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ON INTERPOLATION ERROR ON DEGENERATING PRISMATIC ELEMENTS

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Dedicated to the 50th anniversary of Jan Brandts

Abstract. We propose an analogue of the maximum angle condition (commonly used in finite element analysis for triangular and tetrahedral meshes) for the case of prismatic elements. Under this condition, prisms in the meshes may degenerate in certain ways, violating the so-called inscribed ball condition presented by P. G. Ciarlet (1978), but the interpolation error remains of the order $O(h)$ in the H^1-norm for sufficiently smooth functions.

Keywords: prismatic finite element; interpolation error; semiregular family of prismatic partitions

MSC 2010: 65N50, 65N30, 65N12, 65N15

1. Introduction

The regularity of families of computational meshes (i.e. limitations on the shape parameters of mesh elements) is a very important issue for performing controllable numerical simulations. It influences the interpolation properties of finite elements and through Cea's lemma also the convergence of the finite element method [9]; various regularity mesh properties are required in derivation of a posteriori error estimates for various finite element-type approximations, in the discrete maximum principles (see e.g. [13]), for acceleration of convergence of finite element approximations, also in computer graphics (see [18] and references therein), etc.

For simplicial meshes, the most famous regularity condition (commonly used in various convergence proofs, see e.g. [30], [29], [9], [8]) is the minimum angle condition. (It is also known as the Zlámal minimum angle condition for the case of triangulations.) The definition of this condition in $2d$ is as follows. Consider a family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of face-to-face triangulations $\mathcal{T}_h$ of a bounded polygonal domain.

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We say that the *minimum angle condition* is satisfied if there exists a constant $\alpha_0 > 0$ such that for any triangulation $\mathcal{T}_h \in \mathcal{F}$ and any triangle $T \in \mathcal{T}_h$ one has (see [30])

$$\alpha_T \geqslant \alpha_0, \tag{1}$$

where α_T is the minimum angle of T. Many algorithms for constructing families of triangulations satisfying (1) can be found e.g. in the review paper [18].

The weaker limitation on angles of triangles, called the *maximum angle condition* (see [28]), reads as follows: there exists a constant $\gamma_0 < \pi$ such that for any triangulation $\mathcal{T}_h \in \mathcal{F}$ and any triangle $T \in \mathcal{T}_h$ one has

$$\gamma_T \leqslant \gamma_0, \tag{2}$$

where γ_T is the maximum angle of T. Obviously, condition (1) implies condition (2), but not vice versa. In the mid of 70th, Babuška and Aziz [4], Barnhill and Gregory [5], and Jamet [14] independently derived the optimal interpolation order in the energy norm of finite element approximations under condition (2). Later the maximum angle condition was investigated in various norms in [1], [2], [16], [15], [17], [19], [25], [26], [27]. In 1992, condition (2) was generalized by Křížek [20] to the case of tetrahedral elements as follows: there exists a constant $\gamma_0 < \pi$ such that for any face-to-face tetrahedralization $\mathcal{T}_h \in \mathcal{F}$ and any tetrahedron $T \in \mathcal{T}_h$ one has

$$\gamma_T^{\mathrm{D}} \leqslant \gamma_0 \quad \text{and} \quad \gamma_T^{\mathrm{F}} \leqslant \gamma_0, \tag{3}$$

where γ_T^{D} is the maximum dihedral angle between the faces of T and γ_T^{F} is the maximum angle in all four triangular faces of T. According to [21], the associated finite element approximations preserve the optimal interpolation order in the H^1-norm under condition (3), which allows to use meshes with tetrahedra having some types of degeneracy [10]. A new generalization of (2) and (3) in the case of simplices of any dimension has been recently proposed in [12].

However, the case of prismatic meshes in the above context has been studied quite weakly. Besides the standard case when prismatic elements (among all other available finite elements) satisfy the so-called inscribed ball condition from [9] (see Definition 1 below), which is equivalent to the Zlámal-type angle conditions in the case of simplices [6], [7], [8], we are not aware of any results on interpolation and convergence properties on degenerating prismatic meshes. To fill this gap, we propose here an analogue of the maximum angle conditions (2) and (3) for the case of prismatic elements, which is nothing else but the only requirement for all the triangular bases of all prisms in all partitions to satisfy the maximum angle condition (2). Under this condition, prisms in the meshes may degenerate in certain ways, violating e.g. the inscribed ball condition, but the interpolation error remains of order $O(h)$ in the H^1-norm for sufficiently smooth functions. It is worth to noticing that, due to the special shape of prisms, the maximum angle condition (2) on angles of their triangular bases immediately leads to upper estimation of all dihedral angles of prisms (those between adjacent faces) and upper estimation of all (interior) angles in the faces of prisms, i.e. to the estimation in the spirit of (3).

Note that the degenerated finite (e.g. tetrahedral and prismatic) elements can be of use in many real-life applications, for example, in calculation of physical fields in electrical rotary machines, see [21], [22]. Flat tetrahedral and prismatic elements can be also used to approximate thin slots, layers, or gaps. Moreover, they are highly desired when the true solution of some problem changes more rapidly in one direction than in another direction (e.g. in anisotropic materials) [1].

2. Main definitions and denotations

Let $\Omega \subset \mathbb{R}^3$ be a bounded polyhedral domain with Lipschitz boundary $\partial\Omega$, which can be partitioned (face-to-face) into triangular prisms (for instance, Ω can be a union of several cylindrical domains). Let $\mathcal{T}_h$ denote a face-to-face partition of $\overline{\Omega}$ into (closed) triangular prisms P. This means that the union of all $P \in \mathcal{T}_h$ is $\overline{\Omega}$, the interiors of all $P \in \mathcal{T}_h$ are mutually disjoint, and any face of any $P \in \mathcal{T}_h$ is either a face of another prism from $\mathcal{T}_h$, or a subset of the boundary $\partial\Omega$. As usual, we set $h_P = \operatorname{diam} P$ and the discretization parameter h will be the maximum of h_P over all $P \in \mathcal{T}_h$.

We assume that each triangular prism considered in this work is of the form $P = K \times I$, where K is a triangular face (or base) of P and I is an interval of the length z_I. The angles of K (also called angles of P later on) are denoted by α_K, β_K and γ_K, where

$$0 < \alpha_K \leqslant \beta_K \leqslant \gamma_K. \tag{4}$$

A set of prismatic face-to-face partitions $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of $\overline{\Omega}$ is called a family of prismatic partitions if for every $\varepsilon > 0$ there exists $\mathcal{T}_h \in \mathcal{F}$ with $h < \varepsilon$.

Definition 1. A family of prismatic partitions $\mathcal{F}$ is said to be regular if there exists a constant $m > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and for any $P \in \mathcal{T}_h$ there exists a ball b_P of radius r_P such that $b_P \subset P$ and

$$mh_P \leqslant r_P. \tag{5}$$

R e m a r k 1. Condition (5) is often called the inscribed ball condition in the finite element community, see [9]. This actually means that the regular families of prismatic partitions do not contain "shrinking" or "short" or "long" prisms.

Definition 2. A family of prismatic partitions $\mathcal{F}$ is said to be semiregular if there exists a constant $c > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $P \in \mathcal{T}_h$ all angles of P are bounded from above by $\pi - c$, i.e.

$$0 < \alpha_K \leqslant \beta_K \leqslant \gamma_K \leqslant \pi - c. \tag{6}$$

R e m a r k 2. Semiregular families of prismatic partitions may contain prisms degenerating in many ways. In particular, there is no restriction on heights of the prisms involved in (6). Also, prisms in semiregular families may have arbitrarily small (but not arbitrarily large) angles in their triangular faces.

In what follows, we use the standard denotation $W_p^k(\Omega)$, $k = 0, 1, \ldots$, $p \geqslant 1$, for Sobolev spaces with norms $\|\cdot\|_{k,p} = \|\cdot\|_{k,p,\Omega}$ and seminorms $|\cdot|_{k,p} = |\cdot|_{k,p,\Omega}$. The symbol $C(\overline{\Omega})$ stands for the space of continuous functions over $\overline{\Omega}$.

The following denotation for certain sums of partial derivatives for the functions of three variables $u = u(x, y, z)$ is often used later on:

$$[u]_{m,n}^2 = \sum_{\substack{i+j=m \\ l=1,\ldots,n}} \left| \frac{\partial^{m+n} u}{\partial x^i \partial y^j \partial z^l} \right|^2. \tag{7}$$

With any prismatic mesh $\mathcal{T}_h$ we associate the finite element space

$$V_h = \{v \in C(\overline{\Omega});\ v|_P \in Q(P)\ \forall\, P \in \mathcal{T}_h\}, \tag{8}$$

where $Q(P) = \mathbf{P}_1(K) \times \mathbf{P}_1(I)$, and $\mathbf{P}_1(K)$ and $\mathbf{P}_1(I)$ are the spaces of linear functions defined in the triangle K and in the interval I, respectively.

