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STRICT MITTAG-LEFFLER CONDITIONS AND LOCALLY SPLIT MORPHISMS

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Abstract. In this paper, we prove that any pure submodule of a strict Mittag-Leffler module is a locally split submodule. As applications, we discuss some relations between locally split monomorphisms and locally split epimorphisms and give a partial answer to the open problem whether Gorenstein projective modules are Ding projective.

Keywords: strict Mittag-Leffler condition; locally split morphism; Gorenstein projective module; Ding projective module

MSC 2010: 13D02, 13D07, 13E05, 16D10, 16D80, 16D90

1. Introduction

An inverse system of abelian groups $(A_\alpha, u_{\alpha\beta})_{\alpha,\beta\in I}$ is said to satisfy the Mittag-Leffler condition if, for each $\alpha \in I$, there exists an index $\gamma(\alpha)$ with $\gamma \geqslant \alpha$, such that $u_{\alpha\beta}(A_\beta) = u_{\alpha\gamma}(A_\gamma)$ for any $\beta \geqslant \gamma$. Such condition was first introduced by Grothendieck in [13]. For a countable inverse system, Grothendieck showed that the Mittag-Leffler condition is a sufficient condition for the exactness of the inverse limit functor (cf. [13], Section 13.1.2). Raynaud and Gruson in [15] showed that this condition is closely related to the injectivity of the following canonical map:

$$\iota\colon \Bigl(\prod_{i\in I} Q_i\Bigr) \otimes_R M \to \prod_{i\in I}(Q_i \otimes_R M),$$

where $\{Q_i\}_{i\in I}$ is a family of right R-modules and M is a left R-module. They proved that ι is monomorphic for any family $\{Q_i\}_{i\in I}$ of right R-modules if and only if M can

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be expressed as the direct limit of a direct system of finitely presented left R-modules $(F_\alpha, u_{\beta\alpha})_{\alpha,\beta\in I}$ such that the inverse system $(\mathrm{Hom}_R(F_\alpha, N), \mathrm{Hom}_R(u_{\beta\alpha}, N))_{\alpha,\beta\in I}$ satisfies the Mittag-Leffler condition for any left R-module N. So they called such module Mittag-Leffler. Moreover, they investigated the so-called strict Mittag-Leffler module, a stronger version of Mittag-Leffler module, which is defined via strict Mittag-Leffler condition (see Definition 2.1). In the past few years, (strict) Mittag-Leffler condition and modules were employed to solve many different problems in homological algebra and representation theory, see [1], [4], [5], [9], so these notions have caught many authors' attentions. For example, in [2], Angeleri Hügel and Herbera undertook a systematic study on (strict) Mittag-Leffler conditions, especially they gave some new characterizations of strict Mittag-Leffler modules, which they called strict stationary modules in their paper. And in [10], Emmanouil and Talelli investigated the flat length of injective modules by employing strict Mittag-Leffler modules.

Motivated by the above results, in Section 2 of this paper, we prove that any pure submodule of a strict Mittag-Leffler module is a locally split submodule (see Theorem 2.7), then we investigate the relations between locally split monomorphisms and locally split epimorphisms (see Corollary 2.8). Finally, in Section 3, based on the results obtained in Section 2, we give a partial answer to the open problem when Gorenstein projective modules are Ding projective.

Notation. Throughout this paper, R always denotes an associative ring with an identity. All modules are left R-modules unless stated otherwise. We denote by $\mathcal{I}(R)$, $\mathcal{P}(R)$ and $\mathcal{F}(R)$ the class of injective, projective and flat left R-modules, respectively. The category of all left R-modules is denoted by R-Mod. We assume that all direct and inverse systems are indexed by a directed set.

2. Locally split morphisms

In this section, we first study the closure properties of the class of strict Mittag-Leffler modules, and prove that the pure submodules of a strict Mittag-Leffler module are locally split submodules. Then we discuss the relations between locally split monomorphisms and locally split epimorphisms, which can be viewed as an extension of the relations between split monomorphisms and split epimorphisms. First of all, we recall some properties of inverse systems of abelian groups that are related to (strict) Mittag-Leffler conditions.

