of matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 2/3 & -2/3 \\ 2/3 & 7/3 \end{pmatrix} .$$

The linear system defined by A_1 has a saddle configuration at the origin, while the linear system defined by A_2 has a unstable node configuration at the origin (see Figure 2).

Notice that there are no values of α_1, α_2 for which the matrix $\alpha_1 A_1 + \alpha_2 A_2$ is Hurwitz. However, $\operatorname{tr} A_1 < 0$, and by numerical experiments it is not difficult to select switched solutions with a "rotational" behavior. One such solution can be obtained for instance by iterating the matrix $E(\mathbf{t}_1, \mathbf{t}_2) = e^{t_{21}A_2} \cdot e^{t_{11}A_1}$ with $t_{11} = 2$ and $t_{21} = 1.5$ (the corresponding orbit is shown in Figure 3, on the left), whose eigenvalues are complex

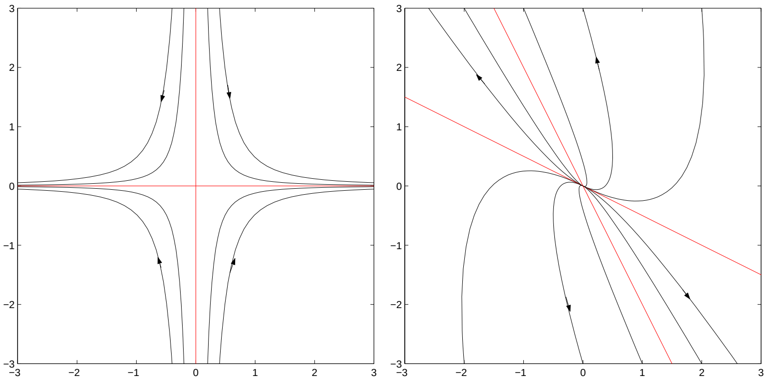

Fig. 2. Trajectories of the single subsystems involved in Example 6.1 defined by A_1 (on the left) and A_2 (on the right).

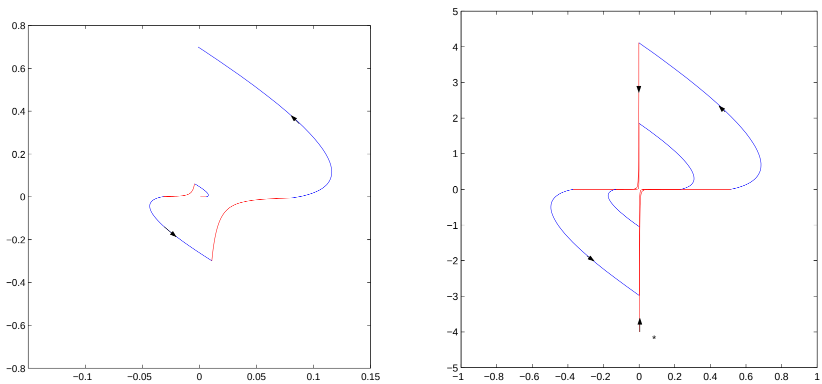

Fig. 3. A switched trajectory of Example 6.1 defined by a periodic switching signal with $k = 1$, $t_{11} = 2$ and $t_{21} = 1.5$ on the left. Switched trajectories generated with $k = 1$, $t_{11} = 4$, $t_{21} = 1$ and $T = 1.387$ on the right (complex eigenvalues).

conjugate: this shows that the radial controllability assumption is met. Unfortunately, the orbit displayed in Figure 3 is divergent: indeed, the modulus of the eigenvalues is approximately 3.49. On the other hand, according to Theorem 4.3, we know that it is possible to construct periodic switching signals for which all the solutions converge to the origin. Since in this case $\alpha_0 = 3$, a natural attempt is to take $k = 1$, $t_{11} = 4$, $t_{21} = 1$, that is

$$E(T\mathbf{t}_1, T\mathbf{t}_2) = e^{TA_2} \cdot e^{4TA_1} \ . \tag{13}$$

The eigenvalues of this matrix can be computed with the aid of symbolic and/or numeric packages. Actually, the interval of the values of T for which both the eigenvalues lie in the open unit disc of the complex plane is very small. More precisely, it is possible to identify four positive numbers $a < c < d < b$ such that (13) is Schur with real eigenvalues if $a < T \leq c$ and $d \leq T < b$, and Schur with complex eigenvalues if $c < T < d$. Approximately, $1.385374 < a < 1.385375$, $1.385558 < c < 1.385559$, $1.387023 < d < 1.387024$, $1.387205 < b < 1.387206$.