The interpolation operator $\pi_h\colon\ C(\overline{\Omega}) \to V_h$ is uniquely determined by the requirement

$$\pi_h v(x) = v(x) \quad \text{for all vertices } x \text{ of all } P \in \mathcal{T}_h. \tag{9}$$

To prove the main result of the paper we will employ the technique using transfer of the prism $P \in \mathcal{T}_h$ onto the reference prism $\widehat{P} = \widehat{K} \times \hat{I}$, where $\widehat{K}$ is the triangular base and $\hat{I}$ is the altitude of $\widehat{P}$.

Let $\widehat{P}$ have the vertices $\hat{A}_0, \ldots, \hat{A}_5$ as indicated in Figure 1. The associated basis functions $\widehat{\varphi}_0, \ldots, \widehat{\varphi}_5$ are

$$\begin{gathered}\widehat{\varphi}_0(\hat{x}, \hat{y}, \hat{z}) = (1 - \hat{x} - \hat{y})(1 - \hat{z}),\ \widehat{\varphi}_1(\hat{x}, \hat{y}, \hat{z}) = \hat{x}(1 - \hat{z}),\ \widehat{\varphi}_2(\hat{x}, \hat{y}, \hat{z}) = \hat{y}(1 - \hat{z}),\\ \widehat{\varphi}_3(\hat{x}, \hat{y}, \hat{z}) = (1 - \hat{x} - \hat{y})\hat{z},\ \widehat{\varphi}_4(\hat{x}, \hat{y}, \hat{z}) = \hat{x}\hat{z},\ \widehat{\varphi}_5(\hat{x}, \hat{y}, \hat{z}) = \hat{y}\hat{z}.\end{gathered} \tag{10}$$

The prismatic interpolant $\hat{\pi}_{\widehat{P}}$ of the function $\hat{u}$ defined on $\widehat{P}$ is constructed then as

$$\hat{\pi}_{\widehat{P}}\hat{u} = \sum_{i=0}^{5} \hat{u}(\hat{A}_i)\widehat{\varphi}_i. \tag{11}$$

By definition, $\hat{\pi}_{\widehat{P}}\hat{u}(\hat{A}_i) = \hat{u}(\hat{A}_i)$, $i = 0, \ldots, 5$, for any $\hat{u} \in C(\widehat{P})$.

Assume that a given prismatic element $P = K \times I$ has the vertices $A_i = (A_{i,x}, A_{i,y}, A_{i,z})^\top$, $i = 0, \ldots, 5$, as marked in Figure 1 (right), where the largest angle γ_K of K is at the vertex A_0.

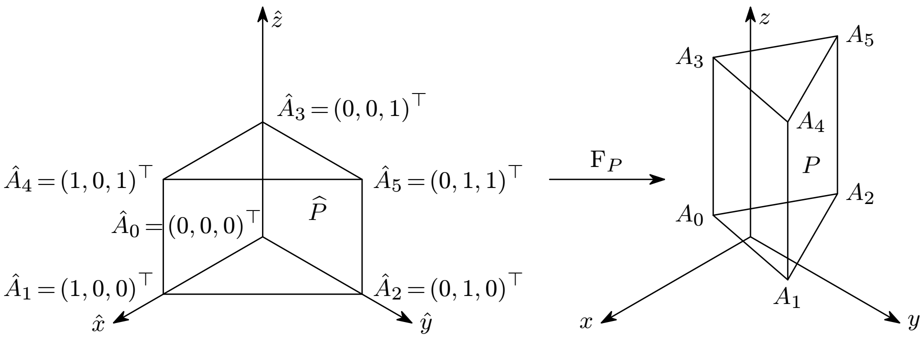

Figure 1. The reference prism $\widehat{P}$ (left) and an arbitrary prismatic element P (right).

We define an affine one-to-one mapping $\mathrm{F}_P\colon\ \widehat{P} \to P$ as

$$\mathrm{F}_P(\widehat{X}) = \mathrm{B}_P\widehat{X} + A_0 \quad \text{with} \quad \widehat{X} = (\hat{x}, \hat{y}, \hat{z})^\top \in \widehat{P}, \tag{12}$$

where B_P is a (3×3) matrix with entries denoted by B_{ij}. The matrix

$$\mathrm{B}_P = (A_1 - A_0 \mid A_2 - A_0 \mid A_3 - A_0)$$

is of a block structure:

$$\mathrm{B}_P = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{B_K}} & 0 \\ & & 0 \\ \hline 0 & 0 & A_{3,z} - A_{0,z} \end{array}\right], \tag{13}$$

where

$$\mathrm{B}_K = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{1,x} - A_{0,x} & A_{2,x} - A_{0,x} \\ A_{1,y} - A_{0,y} & A_{2,y} - A_{0,y} \end{bmatrix}. \tag{14}$$

The matrix B_P is nonsingular, since the vertices A_i's are not coplanar.

We observe that

$$|B_{11}| \leqslant f_K, \quad |B_{21}| \leqslant f_K, \quad |B_{12}| \leqslant g_K, \quad |B_{22}| \leqslant g_K, \quad |B_{33}| = z_I, \tag{15}$$

where f_K and g_K are the lengths of the edges A_0A_1 and A_0A_2, respectively.

Let C_{ij} denote the entries of the inverse matrix B_P^{-1}. Obviously, B_P^{-1} has also a block structure, namely,

$$\mathrm{B}_P^{-1} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{B_K^{-1}}} & 0 \\ & & 0 \\ \hline 0 & 0 & (A_{3,z} - A_{0,z})^{-1} \end{array}\right]. \tag{16}$$

In what follows, the functions $\hat{u}$ and u are always related by

$$\hat{u}(\widehat{X}) = u(X), \tag{17}$$

where

$$X = (x, y, z)^\top = \mathrm{F}_P(\widehat{X}). \tag{18}$$

Further, we observe for $u \in W_2^1(P)$ that

$$\Big(\frac{\partial \hat{u}}{\partial \hat{x}}, \frac{\partial \hat{u}}{\partial \hat{y}}, \frac{\partial \hat{u}}{\partial \hat{z}}\Big)^\top = \mathrm{B}_P^\top \Big(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\Big)^\top, \tag{19}$$

and for $u \in W_2^2(P)$ that

$$\begin{bmatrix} \dfrac{\partial^2 \hat{u}}{\partial \hat{x}^2} & \dfrac{\partial^2 \hat{u}}{\partial \hat{x}\partial \hat{y}} & \dfrac{\partial^2 \hat{u}}{\partial \hat{x}\partial \hat{z}} \\ \dfrac{\partial^2 \hat{u}}{\partial \hat{y}\partial \hat{x}} & \dfrac{\partial^2 \hat{u}}{\partial \hat{y}^2} & \dfrac{\partial^2 \hat{u}}{\partial \hat{y}\partial \hat{z}} \\ \dfrac{\partial^2 \hat{u}}{\partial \hat{z}\partial \hat{x}} & \dfrac{\partial^2 \hat{u}}{\partial \hat{z}\partial \hat{y}} & \dfrac{\partial^2 \hat{u}}{\partial \hat{z}^2} \end{bmatrix} = \mathrm{B}_P^\top \begin{bmatrix} \dfrac{\partial^2 u}{\partial x^2} & \dfrac{\partial^2 u}{\partial x\partial y} & \dfrac{\partial^2 u}{\partial x\partial z} \\ \dfrac{\partial^2 u}{\partial y\partial x} & \dfrac{\partial^2 u}{\partial y^2} & \dfrac{\partial^2 u}{\partial y\partial z} \\ \dfrac{\partial^2 u}{\partial z\partial x} & \dfrac{\partial^2 u}{\partial z\partial y} & \dfrac{\partial^2 u}{\partial z^2} \end{bmatrix} \mathrm{B}_P. \tag{20}$$

From (20) and (15) we get the estimate

$$\begin{aligned} \Big|\frac{\partial^2 \hat{u}}{\partial \hat{x}^2}\Big|^2 &= \Big|B_{11}^2 \frac{\partial^2 u}{\partial x^2} + 2B_{11}B_{21}\frac{\partial^2 u}{\partial x\partial y} + B_{21}^2 \frac{\partial^2 u}{\partial y^2}\Big|^2 \\ &\leqslant 12 f_K^4 \Big(\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2 + \Big|\frac{\partial^2 u}{\partial x\partial y}\Big|^2 + \Big|\frac{\partial^2 u}{\partial y^2}\Big|^2\Big), \end{aligned} \tag{21}$$

where we used the so-called sum of squares inequality (a special case of the Jensen inequality)

$$\Big(\sum_{i=1}^{s} a_i\Big)^2 \leqslant s \sum_{i=1}^{s} a_i^2 \tag{22}$$

with $s = 3$. Similarly, we can show that

$$
\begin{aligned}
&\Big|\frac{\partial^2 \hat{u}}{\partial \hat{x} \partial \hat{y}}\Big|^2 \leqslant 12 f_K^2 g_K^2 \Big(\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2 + \Big|\frac{\partial^2 u}{\partial x \partial y}\Big|^2 + \Big|\frac{\partial^2 u}{\partial y^2}\Big|^2\Big), \\
&\Big|\frac{\partial^2 \hat{u}}{\partial \hat{x} \partial \hat{z}}\Big|^2 \leqslant 2 f_K^2 z_I^2 \Big(\Big|\frac{\partial^2 u}{\partial x \partial z}\Big|^2 + \Big|\frac{\partial^2 u}{\partial y \partial z}\Big|^2\Big).
\end{aligned} \tag{23}
$$