Definition 2.1. An inverse system of abelian groups $(A_\alpha, u_{\alpha\beta})_{\alpha,\beta\in I}$ with $A = \varprojlim A_\alpha$ is said to satisfy the strict Mittag-Leffler condition if for each $\alpha \in I$, there exists an index $\gamma(\alpha)$ with $\gamma \geqslant \alpha$, such that $u_{\alpha\beta}(A_\beta) = u_\alpha(A)$ for any $\beta \geqslant \gamma$, where u_α denotes the canonical map $A \to A_\alpha$.

Remark 2.2. Note that an inverse system satisfies the Mittag-Leffler condition if it satisfies the strict Mittag-Leffler condition. And by [2], Lemma 3.3, if the index set I is countable, then an inverse system $(A_\alpha, u_{\alpha\beta})_{\alpha,\beta\in I}$ satisfies the Mittag-Leffler condition if and only if it satisfies the strict Mittag-Leffler condition.

Strict Mittag-Leffler modules were studied by several authors under different names. We collect some characterizations of such modules in the following theorem.

Theorem 2.3 ([2], Proposition 8.1, Theorem 8.11)**.** *Let M and N be left R-modules. The following statements are equivalent:*

(1) *There is a direct system of finitely presented modules $(F_\alpha, u_{\beta\alpha})_{\alpha,\beta\in I}$ with $M = \varinjlim F_\alpha$, such that the inverse system*

$$(\mathrm{Hom}_R(F_\alpha, N), \mathrm{Hom}_R(u_{\beta\alpha}, N))_{\alpha,\beta\in I}$$

satisfies the strict Mittag-Leffler condition.

(2) *Every direct system of finitely presented modules $(F_\alpha, u_{\beta\alpha})_{\alpha,\beta\in I}$ with $M = \varinjlim F_\alpha$ has the property that the inverse system*

$$(\mathrm{Hom}_R(F_\alpha, N), \mathrm{Hom}_R(u_{\beta\alpha}, N))_{\alpha,\beta\in I}$$

satisfies the strict Mittag-Leffler condition.

(3) *There is a direct system of finitely presented modules $(F_\alpha, u_{\beta\alpha})_{\alpha,\beta\in I}$ with $M = \varinjlim F_\alpha$ having the property that for any $\alpha \in I$ there exists $\beta \geqslant \alpha$ such that any homomorphism $f\colon F_\beta \to N$ satisfies the condition that $fu_{\beta\alpha}$ factors through the canonical map $u_\alpha\colon F_\alpha \to M$.*

(4) *Every direct system of finitely presented modules $(F_\alpha, u_{\beta\alpha})_{\alpha,\beta\in I}$ with $M = \varinjlim F_\alpha$ has the property that for any $\alpha \in I$ there exists $\beta \geqslant \alpha$ such that any homomorphism $f\colon F_\beta \to N$ satisfies the condition that $fu_{\beta\alpha}$ factors through the canonical map $u_\alpha\colon F_\alpha \to M$.*

(5) *For every finitely presented module F and every homomorphism $u\colon F \to M$, there exist a finitely presented module F' and a homomorphism $v\colon F \to F'$ such that u factors through v, and moreover, for any homomorphism $f\colon F' \to N$, fv factors through u.*

(6) *For any divisible abelian group D, the natural transformation*

$$\Phi\colon \mathrm{Hom}_Z(N, D) \otimes_R M \longrightarrow \mathrm{Hom}_Z(\mathrm{Hom}_R(M, N), D)$$

defined by $\Phi(f \otimes m)\colon g \mapsto f(g(m))$, $f \in \mathrm{Hom}_Z(N, D)$, $m \in M$ *and* $g \in \mathrm{Hom}_R(M, N)$ *is a monomorphism.*

Following Emmanouil's terminology from [10], we say that M is a strict Mittag-Leffler module over N if it satisfies the equivalent conditions in the above theorem, and denote by $\mathrm{SML}(N)$ the class of strict Mittag-Leffler modules over N. This is precisely the class of strict N-stationary modules in [2]. Let $\mathcal{N}$ be a class of left R-modules. If, for any $N \in \mathcal{N}$, M is a strict Mittag-Leffler module over N, we say M is a strict Mittag-Leffler module over $\mathcal{N}$ and denote it by $M \in \mathrm{SML}(\mathcal{N})$. M is called a strict Mittag-Leffler module if $\mathcal{N}$=R-Mod.