Figure 3 (on the right) shows a simulation for $T = 1.387$. An alternative way to achieve these conclusions is the well known Schur criterion.

Example 6.2. Consider the pair of matrices

$$A_1 = \begin{pmatrix} -3 & -2 \\ 2 & 2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} .$$

The linear systems defined by A_1 and A_2 both have saddle configurations at the origin, as shown in Figure 4. Notice that there are no positively invariant sectors, so that the switched process is RC. The orbit of a divergent solution exhibiting rotational behavior is drawn in Figure 5: it is obtained by iterating the matrix $e^{A_2}e^{A_1}$ whose eigenvalues are real (approximately, $\mu_1 = -3.7805$ and $\mu_2 = -0.2645$).

In this case, there exist positive values of α_1, α_2 for which the matrix $\alpha_1 A_1 + \alpha_2 A_2$ is Hurwitz, but this does not happen for instance if we chose $\alpha_1 = 7, \alpha_2 = 1$. We therefore focus on the matrix

$$E(T\mathbf{t}_1, T\mathbf{t}_2) = e^{TA_2} \cdot e^{7TA_1} \ . \tag{14}$$

Note that this choice is compatible with (8), since $\alpha_0 = 1$. We find a situation similar to that of Example 6.1. There are four positive numbers $a < c < d < b$ such that (14) is Schur with positive real eigenvalues if $a < T \leq c$, Schur with negative real eigenvalues if $d \leq T < b$, and Schur with complex eigenvalues if $c < T < d$. This time, approximately, $a = 0.4428$, $c = 0.4536$, $d = 0.4694$, $b = 0.4766$.

Some orbits are plotted, for the different cases, in Figure 6.

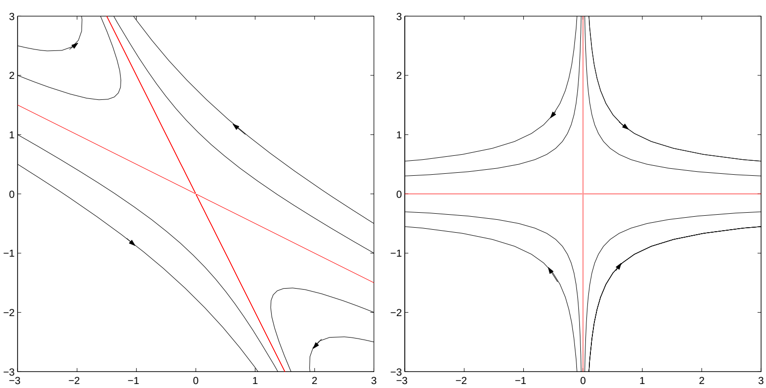

Fig. 4. Trajectories of the single subsystems involved in Example 6.2 defined by A_1 (on the left) and A_2 (on the right).

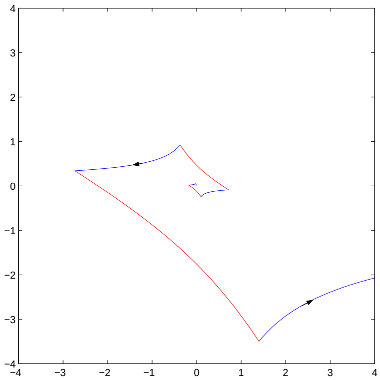

Fig. 5. A trajectory of Example 6.2 with $t_{11} = t_{21} = 1$.

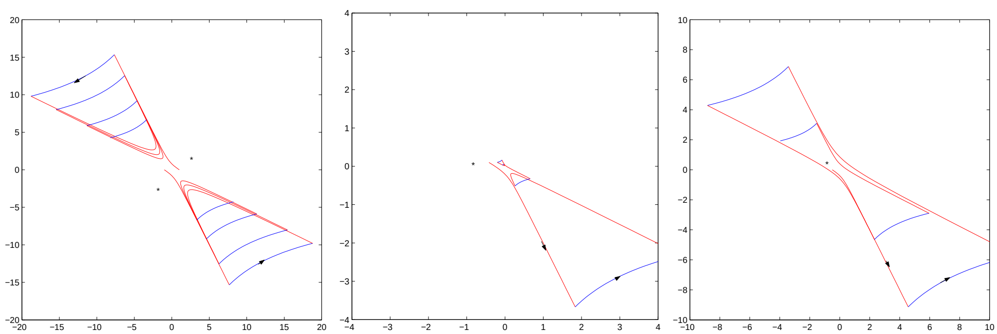

Fig. 6. From left to right, trajectories of the switched system of Example 6.2 with $T = 0.448$ (real positive eigenvalues), with $T = 0.46$ (complex eigenvalues), with $T = 0.473$ (real negative eigenvalues).