The third order derivatives $\partial^3 \hat{u}/\partial \hat{x}^2 \partial \hat{z}$, $\partial^3 \hat{u}/\partial \hat{x} \partial \hat{y} \partial \hat{z}$ and $\partial^3 \hat{u}/\partial \hat{y}^2 \partial \hat{z}$ will be required in Section 3. For instance, $\partial^3 \hat{u}/\partial \hat{x} \partial \hat{y} \partial \hat{z}$ is calculated as follows. From (20) we have

$$
\frac{\partial^2 \hat{u}}{\partial \hat{y} \partial \hat{z}} = B_{12} B_{33} \frac{\partial^2 u}{\partial x \partial z} + B_{22} B_{33} \frac{\partial^2 u}{\partial y \partial z}. \tag{24}
$$

Taking the partial derivative with respect to $\hat{x}$ at both sides of (24), we get

$$
\begin{aligned}
\frac{\partial^3 \hat{u}}{\partial \hat{x} \partial \hat{y} \partial \hat{z}} = {}& \frac{\partial}{\partial x}\Big(B_{12} B_{33} \frac{\partial^2 u}{\partial x \partial z}\Big) \frac{\partial x}{\partial \hat{x}} + \frac{\partial}{\partial y}\Big(B_{12} B_{33} \frac{\partial^2 u}{\partial x \partial z}\Big) \frac{\partial y}{\partial \hat{x}} \\
&+ \frac{\partial}{\partial z}\Big(B_{12} B_{33} \frac{\partial^2 u}{\partial x \partial z}\Big) \frac{\partial z}{\partial \hat{x}} + \frac{\partial}{\partial x}\Big(B_{22} B_{33} \frac{\partial^2 u}{\partial y \partial z}\Big) \frac{\partial x}{\partial \hat{x}} \\
&+ \frac{\partial}{\partial y}\Big(B_{22} B_{33} \frac{\partial^2 u}{\partial y \partial z}\Big) \frac{\partial y}{\partial \hat{x}} + \frac{\partial}{\partial z}\Big(B_{22} B_{33} \frac{\partial^2 u}{\partial y \partial z}\Big) \frac{\partial z}{\partial \hat{x}}.
\end{aligned}
$$

From estimates (15), (18), and the fact that $\partial z/\partial \hat{x} = 0$, we come to

$$
\Big|\frac{\partial^3 \hat{u}}{\partial \hat{x} \partial \hat{y} \partial \hat{z}}\Big| \leqslant g_K f_K z_I \Big(\Big|\frac{\partial^3 u}{\partial x^2 \partial z}\Big| + 2\Big|\frac{\partial^3 u}{\partial x \partial y \partial z}\Big| + \Big|\frac{\partial^3 u}{\partial y^2 \partial z}\Big|\Big).
$$

Using (22) with $s = 3$, we get the estimate

$$
\Big|\frac{\partial^3 \hat{u}}{\partial \hat{x} \partial \hat{y} \partial \hat{z}}\Big|^2 \leqslant 12 g_K^2 f_K^2 z_I^2 \Big(\Big|\frac{\partial^3 u}{\partial x^2 \partial z}\Big|^2 + \Big|\frac{\partial^3 u}{\partial x \partial y \partial z}\Big|^2 + \Big|\frac{\partial^3 u}{\partial y^2 \partial z}\Big|^2\Big). \tag{25}
$$

Analogously, we can show that

$$
\begin{aligned}
&\Big|\frac{\partial^3 \hat{u}}{\partial \hat{x}^2 \partial \hat{z}}\Big|^2 \leqslant 12 z_I^2 f_K^4 \Big(\Big|\frac{\partial^3 u}{\partial x^2 \partial z}\Big|^2 + \Big|\frac{\partial^3 u}{\partial x \partial y \partial z}\Big|^2 + \Big|\frac{\partial^3 u}{\partial y^2 \partial z}\Big|^2\Big), \\
&\Big|\frac{\partial^3 \hat{u}}{\partial \hat{y}^2 \partial \hat{z}}\Big|^2 \leqslant 12 z_I^2 g_K^4 \Big(\Big|\frac{\partial^3 u}{\partial x^2 \partial z}\Big|^2 + \Big|\frac{\partial^3 u}{\partial x \partial y \partial z}\Big|^2 + \Big|\frac{\partial^3 u}{\partial y^2 \partial z}\Big|^2\Big).
\end{aligned} \tag{26}
$$

Theorem 1. *Any regular family of prismatic partitions of a polyhedron is semiregular.*

P r o o f. Assume that condition (6) is not satisfied. Then there exists an infinite sequence of prisms $\{P_i\}_{i=1}^{\infty}$ in partitions from $\mathcal{F}$ such that their corresponding maximal angles tend to π as $i \to \infty$. Now, consider the triangular base K_i of each P_i with three edges $a_{K_i} \leqslant b_{K_i} \leqslant c_{K_i}$ so that the maximum angle γ_{K_i} is the angle between edges a_{K_i} and b_{K_i}. We denote the perimeter and the area of the triangle K_i by p_Δ and S_Δ, respectively. It is clear that r_{P_i}, p_Δ, and S_Δ satisfy the relation $r_{P_i} \leqslant 2S_\Delta/p_\Delta$, where r_{P_i} is the radius of the (inscribed) ball $b_{P_i} \subset P_i$. Now, using (5) we observe that

$$(27)\quad 0 < m \leqslant \frac{r_{P_i}}{h_{P_i}} \leqslant \frac{2S_\Delta}{h_{P_i} p_\Delta} \leqslant \frac{2S_\Delta}{c_{K_i} p_\Delta} \leqslant \frac{a_{K_i} b_{K_i} \sin\gamma_{K_i}}{c_{K_i} 3 a_{K_i}} < \sin\gamma_{K_i} \to 0 \quad \text{as } i \to \infty,$$

which contradicts the fact that m is a fixed positive constant. $\square$

R e m a r k 3. The statement converse to Theorem 1 is not true. For example, prisms with arbitrarily small or arbitrarily large heights compared to the size of the triangles may belong to semiregular families, but not to regular ones.

We are interested in estimation of the following interpolation error:

$$(28)\quad |u - \pi_P u|_{1,2,P}^2 = \int_P \Big(\Big| \frac{\partial}{\partial x}(u - \pi_P u) \Big|^2 + \Big| \frac{\partial}{\partial y}(u - \pi_P u) \Big|^2 + \Big| \frac{\partial}{\partial z}(u - \pi_P u) \Big|^2 \Big) \, \mathrm{d}X,$$

where $\pi_P u$ denotes the prismatic interpolant for u on P.

For this purpose we first prove two lemmas.

Lemma 1. *Let $\hat{u} \in \mathrm{W}_2^3(\widehat{P})$. Then the following estimates hold:*

$$(29)\quad \int_{\widehat{P}} \Big| \frac{\partial}{\partial \hat{x}}(\hat{u} - \hat{\pi}_{\widehat{P}} \hat{u}) \Big|^2 \, \mathrm{d}\widehat{X} \leqslant \widehat{C} |\det \mathrm{B}_P^{-1}| f_K^2 \int_P ((f_K^2 + g_K^2)[u]_{2,0}^2 + z_I^2([u]_{1,1}^2 + g_K^2 [u]_{2,1}^2)) \, \mathrm{d}X$$

and

$$(30)\quad \int_{\widehat{P}} \Big| \frac{\partial}{\partial \hat{y}}(\hat{u} - \hat{\pi}_{\widehat{P}} \hat{u}) \Big|^2 \, \mathrm{d}\widehat{X} \leqslant \widehat{C} |\det \mathrm{B}_P^{-1}| g_K^2 \int_P ((f_K^2 + g_K^2)[u]_{2,0}^2 + z_I^2([u]_{1,1}^2 + f_K^2 [u]_{2,1}^2)) \, \mathrm{d}X,$$

where $\widehat{C} > 0$ is a constant and $\hat{u}$ and u are related via (17).