Remark 2.4. (1) By Theorem 2.3 (6), it is easy to see that the class $\mathrm{SML}(N)$ is closed under direct sums and direct summands. Furthermore, $\mathrm{SML}(N)$ is closed under pure submodules.

(2) Note that the map in Theorem 2.3 (6) is an isomorphism if M is finitely presented. Thus all pure projective modules are strict Mittag-Leffler over the class of all left R-modules, i.e., $M \in \mathrm{SML}(R\text{-Mod})$ holds for any pure projective left R-module M. This follows from the fact that every pure projective module is a direct summand of a direct sum of finitely presented modules. So $\mathcal{P}(R) \subseteq \mathrm{SML}(R\text{-Mod})$.

We now recall the definitions of locally split morphisms.

Definition 2.5.

(1) A monomorphism $\varepsilon\colon A \to B$ is called locally split if for each element $x \in A$ there is a homomorphism $f\colon B \to A$ such that $f\varepsilon(x) = x$.

(2) An epimorphism $\pi\colon B \to C$ is called locally split if for each element $x \in C$ there is a homomorphism $g\colon C \to B$ such that $\pi g(x) = x$.

An R-module M is called locally pure projective if any pure epimorphism onto it is locally split. Azumaya proved in [3] that the class of locally pure projective modules coincides with the class of strict Mittag-Leffler modules.

Locally split monomorphism is called strongly pure monomorphism by Zimmermann in [16]. Using the technique given by Villamayor, which is reproduced in [7], Zimmermann showed in [16], Proposition 1.2, that a monomorphism is strongly pure if and only if for each finite subset X of A, there exists $f\colon B \to A$ such that $f\varepsilon(x) = x$ for any $x \in X$. It is easily seen that locally split morphisms are generalizations of split morphisms. It is well known that in an exact sequence $0 \to M' \xrightarrow{\varepsilon} M \xrightarrow{\pi} M'' \to 0$, ε is a split monomorphism if and only if π is a split epimorphism. But this does not hold true for locally split morphisms. In the rest of this section, we first investigate more closure properties of strict Mittag-Leffler modules with locally split morphisms, then, we discuss the relations between locally split monomorphisms and locally split epimorphisms by strict Mittag-Leffler modules.

Note that the first part of the following proposition has been proven by the authors in [2], Corollary 8.5, but we here give a simpler proof.

Proposition 2.6. *Let R be a ring.*

(1) ([2], Corollary 8.5) *Let $0 \to A \xrightarrow{\varepsilon} B \to C \to 0$ be an exact sequence of modules with ε a locally split monomorphism and M an R-module. Then $M \in \mathrm{SML}(B)$ implies $M \in \mathrm{SML}(A)$.*

(2) *Let $0 \to M' \to M \xrightarrow{\pi} M'' \to 0$ be an exact sequence of modules with π a locally split epimorphic and N an R-module. Then $M \in \mathrm{SML}(N)$ implies $M'' \in \mathrm{SML}(N)$.*

P r o o f. (1) Given a direct system of finitely presented R-modules $(F_\alpha, u_{\beta\alpha})_{\alpha,\beta\in I}$ with $M = \varinjlim F_\alpha$, we denote the canonical maps $F_\alpha \to M$ by u_α. Consider the following diagram:

$$\begin{tikzcd}
F_\alpha \arrow[r, "u_{\beta\alpha}"] \arrow[d, "u_\alpha"'] & F_\beta \arrow[r, "f"] \arrow[dl, "u_\beta"] & A \arrow[d, shift left, "\varepsilon"] \\
M \arrow[rr, dashed, "g"'] \arrow[urr, dashed, "h"] & & B. \arrow[u, dashed, shift left, "\varphi"]
\end{tikzcd}$$