Example 6.3. Consider finally the pair of matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1/2 & -1 \\ 1 & 1/2 \end{pmatrix} .$$

The linear system defined by A_1 has again a saddle configuration, while the system defined by A_2 has an unstable focus configuration at the origin. There are no Hurwitz convex combinations of A_1 and A_2. The process is trivially RC. Since $\alpha_0 = 1$, in order to construct converging trajectories we can try

$$E(T\mathbf{t}_1, T\mathbf{t}_2) = e^{TA_2} \cdot e^{2TA_1} . \tag{15}$$

The orbit of a solution exhibiting a rotational behavior is drawn in Figure 7, on the left. It corresponds to the choice $T = 1.6$. Such solution is divergent. Indeed,

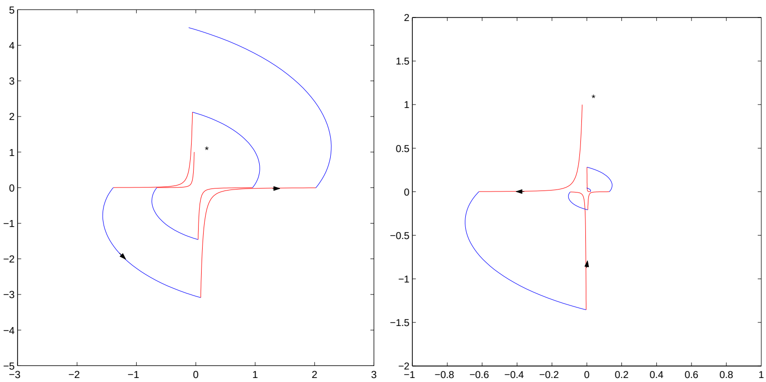

Fig. 7. A divergent trajectory of Example 6.3 for $T = 1.6$ on the left; a convergent trajectory for $T = \pi/2$ on the right.

for $T = 1.6$ the eigenvalues of the matrix $E(T\mathbf{t}_1, T\mathbf{t}_2) = e^{TA_2}e^{2TA_1}$ are -0.1387 and -1.4556, approximately. An eigenvector corresponding to the unstable eigenvalue is $(-0.0267, 0.9996)$.

It can be seen that the matrix (15) is Schur, for instance, when $T = \frac{\pi}{2} + m\pi$, for each integer m (see Figure 7, on the right).

7. CONCLUSION

When a periodic switching signal σ is applied to (2), the matrix $B(t)$ appearing in (3) is clearly bounded. Theorems 3.2 and 4.3 together lead to the following conclusion.

Theorem 7.1. If $n = 2$ and the pair of matrices A_1, A_2 satisfies conditions $(i), (ii)$ of Theorem 4.3, then the switched process $(\mathcal{F}, \mathcal{G})$ is BIBS stabilizable.

Example 7.2. Let us consider a process $(\mathcal{F}, \mathcal{G})$ with $n = 2, m = 1$, where $\mathcal{F}$ is the same pair of matrices studied in Example 6.1, and $\mathcal{G}$ is formed by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let us apply the external input $u(t) = \sin t$ and the periodic switched signal corresponding to (13) with $T = 1.387$ which, as we know, provides asymptotic stability for the unforced associated system. The orbit plotted in Figure 8 does not converge to the origin, but it is bounded, as predicted by Theorem 7.1.

The extension of Theorem 7.1 to switched process with $N > 2$ is straightforward. The problem is open for $n > 2$. As far as the case $n = 2$ is concerned, we conjecture that if a pair of matrices has the UPAC property, then either the conditions of Theorem 4.3 are satisfied, or there exists a Hurwitz convex combination $\alpha_1 A_1 + \alpha_2 A_2$. In other words, the conditions known so far should cover all the possible cases.

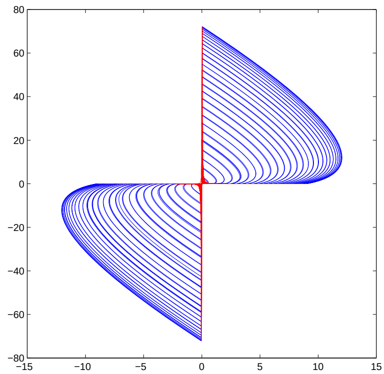

Fig. 8. A trajectory of Example 6.1, with sinusoidal external input.

(Received August 25, 2016)

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