Proof. Using the definition of the basis functions (10), we have

$$
\begin{aligned}
(31)\quad & \int_{\widehat{P}} \Big|\frac{\partial}{\partial \hat{x}}(\hat{u}-\hat{\pi}_{\widehat{P}}\hat{u})\Big|^2 \,\mathrm{d}\widehat{X} \\
&= \int_{\widehat{P}} \Big|\frac{\partial}{\partial \hat{x}}\hat{u}(\hat{x},\hat{y},\hat{z}) - ((1-\hat{z})(\hat{u}(1,0,0)-\hat{u}(0,0,0)) \\
&\qquad\qquad + \hat{z}(\hat{u}(1,0,1)-\hat{u}(0,0,1)))\Big|^2 \,\mathrm{d}\widehat{X} \\
&= \int_{\widehat{P}} \Big|\frac{\partial}{\partial \hat{x}}\hat{u}(\hat{x},\hat{y},\hat{z}) - \Big((1-\hat{z})\int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,0,0)\,\mathrm{d}\theta + \hat{z}\int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,0,1)\,\mathrm{d}\theta\Big)\Big|^2 \,\mathrm{d}\widehat{X} \\
&= \int_{\widehat{P}} \Big|(1-\hat{z})\Big(\frac{\partial}{\partial \hat{x}}\hat{u}(\hat{x},\hat{y},\hat{z}) - \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,0,0)\,\mathrm{d}\theta\Big) \\
&\qquad\qquad + \hat{z}\Big(\frac{\partial}{\partial \hat{x}}\hat{u}(\hat{x},\hat{y},\hat{z}) - \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,0,1)\,\mathrm{d}\theta\Big)\Big|^2 \,\mathrm{d}\widehat{X}.
\end{aligned}
$$

Further, we rewrite the integrand on the right-hand side of (31) in a more suitable form. First, we have

$$
\begin{aligned}
(32)\quad & \frac{\partial}{\partial \hat{x}}\hat{u}(\hat{x},\hat{y},\hat{z}) - \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,0,0)\,\mathrm{d}\theta \\
&= \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\hat{x},\hat{y},\hat{z})\,\mathrm{d}\theta - \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,0,0)\,\mathrm{d}\theta \\
&= \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\hat{x},\hat{y},\hat{z})\,\mathrm{d}\theta - \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,\hat{y},\hat{z})\,\mathrm{d}\theta \\
&\quad + \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,\hat{y},\hat{z})\,\mathrm{d}\theta - \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,0,0)\,\mathrm{d}\theta \\
&= \int_0^1\int_\theta^{\hat{x}} \frac{\partial^2}{\partial \hat{x}^2}\hat{u}(t,\hat{y},\hat{z})\,\mathrm{d}t\,\mathrm{d}\theta + \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,\hat{y},\hat{z})\,\mathrm{d}\theta - \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,0,0)\,\mathrm{d}\theta.
\end{aligned}
$$

The last two terms on the right-hand side of (32) can be further rewritten as

$$
\begin{aligned}
(33)\quad & \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,\hat{y},\hat{z})\,\mathrm{d}\theta - \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,0,0)\,\mathrm{d}\theta \\
&= \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,\hat{y},\hat{z})\,\mathrm{d}\theta - \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,0,\hat{z})\,\mathrm{d}\theta \\
&\quad + \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,0,\hat{z})\,\mathrm{d}\theta - \int_0^1 \frac{\partial}{\partial \hat{x}}\hat{u}(\theta,0,0)\,\mathrm{d}\theta \\
&= \int_0^1\int_0^{\hat{y}} \frac{\partial^2}{\partial \hat{x}\partial \hat{y}}\hat{u}(\theta,\sigma,\hat{z})\,\mathrm{d}\sigma\,\mathrm{d}\theta + \int_0^1\int_0^{\hat{z}} \frac{\partial^2}{\partial \hat{x}\partial \hat{z}}\hat{u}(\theta,0,\xi)\,\mathrm{d}\xi\,\mathrm{d}\theta \\
&\quad - \int_0^1\int_0^{\hat{z}} \frac{\partial^2}{\partial \hat{x}\partial \hat{z}}\hat{u}(\theta,\hat{y},\xi)\,\mathrm{d}\xi\,\mathrm{d}\theta + \int_0^1\int_0^{\hat{z}} \frac{\partial^2}{\partial \hat{x}\partial \hat{z}}\hat{u}(\theta,\hat{y},\xi)\,\mathrm{d}\xi\,\mathrm{d}\theta
\end{aligned}
$$

$$= \int_0^1 \int_0^{\hat{y}} \frac{\partial^2}{\partial\hat{x}\partial\hat{y}} \hat{u}(\theta,\sigma,\hat{z}) \,\mathrm{d}\sigma \,\mathrm{d}\theta + \int_0^1 \int_0^{\hat{z}} \frac{\partial^2}{\partial\hat{x}\partial\hat{z}} \hat{u}(\theta,\hat{y},\xi) \,\mathrm{d}\xi \,\mathrm{d}\theta - \int_0^1 \int_0^{\hat{z}} \int_0^{\hat{y}} \frac{\partial^3}{\partial\hat{x}\partial\hat{z}\partial\hat{y}} \hat{u}(\theta,\sigma,\xi) \,\mathrm{d}\sigma \,\mathrm{d}\xi \,\mathrm{d}\theta.$$

Now, substituting (33) into (32), we see that

$$\begin{aligned} &\frac{\partial}{\partial\hat{x}} \hat{u}(\hat{x},\hat{y},\hat{z}) - \int_0^1 \frac{\partial}{\partial\hat{x}} \hat{u}(\theta,0,0) \,\mathrm{d}\theta \\ &= \int_0^1 \int_\theta^{\hat{x}} \frac{\partial^2}{\partial\hat{x}^2} \hat{u}(t,\hat{y},\hat{z}) \,\mathrm{d}t \,\mathrm{d}\theta + \int_0^1 \int_0^{\hat{y}} \frac{\partial^2}{\partial\hat{x}\partial\hat{y}} \hat{u}(\theta,\sigma,\hat{z}) \,\mathrm{d}\sigma \,\mathrm{d}\theta \\ &\quad + \int_0^1 \int_0^{\hat{z}} \frac{\partial^2}{\partial\hat{x}\partial\hat{z}} \hat{u}(\theta,\hat{y},\xi) \,\mathrm{d}\xi \,\mathrm{d}\theta - \int_0^1 \int_0^{\hat{z}} \int_0^{\hat{y}} \frac{\partial^3}{\partial\hat{x}\partial\hat{z}\partial\hat{y}} \hat{u}(\theta,\sigma,\xi) \,\mathrm{d}\sigma \,\mathrm{d}\xi \,\mathrm{d}\theta. \end{aligned} \tag{34}$$

Analogously, we can show that

$$\begin{aligned} &\frac{\partial}{\partial\hat{x}} \hat{u}(\hat{x},\hat{y},\hat{z}) - \int_0^1 \frac{\partial}{\partial\hat{x}} \hat{u}(\theta,0,1) \,\mathrm{d}\theta \\ &= \int_0^1 \int_\theta^{\hat{x}} \frac{\partial^2}{\partial\hat{x}^2} \hat{u}(t,\hat{y},\hat{z}) \,\mathrm{d}t \,\mathrm{d}\theta + \int_0^1 \int_0^{\hat{y}} \frac{\partial^2}{\partial\hat{x}\partial\hat{y}} \hat{u}(\theta,\sigma,\hat{z}) \,\mathrm{d}\sigma \,\mathrm{d}\theta \\ &\quad - \int_0^1 \int_{\hat{z}}^1 \frac{\partial^2}{\partial\hat{x}\partial\hat{z}} \hat{u}(\theta,\hat{y},\xi) \,\mathrm{d}\xi \,\mathrm{d}\theta + \int_0^1 \int_{\hat{z}}^1 \int_0^{\hat{y}} \frac{\partial^3}{\partial\hat{x}\partial\hat{z}\partial\hat{y}} \hat{u}(\theta,\sigma,\xi) \,\mathrm{d}\sigma \,\mathrm{d}\xi \,\mathrm{d}\theta. \end{aligned} \tag{35}$$

Applying (34) and (35) for (31), and using inequality (22) with $s = 6$, after straighforward calculation, we get the following relation:

$$\begin{aligned} &\int_{\widehat{P}} \Big| (1-\hat{z}) \Big(\frac{\partial}{\partial\hat{x}} \hat{u}(\hat{x},\hat{y},\hat{z}) - \int_0^1 \frac{\partial}{\partial\hat{x}} \hat{u}(\theta,0,0) \,\mathrm{d}\theta \Big) \\ &\quad + \hat{z} \Big(\frac{\partial}{\partial\hat{x}} \hat{u}(\hat{x},\hat{y},\hat{z}) - \int_0^1 \frac{\partial}{\partial\hat{x}} \hat{u}(\theta,0,1) \,\mathrm{d}\theta \Big) \Big|^2 \,\mathrm{d}\widehat{X} \\ &= \int_{\widehat{P}} \Big| \int_0^1 \int_\theta^{\hat{x}} \frac{\partial^2}{\partial\hat{x}^2} \hat{u}(t,\hat{y},\hat{z}) \,\mathrm{d}t \,\mathrm{d}\theta + \int_0^1 \int_0^{\hat{y}} \frac{\partial^2}{\partial\hat{x}\partial\hat{y}} \hat{u}(\theta,\sigma,\hat{z}) \,\mathrm{d}\sigma \,\mathrm{d}\theta \\ &\quad + \int_0^1 \int_0^{\hat{z}} \frac{\partial^2}{\partial\hat{x}\partial\hat{z}} \hat{u}(\theta,\hat{y},\xi) \,\mathrm{d}\xi \,\mathrm{d}\theta - \hat{z} \int_0^1 \int_0^1 \frac{\partial^2}{\partial\hat{x}\partial\hat{z}} \hat{u}(\theta,\hat{y},\xi) \,\mathrm{d}\xi \,\mathrm{d}\theta \\ &\quad - \int_0^1 \int_0^{\hat{z}} \int_0^{\hat{y}} \frac{\partial^3}{\partial\hat{x}\partial\hat{z}\partial\hat{y}} \hat{u}(\theta,\sigma,\xi) \,\mathrm{d}\sigma \,\mathrm{d}\xi \,\mathrm{d}\theta \\ &\quad + \hat{z} \int_0^1 \int_0^1 \int_0^{\hat{y}} \frac{\partial^3}{\partial\hat{x}\partial\hat{z}\partial\hat{y}} \hat{u}(\theta,\sigma,\xi) \,\mathrm{d}\sigma \,\mathrm{d}\xi \,\mathrm{d}\theta \Big|^2 \,\mathrm{d}\widehat{X} \end{aligned} \tag{36}$$

$$\leqslant 6\int_{\widehat{P}}\int_0^1\int_0^{\hat{x}}\Big|\frac{\partial^2}{\partial\hat{x}^2}\hat{u}(t,\hat{y},\hat{z})\Big|^2\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\widehat{X}+6\int_{\widehat{P}}\int_0^1\int_0^{\hat{y}}\Big|\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}\hat{u}(\theta,\sigma,\hat{z})\Big|^2\,\mathrm{d}\sigma\,\mathrm{d}\theta\,\mathrm{d}\widehat{X}$$
$$+6\int_{\widehat{P}}\int_0^1\int_0^{\hat{z}}\Big|\frac{\partial^2}{\partial\hat{x}\partial\hat{z}}\hat{u}(\theta,\hat{y},\xi)\Big|^2\,\mathrm{d}\xi\,\mathrm{d}\theta\,\mathrm{d}\widehat{X}$$
$$+6\int_{\widehat{P}}\int_0^1\int_0^1\hat{z}^2\Big|\frac{\partial^2}{\partial\hat{x}\partial\hat{z}}\hat{u}(\theta,\hat{y},\xi)\Big|^2\,\mathrm{d}\xi\,\mathrm{d}\theta\,\mathrm{d}\widehat{X}$$
$$+6\int_{\widehat{P}}\int_0^1\int_0^{\hat{z}}\int_0^{\hat{y}}\Big|\frac{\partial^3}{\partial\hat{x}\partial\hat{z}\partial\hat{y}}\hat{u}(\theta,\sigma,\xi)\Big|^2\,\mathrm{d}\sigma\,\mathrm{d}\xi\,\mathrm{d}\theta\,\mathrm{d}\widehat{X}$$
$$+6\int_{\widehat{P}}\int_0^1\int_0^1\int_0^{\hat{y}}\hat{z}^2\Big|\frac{\partial^3}{\partial\hat{x}\partial\hat{z}\partial\hat{y}}\hat{u}(\theta,\sigma,\xi)\Big|^2\,\mathrm{d}\sigma\,\mathrm{d}\xi\,\mathrm{d}\theta\,\mathrm{d}\widehat{X}=I_1+\ldots+I_6.$$

Now, we estimate the terms I_i, $i=1,\ldots,6$. First, we observe that

$$I_1=6\int_0^1\int_0^1\int_0^{1-\hat{y}}\int_0^{\hat{x}}\Big|\frac{\partial^2}{\partial\hat{x}^2}\hat{u}(t,\hat{y},\hat{z})\Big|^2\,\mathrm{d}t\,\mathrm{d}\hat{x}\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{z}.$$

Since $0\leqslant\hat{x}\leqslant 1-\hat{y}$, we obtain

$$I_1\leqslant 6\int_0^1\int_0^1(1-\hat{y})\int_0^{1-\hat{y}}\Big|\frac{\partial^2}{\partial\hat{x}^2}\hat{u}(t,\hat{y},\hat{z})\Big|^2\,\mathrm{d}t\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{z}.\tag{37}$$

The maximum of the positive function $(1-\hat{y})$, where $\hat{y}\in[0,1]$, is equal to 1, and replacing t by $\hat{x}$, we get

$$I_1\leqslant 6\int_0^1\int_0^1(1-\hat{y})\int_0^{1-\hat{y}}\Big|\frac{\partial^2}{\partial\hat{x}^2}\hat{u}(t,\hat{y},\hat{z})\Big|^2\,\mathrm{d}t\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{z}\leqslant 6\int_{\widehat{P}}\Big|\frac{\partial^2}{\partial\hat{x}^2}\hat{u}(\hat{x},\hat{y},\hat{z})\Big|^2\,\mathrm{d}\widehat{X}.\tag{38}$$

Now, I_2 can be expressed as

$$I_2=6\int_0^1\int_0^1(1-\hat{y})\int_0^1\int_0^{\hat{y}}\Big|\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}\hat{u}(\theta,\sigma,\hat{z})\Big|^2\,\mathrm{d}\sigma\,\mathrm{d}\theta\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{z}.$$

Since $0\leqslant\hat{y}\leqslant 1-\hat{x}$ and $0\leqslant 1-\hat{y}\leqslant 1$, we get

$$I_2\leqslant 6\int_0^1\int_0^1\int_0^{1-\hat{x}}\Big|\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}\hat{u}(\theta,\sigma,\hat{z})\Big|^2\,\mathrm{d}\sigma\,\mathrm{d}\theta\,\mathrm{d}\hat{z}.\tag{39}$$

Substituting θ and σ with $\hat{x}$ and $\hat{y}$, respectively, implies

$$I_2\leqslant 6\int_{\widehat{P}}\Big|\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}\hat{u}(\hat{x},\hat{y},\hat{z})\Big|^2\,\mathrm{d}\widehat{X}.$$

Due to $0 \leqslant \theta \leqslant 1-\hat{y}$ and $0 \leqslant \xi \leqslant 1$, we observe that

$$\begin{aligned} I_3 &= 6\int_0^1\int_0^1 (1-\hat{y})\int_0^1\int_0^{\hat{z}} \Big|\frac{\partial^2}{\partial\hat{x}\partial\hat{z}}\hat{u}(\theta,\hat{y},\xi)\Big|^2\,\mathrm{d}\xi\,\mathrm{d}\theta\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{z} \\ &\leqslant 6\int_0^1\int_0^{1-\hat{y}}\int_0^1 \Big|\frac{\partial^2}{\partial\hat{x}\partial\hat{z}}\hat{u}(\theta,\hat{y},\xi)\Big|^2\,\mathrm{d}\xi\,\mathrm{d}\theta\,\mathrm{d}\hat{y} \\ &= 6\int_0^1\int_0^1\int_0^{1-\hat{y}} \Big|\frac{\partial^2}{\partial\hat{x}\partial\hat{z}}\hat{u}(\theta,\hat{y},\xi)\Big|^2\,\mathrm{d}\theta\,\mathrm{d}\hat{y}\,\mathrm{d}\xi. \end{aligned}$$

Replacing θ and ξ with $\hat{x}$ and $\hat{z}$, respectively, implies

$$I_3 \leqslant 6\int_{\widehat{P}} \Big|\frac{\partial^2}{\partial\hat{x}\partial\hat{z}}\hat{u}(\hat{x},\hat{y},\hat{z})\Big|^2\,\mathrm{d}\widehat{X}.$$

For I_4, when using $\int_0^1 \hat{z}^2\,\mathrm{d}\hat{z} < 1$ and repeating the same process as we have carried out for I_3, the following result holds:

$$I_4 \leqslant 6\int_{\widehat{P}} \Big|\frac{\partial^2}{\partial\hat{x}\partial\hat{z}}\hat{u}(\hat{x},\hat{y},\hat{z})\Big|^2\,\mathrm{d}\widehat{X}.$$

For I_5 we can write

$$\begin{aligned} I_5 &\leqslant 6\int_0^1 (1-\hat{y})\int_0^1\int_0^1\int_0^{\hat{y}} \Big|\frac{\partial^3}{\partial\hat{x}\partial\hat{z}\partial\hat{y}}\hat{u}(\theta,\sigma,\xi)\Big|^2\,\mathrm{d}\sigma\,\mathrm{d}\xi\,\mathrm{d}\theta\,\mathrm{d}\hat{y} \\ &\leqslant 6\int_0^1\int_0^1\int_0^{1-\hat{x}} \Big|\frac{\partial^3}{\partial\hat{x}\partial\hat{z}\partial\hat{y}}\hat{u}(\theta,\sigma,\xi)\Big|^2\,\mathrm{d}\sigma\,\mathrm{d}\xi\,\mathrm{d}\theta \\ &= 6\int_0^1\int_0^1\int_0^{1-\hat{x}} \Big|\frac{\partial^3}{\partial\hat{x}\partial\hat{z}\partial\hat{y}}\hat{u}(\theta,\sigma,\xi)\Big|^2\,\mathrm{d}\sigma\,\mathrm{d}\theta\,\mathrm{d}\xi. \end{aligned}$$

Substituting θ, σ, and ξ with $\hat{x}$, $\hat{y}$, and $\hat{z}$, respectively, we get

$$I_5 \leqslant 6\int_{\widehat{P}} \Big|\frac{\partial^3}{\partial\hat{x}\partial\hat{z}\partial\hat{y}}\hat{u}(\hat{x},\hat{y},\hat{z})\Big|^2\,\mathrm{d}\widehat{X}.$$

Similarly to I_5, we have

$$I_6 \leqslant 6\int_{\widehat{P}} \Big|\frac{\partial^3}{\partial\hat{x}\partial\hat{z}\partial\hat{y}}\hat{u}(\hat{x},\hat{y},\hat{z})\Big|^2\,\mathrm{d}\widehat{X}.$$

Considering the estimates of I_i, $i=1,\ldots,6$, relation (31) can be written now as

$$\int_{\widehat{P}} \Big|\frac{\partial}{\partial\hat{x}}(\hat{u}-\hat{\pi}_{\widehat{P}}\hat{u})\Big|^2\,\mathrm{d}\widehat{X} \leqslant 12\int_{\widehat{P}} \Big(\Big|\frac{\partial^2\hat{u}}{\partial\hat{x}^2}\Big|^2+\Big|\frac{\partial^2\hat{u}}{\partial\hat{x}\partial\hat{y}}\Big|^2+\Big|\frac{\partial^2\hat{u}}{\partial\hat{x}\partial\hat{z}}\Big|^2+\Big|\frac{\partial^3\hat{u}}{\partial\hat{x}\partial\hat{y}\partial\hat{z}}\Big|^2\Big)\,\mathrm{d}\widehat{X}. \tag{40}$$

From (21), (23) and (25), substituting partial derivatives of $\hat{u}$ on the right-hand side of (40) by the corresponding partial derivatives of u, we obtain

$$
\begin{aligned}
(41)\qquad & \int_{\widehat{P}} \Big|\frac{\partial}{\partial \hat{x}}(\hat{u}-\hat{\pi}_{\widehat{P}}\hat{u})\Big|^2 \,\mathrm{d}\widehat{X} \\
& \leqslant \widehat{C}|\det \mathrm{B}_P^{-1}| \int_P \Big(f_K^4 \Big(\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2 + \Big|\frac{\partial^2 u}{\partial x \partial y}\Big|^2 + \Big|\frac{\partial^2 u}{\partial y^2}\Big|^2 \Big) \\
& \quad + f_K^2 g_K^2 \Big(\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2 + \Big|\frac{\partial^2 u}{\partial x \partial y}\Big|^2 + \Big|\frac{\partial^2 u}{\partial y^2}\Big|^2 \Big) + z_I^2 f_K^2 \Big(\Big|\frac{\partial^2 u}{\partial x \partial z}\Big|^2 + \Big|\frac{\partial^2 u}{\partial y \partial z}\Big|^2 \Big) \\
& \quad + g_K^2 f_K^2 z_I^2 \Big(\Big|\frac{\partial^3 u}{\partial x^2 \partial z}\Big|^2 + \Big|\frac{\partial^3 u}{\partial x \partial y \partial z}\Big|^2 + \Big|\frac{\partial^3 u}{\partial y^2 \partial z}\Big|^2 \Big)\Big) \,\mathrm{d}X,
\end{aligned}
$$

where $\widehat{C} > 0$ is a constant. Using denotation (7), estimate (29) is thus obtained.

The derivation of estimate (30) is similar. □

Lemma 2. *Let $\hat{u} \in \mathrm{W}_2^3(\widehat{P})$. Then*

$$
\begin{aligned}
& \int_{\widehat{P}} \Big|\frac{\partial}{\partial \hat{z}}(\hat{u}-\hat{\pi}_{\widehat{P}}\hat{u})\Big|^2 \,\mathrm{d}\widehat{X} \\
& \leqslant |\det \mathrm{B}_P^{-1}| \Big(2 z_I^4 \int_P \Big|\frac{\partial^2 u}{\partial z^2}\Big|^2 \,\mathrm{d}X + C_1 z_I^2 (f_K^4 + f_K^2 g_K^2 + g_K^4) \int_P [u]_{2,1}^2 \,\mathrm{d}X \Big),
\end{aligned}
$$

where C_1 is a positive constant and $\hat{u}$ and u are related via (17).

P r o o f. We have

$$
\begin{aligned}
(42)\qquad & \int_{\widehat{P}} \Big|\frac{\partial}{\partial \hat{z}}(\hat{u}-\hat{\pi}_{\widehat{P}}\hat{u})\Big|^2 \,\mathrm{d}\widehat{X} \\
& = \int_{\widehat{P}} \Big|\frac{\partial}{\partial \hat{z}}\hat{u}(\hat{x},\hat{y},\hat{z}) - (\hat{x}(\hat{u}(1,0,1)-\hat{u}(1,0,0)) + \hat{y}(\hat{u}(0,1,1)-\hat{u}(0,1,0)) \\
& \qquad\qquad + (1-\hat{x}-\hat{y})(\hat{u}(0,0,1)-\hat{u}(0,0,0)))\Big|^2 \,\mathrm{d}\widehat{X} \\
& = \int_{\widehat{P}} \Big|\frac{\partial}{\partial \hat{z}}\hat{u}(\hat{x},\hat{y},\hat{z}) - \Big(\hat{x}\int_0^1 \frac{\partial}{\partial \hat{z}}\hat{u}(1,0,\xi)\,\mathrm{d}\xi + \hat{y}\int_0^1 \frac{\partial}{\partial \hat{z}}\hat{u}(0,1,\xi)\,\mathrm{d}\xi \\
& \qquad\qquad + (1-\hat{x}-\hat{y})\int_0^1 \frac{\partial}{\partial \hat{z}}\hat{u}(0,0,\xi)\,\mathrm{d}\xi \Big)\Big|^2 \,\mathrm{d}\widehat{X}.
\end{aligned}
$$

Assume that $\xi \in [0,1]$ is given, $g(\hat{x},\hat{y},\xi) = \frac{\partial}{\partial \hat{z}}\hat{u}(\hat{x},\hat{y},\xi)$, and $\hat{\pi}_{\widehat{K}}$ is the standard linear interpolation operator over the reference triangle $\widehat{K}$ with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Then

$$
\hat{\pi}_{\widehat{K}} g = \hat{x}\frac{\partial}{\partial \hat{z}}\hat{u}(1,0,\xi) + \hat{y}\frac{\partial}{\partial \hat{z}}\hat{u}(0,1,\xi) + (1-\hat{x}-\hat{y})\frac{\partial}{\partial \hat{z}}\hat{u}(0,0,\xi).
$$

Now, (42) can be written as

$$\int_{\widehat{P}} \Big|\frac{\partial}{\partial \hat z}(\hat u - \hat\pi_{\widehat{P}}\hat u)\Big|^2 \,\mathrm{d}\widehat{X}$$
$$= \int_{\widehat{P}}\Big|\int_0^1 \frac{\partial}{\partial \hat z}\hat u(\hat x,\hat y,\hat z)\,\mathrm{d}\xi - \int_0^1 \frac{\partial}{\partial \hat z}\hat u(\hat x,\hat y,\xi)\,\mathrm{d}\xi + \int_0^1 \frac{\partial}{\partial \hat z}\hat u(\hat x,\hat y,\xi) d\xi - \int_0^1 \hat\pi_{\widehat{K}} g\Big|^2 \,\mathrm{d}\xi$$
$$= \int_{\widehat{P}}\Big|\int_0^1\int_\xi^{\hat z} \frac{\partial^2}{\partial \hat z^2}\hat u(\hat x,\hat y,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi + \int_0^1 (g - \hat\pi_{\widehat{K}} g)\,\mathrm{d}\xi\Big|^2 \,\mathrm{d}\widehat{X}$$
$$\leqslant 2\Big(\int_0^1\int_0^{1-\hat y}\int_0^1 \Big|\frac{\partial^2}{\partial \hat z^2}\hat u(\hat x,\hat y,\eta)\Big|^2 \,\mathrm{d}\eta\,\mathrm{d}\hat x\,\mathrm{d}\hat y + \int_0^1\int_0^{1-\hat y}\int_0^1 |g - \hat\pi_{\widehat{K}} g|^2 \,\mathrm{d}\xi\,\mathrm{d}\hat x\,\mathrm{d}\hat y\Big).$$

Replacing η with $\hat z$ implies

$$\int_{\widehat{P}} \Big|\frac{\partial}{\partial \hat z}(\hat u - \hat\pi_{\widehat{P}}\hat u)\Big|^2 \,\mathrm{d}\widehat{X} \leqslant 2\Big(\int_{\widehat{P}}\Big|\frac{\partial^2}{\partial \hat z^2}\hat u(\hat x,\hat y,\hat z)\Big|^2 \,\mathrm{d}\widehat{X} + \int_0^1 \Big(\int_{\widehat{K}} |g - \hat\pi_{\widehat{K}} g|^2 \,\mathrm{d}\hat x\,\mathrm{d}\hat y\Big)\,\mathrm{d}\xi\Big). \tag{43}$$

By [9], pp. 118–120, we have

$$\|g - \hat\pi_{\widehat{K}} g\|_{0,2,\widehat{K}} \leqslant C|g|_{2,2,\widehat{K}}, \tag{44}$$

where C is a positive constant. Therefore

$$\begin{aligned}
\int_{\widehat{P}} \Big|\frac{\partial}{\partial \hat z}(\hat u - \hat\pi_{\widehat{P}}\hat u)\Big|^2 \,\mathrm{d}\widehat{X}
&\leqslant 2\int_{\widehat{P}}\Big|\frac{\partial^2}{\partial \hat z^2}\hat u(\hat x,\hat y,\hat z)\Big|^2 \,\mathrm{d}\widehat{X} + C_1\int_0^1\int_{\widehat{K}}\Big(\Big|\frac{\partial}{\partial \hat x^2\partial \hat z}\hat u(\hat x,\hat y,\xi)\Big|^2 \\
&\qquad + \Big|\frac{\partial}{\partial \hat x\partial \hat y\partial \hat z}\hat u(\hat x,\hat y,\xi)\Big|^2 + \Big|\frac{\partial}{\partial \hat y^2\partial \hat z}\hat u(\hat x,\hat y,\xi)\Big|^2\Big)\,\mathrm{d}\hat x\,\mathrm{d}\hat y\,\mathrm{d}\xi \\
&= 2\int_{\widehat{P}}\Big|\frac{\partial^2}{\partial \hat z^2}\bar u(\hat x,\hat y,\hat z)\Big|^2 \,\mathrm{d}\widehat{X} \\
&\quad + C_1\int_{\widehat{P}}\Big(\Big|\frac{\partial}{\partial \hat x^2\partial \hat z}\hat u(\hat x,\hat y,\hat z)\Big|^2 + \Big|\frac{\partial}{\partial \hat x\partial \hat y\partial \hat z}\hat u(\hat x,\hat y,\hat z)\Big|^2 + \Big|\frac{\partial}{\partial \hat y^2\partial \hat z}\hat u(\hat x,\hat y,\hat z)\Big|^2\Big)\,\mathrm{d}\widehat{X} \\
&\leqslant 2|\det \mathrm{B}_P^{-1}|\int_P z_I^4\Big|\frac{\partial^2 u}{\partial z^2}\Big|^2 \,\mathrm{d}X \\
&\quad + C_1|\det \mathrm{B}_P^{-1}|\int_P\Big(4z_I^2 f_K^4\Big|\frac{\partial^3 u}{\partial x^2\partial z}\Big|^2 + 8z_I^2 f_K^4\Big|\frac{\partial^3 u}{\partial x\partial y\partial z}\Big|^2 + 2z_I^2 f_K^4\Big|\frac{\partial^3 u}{\partial y^2\partial z}\Big|^2
\end{aligned} \tag{45}$$

$$
\begin{aligned}
&+4g_K^2 z_I^2 f_K^2 \frac{\partial^3 u}{\partial x^2 \partial z}\Big|^2 + 8g_K^2 z_I^2 f_K^2 \Big|\frac{\partial^3 u}{\partial x \partial y \partial z}\Big|^2 + 2g_K^2 z_I^2 f_K^2 \Big|\frac{\partial^3 u}{\partial y^2 \partial z}\Big|^2 \\
&+4z_I^2 g_K^4 \Big|\frac{\partial^3 u}{\partial x^2 \partial z}\Big|^2 + 8z_I^2 g_K^4 \Big|\frac{\partial^3 u}{\partial x \partial y \partial z}\Big|^2 + 2z_I^2 g_K^4 \Big|\frac{\partial^3 u}{\partial y^2 \partial z}\Big|^2\Big) \,\mathrm{d}X \\
\leqslant{}& |\det \mathrm{B}_P^{-1}| \Big(2z_I^4 \int_P \Big|\frac{\partial^2 u}{\partial z^2}\Big|^2 \,\mathrm{d}X \\
&+ C_1 z_I^2 (f_K^4 + f_K^2 g_K^2 + g_K^4) \int_P \Big(\Big|\frac{\partial^3 u}{\partial x^2 \partial z}\Big|^2 + \Big|\frac{\partial^3 u}{\partial x \partial y \partial z}\Big|^2 + \Big|\frac{\partial^3 u}{\partial y^2 \partial z}\Big|^2\Big) \,\mathrm{d}X \Big),
\end{aligned}
$$

and the proof is completed. □

4. Main result

Theorem 2. *Let $u \in \mathrm{W}_2^3(\Omega)$ and $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ be a semiregular family of prismatic partitions of $\bar{\Omega}$. Then there exists $h_0 > 0$ such that for all $h \in (0, h_0)$ the following estimate holds:*

$$
\|u - \pi_h u\|_{1,2,\Omega} \leqslant \tilde{C} h |u|_{2,2,\Omega}, \tag{46}
$$

where $\widetilde{C} > 0$ is a constant.

Proof. First, due to (19) and inequality (22) with $s = 2$, we have that

$$
\begin{aligned}
(47)\quad & |u - \pi_P u|_{1,2,P}^2 \\
&= \int_P \Big(\Big|\frac{\partial}{\partial x}(u - \pi_P u)\Big|^2 + \Big|\frac{\partial}{\partial y}(u - \pi_P u)\Big|^2 + \Big|\frac{\partial}{\partial z}(u - \pi_P u)\Big|^2 \Big) \,\mathrm{d}X \\
&= |\det \mathrm{B}_P| \int_{\widehat{P}} \Big(\Big| C_{11} \frac{\partial}{\partial \hat{x}}(\hat{u} - \hat{\pi}_{\widehat{P}} \hat{u}) + C_{21} \frac{\partial}{\partial \hat{y}}(\hat{u} - \hat{\pi}_{\widehat{P}} \hat{u}) \Big|^2 \\
&\quad + \Big| C_{12} \frac{\partial}{\partial \hat{x}}(\hat{u} - \hat{\pi}_{\widehat{P}} \hat{u}) + C_{22} \frac{\partial}{\partial \hat{y}}(\hat{u} - \hat{\pi}_{\widehat{P}} \hat{u}) \Big| r^2 + \Big| C_{33} \frac{\partial}{\partial \hat{z}}(\hat{u} - \hat{\pi}_{\widehat{P}} \hat{u}) \Big|^2 \Big) \,\mathrm{d}\widehat{X} \\
&\leqslant 2|\det \mathrm{B}_P| \int_{\widehat{P}} \Big((|C_{11}|^2 + |C_{12}|^2) \Big| \frac{\partial}{\partial \hat{x}}(\hat{u} - \hat{\pi}_{\widehat{P}} \hat{u}) \Big|^2 \\
&\quad + (|C_{21}|^2 + |C_{22}|^2) \Big| \frac{\partial}{\partial \hat{y}}(\hat{u} - \hat{\pi}_{\widehat{P}} \hat{u}) \Big|^2 \Big) \,\mathrm{d}\widehat{X} \\
&\quad + |\det \mathrm{B}_P| \int_{\widehat{P}} |C_{33} t|^2 \Big| \frac{\partial}{\partial \hat{z}}(\hat{u} - \hat{\pi}_{\widehat{P}} \hat{u}) \Big|^2 \,\mathrm{d}\widehat{X}.
\end{aligned}
$$

By definition, $C_{11} = |\det \mathrm{B}_K|^{-1} B_{22}$ and $C_{12} = -|\det \mathrm{B}_K|^{-1} B_{12}$, therefore

$$
|C_{11}|^2 + |C_{12}|^2 = |\det \mathrm{B}_K|^{-2} (|B_{12}|^2 + |B_{22}|^2) \leqslant \frac{C_2}{f_K^2}, \tag{48}
$$

where we used estimates (15) and the easily provable fact that if the maximum angle condition on partitions holds, then $|\det \mathrm{B}_K|^{-1} = C_3/f_K g_K$. Analogously, we get

$$|C_{21}|^2 + |C_{22}|^2 \leqslant \frac{C_4}{g_K^2}. \tag{49}$$

Using the estimates from Lemma 1, Lemma 2, (48), (49), and the fact that $C_{33} = 1/z_I$ for (47), we get the estimate

$$\begin{aligned} |u - \pi_P u|_{1,2,P}^2 \leqslant & \int_P \Big(\widehat{C}((f_K^2 + g_K^2)[u]_{2,0}^2 + z_I^2[u]_{1,1}^2) + 2z_I^2\Big|\frac{\partial^2 u}{\partial z^2}\Big|^2\Big)\,\mathrm{d}X \\ & + \int_P (\widehat{C}z_I^2(f_K^2 + g_K^2) + C_1(f_K^4 + f_K^2 g_K^2 + g_K^4))[u]_{2,1}^2\,\mathrm{d}X. \end{aligned}$$

Finally,

$$\begin{aligned} |u - \pi_P u|_{1,2,P}^2 & \leqslant h_P^2 \overline{C}_1 \int_P \Big([u]_{2,0}^2 + [u]_{1,1}^2 + \Big|\frac{\partial^2 u}{\partial z^2}\Big|^2\Big)\,\mathrm{d}X + h_P^4 \overline{C}_2 \int_P [u]_{2,1}^2\,\mathrm{d}X \\ & = h_P^2 \overline{C}_1 |u|_{2,2,P}^2 + h_P^4 \overline{C}_2 \int_P [u]_{2,1}^2\,\mathrm{d}X, \end{aligned} \tag{50}$$

where $h_P = \max\{f_K, g_K, z_I\}$.

Summing up the above inequality over all prisms $P \in \mathcal{T}_h$ and using the fact that $h_P \leqslant h$, for sufficiently small h we get that

$$|u - \pi_h u|_{1,2,\Omega} \leqslant \overline{C}_1 h |u|_{2,2,\Omega}. \tag{51}$$

The result of the theorem now follows from the above estimate and the following result from [9], p. 121, which is valid without any regularity assumptions on the meshes:

$$\|u - \pi_h u\|_{0,2,\Omega} \leqslant \overline{C}_2 h^2 |u|_{2,2,\Omega}. \tag{52}$$

□

5. Numerical results

In this section, we present results of several numerical tests obtained for the function $u(x,y,z) = x^2y^3 + xz^2$ interpolated on various (regular and degenerating) face-to-face prismatic partitions of the domain $\Omega = (0,1)^3$.

All prismatic partitions of Ω used in the tests are constructed by means of three natural numbers i, j and k which describe and control the geometric characteristics of the partitions as follows. First, we fix the values of i, j, k (see the tables below). Then we define $h_x = 2^{-i}$, $h_y = 2^{-j}$, $h_z = 2^{-k}$. By three sets of planes defined as $\{x = h_x p,\ p = 0,1,\dots,2^i\}$, $\{y = h_y p,\ p = 0,1,\dots,2^j\}$ and $\{z = h_z p,\ p = 0,1,\dots,2^k\}$, we split the cubic domain $\overline{\Omega}$ into blocks. Further, each block is split into two prisms using a vertical cut parallel to the plane $2^{-j}x + 2^{-i}y = 2^{-(i+j)}$ and in this way we get some face-to-face prismatic partition of $\overline{\Omega}$.

Varying i, j and k we get different families of prismatic partitions with various regularity properties. In order to compute relevant norms and a seminorm (of polynomial functions) exactly, we use the Gaussian quadratures with sufficient number of nodes, see [3].

In Tables $1, 2, \dots, 5$, we present computations associated with justification of interpolation error (51), and in Tables $6, 7, \dots, 10$, with justification of interpolation error in L_2-norm (52). We consider the case of regular families (Tables 1 and 6), also the cases of various degeneracies—short prism/regular bases (Tables 2 and 7), long prisms/regular bases (Tables 3 and 8), and the cases of degenerating bases with various choices of heights (Tables 4, 5, 9, and 10).

Monitoring behaviour of ratios

$$\frac{|u-\pi_h u|_{1,2,\Omega}}{h|u|_{2,2,\Omega}} \quad \text{and} \quad \frac{\|u-\pi_h u\|_{0,2,\Omega}}{h^2|u|_{2,2,\Omega}},$$

we observe that those are always converging in all corresponding series of tests (see the last columns in the tables below) to some values, thus, proving numerically the interpolation estimates (51) and (52).

i	j	k	h	$\lvert u-\pi_h u\rvert_{1,2,\Omega}$	$h^{-1}\lvert u-\pi_h u\rvert_{1,2,\Omega}/\lvert u\rvert_{2,2,\Omega}$
0	0	0	1.732051	0.823508	0.441637
1	1	1	0.866025	0.398421	0.427336
2	2	2	0.433013	0.197105	0.422820
3	3	3	0.216506	0.098282	0.421660
4	4	4	0.108253	0.049107	0.421369

Table 1. Regular prisms ($i = j = k$).

i	j	k	h	$\|u-\pi_h u\|_{1,2,\Omega}$	$h^{-1}\|u-\pi_h u\|_{1,2,\Omega}/\|u\|_{2,2,\Omega}$
0	0	0	1.732051	0.823508	0.441637
1	1	2	0.750000	0.363382	0.450050
2	2	4	0.359035	0.178620	0.462116
3	3	6	0.177466	0.089007	0.465873
4	4	8	0.088475	0.044468	0.466861

Table 2. Short prisms ($k \gg i = j$).

i	j	k	h	$\|u-\pi_h u\|_{1,2,\Omega}$	$h^{-1}\|u-\pi_h u\|_{1,2,\Omega}/\|u\|_{2,2,\Omega}$
0	0	0	1.732051	0.823508	0.441637
2	2	1	0.612372	0.250227	0.379556
4	4	2	0.265165	0.095240	0.333628
6	6	3	0.126938	0.043221	0.316275
8	8	4	0.062744	0.021030	0.311335

Table 3. Long prisms ($i = j \gg k$).

i	j	k	h	$\|u-\pi_h u\|_{1,2,\Omega}$	$h^{-1}\|u-\pi_h u\|_{1,2,\Omega}/\|u\|_{2,2,\Omega}$
0	0	0	1.732051	0.823508	0.441637
2	1	1	0.750000	0.377079	0.467013
4	2	2	0.359035	0.194645	0.503576
6	3	3	0.177466	0.100831	0.527760
8	4	4	0.088475	0.051502	0.540706

Table 4. Degenerating bases and proportional height ($i \gg j = k$).

i	j	k	h	$\|u-\pi_h u\|_{1,2,\Omega}$	$h^{-1}\|u-\pi_h u\|_{1,2,\Omega}/\|u\|_{2,2,\Omega}$
0	0	0	1.732051	0.823508	0.441637
2	1	2	0.612372	0.339846	0.515496
4	2	4	0.265165	0.175902	0.616187
6	3	6	0.126938	0.091813	0.671850
8	4	8	0.062744	0.047099	0.697269

Table 5. Degenerating bases and proportional height ($i = k \gg j$).

i	j	k	h	$\|u-\pi_h u\|_{1,2,\Omega}$	$h^{-1}\|u-\pi_h u\|_{1,2,\Omega}/\|u\|_{2,2,\Omega}$
0	0	0	1.732051	0.227210	0.070350
1	1	1	0.866025	0.046685	0.057819
2	2	2	0.433013	0.010880	0.053899
3	3	3	0.216506	0.002667	0.052855
4	4	4	0.108253	0.000663	0.052587

Table 6. Regular prisms ($i = j = k$).

i	j	k	h	$\lvert u-\pi_h u\rvert_{1,2,\Omega}$	$h^{-1}\lvert u-\pi_h u\rvert_{1,2,\Omega}/\lvert u\rvert_{2,2,\Omega}$
0	0	0	1.732051	0.227210	0.070350
1	1	2	0.750000	0.030964	0.051131
2	2	4	0.359035	0.006211	0.044757
3	3	6	0.177466	0.001461	0.043081
4	4	8	0.088475	0.000359	0.042652

Table 7. Short prisms ($k \gg i = j$).

i	j	k	h	$\lvert u-\pi_h u\rvert_{1,2,\Omega}$	$h^{-1}\lvert u-\pi_h u\rvert_{1,2,\Omega}/\lvert u\rvert_{2,2,\Omega}$
0	0	0	1.732051	0.227210	0.070350
2	2	1	0.750000	0.029772	0.073746
4	4	2	0.359035	0.006768	0.089407
6	6	3	0.177466	0.001658	0.095570
8	8	4	0.088475	0.000412	0.097312

Table 8. Long prisms ($i = j \gg k$).

i	j	k	h	$\lvert u-\pi_h u\rvert_{1,2,\Omega}$	$h^{-1}\lvert u-\pi_h u\rvert_{1,2,\Omega}/\lvert u\rvert_{2,2,\Omega}$
0	0	0	1.732051	0.227210	0.070350
2	1	1	0.750000	0.046224	0.076322
4	2	2	0.359035	0.012566	0.090546
6	3	3	0.177466	0.003336	0.098384
8	4	4	0.088475	0.000861	0.102199

Table 9. Degenerating bases and proportional height ($i \gg j = k$).

i	j	k	h	$\lvert u-\pi_h u\rvert_{1,2,\Omega}$	$h^{-1}\lvert u-\pi_h u\rvert_{1,2,\Omega}/\lvert u\rvert_{2,2,\Omega}$
0	0	0	1.732051	0.227210	0.070350
2	1	2	0.612372	0.028891	0.071563
4	2	4	0.265165	0.007322	0.096735
6	3	6	0.126938	0.001985	0.114401
8	4	8	0.062744	0.000522	0.123220

Table 10. Degenerating bases and proportional height ($i = k \gg j$).

6. Open problem

It would be interesting to validate several results on sufficient and necessary conditions for the convergence of the finite element method obtained in [11], [23], [25], [24], [27] for the case of simplices, also for the case of prismatic partitions.

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