Assume that $M \in \mathrm{SML}(B)$, then for any homomorphism $f\colon F_\beta \to A$, there is a homomorphism $g\colon M \to B$ such that $\varepsilon f u_{\beta\alpha} = g u_\alpha$ by Theorem 2.3 (4). We note that ε is a locally split monomorphism and $f u_{\beta\alpha}(F_\alpha)$ is finitely generated, hence there exists a homomorphism $\varphi\colon B \to A$ satisfying $\varphi\varepsilon f u_{\beta\alpha}(x) = f u_{\beta\alpha}(x)$ for each $x \in F_\alpha$. Set $h = \varphi g\colon M \to A$, then for any $x \in F_\alpha$, $h u_\alpha(x) = \varphi g u_\alpha(x) = \varphi\varepsilon f u_{\beta\alpha}(x) = f u_{\beta\alpha}(x)$. This shows $M \in \mathrm{SML}(A)$.

(2) For any finitely presented module F and any homomorphism $u\colon F \to M''$ there exists a homomorphism $\varphi\colon M'' \to M$ such that $\pi\varphi u(x) = u(x)$ for any $x \in F$. Now we suppose that $M \in \mathrm{SML}(N)$. Following Theorem 2.3 (5) there exist a finitely presented left R-module F', a homomorphism $v\colon F \to F'$ and a homomorphism $g\colon F' \to M$ such that $gv = \varphi u$, and for any homomorphism $f\colon F' \to N$, fv factors through φu, i.e., there is a homomorphism $h\colon M \to N$ such that $fv = h\varphi u$. Consider the following diagram:

$$\begin{tikzcd}
 & N & F' \arrow[l, "f"'] \arrow[d, "g"] & F \arrow[l, "v"'] \arrow[d, "u"] & \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r, shift left, "\pi"] \arrow[ul, dashed, "h"'] & M'' \arrow[r] \arrow[l, dashed, shift left, "\varphi"] & 0.
\end{tikzcd}$$

It is easily seen that $\pi g v = \pi\varphi u = u$. By Theorem 2.3 (5), $M'' \in \mathrm{SML}(N)$. □

Note that an exact sequence $0 \to A \xrightarrow{\varepsilon} B \xrightarrow{\pi} C \to 0$ of R-modules is pure exact if ε is a locally split monomorphism or π is a locally split epimorphism. We now give a kind of invecs of this result with strict Mittag-Leffler condition. The proof of the following theorem employs some results in [3], note that the class of locally pure projective modules studied in [3] coincides with the class of strict Mittag-Leffler modules.

Theorem 2.7. *Let $0 \to M' \xrightarrow{\varepsilon} M \to M'' \to 0$ be a pure exact sequence of modules, such that M is a strict Mittag-Leffler module. Then ε is a locally split monomorphism.*

Proof. We shall prove that for any $x \in M'$, there is a homomorphism $t\colon M \to M'$ with $t\varepsilon(x) = x$. Note that strict Mittag-Leffler modules coincide with locally pure projective modules. By [3], Proposition 4, there exist a finitely presented R-module F, a homomorphism $h\colon M \to F$ and a homomorphism $g\colon F \to M$ such that $gh\varepsilon(x) = \varepsilon(x)$. Let K be the cyclic R-module generated by $h\varepsilon(x)$ and let $i\colon K \hookrightarrow F$ be the inclusion map. Define $f\colon K \to M'$ by $fh\varepsilon(x) = x$. We consider the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "i"] \arrow[d, "f"'] & F \arrow[d, shift left, "g"] \arrow[dl, dashed, "\varphi"'] \\
0 \arrow[r] & M' \arrow[r, "\varepsilon"] & M. \arrow[u, dashed, shift left, "h"]
\end{tikzcd}
$$

It is easy to check that $\varepsilon f(h\varepsilon(x)) = \varepsilon(x) = gh(\varepsilon(x)) = gi(h\varepsilon(x))$. This shows $\varepsilon f = gi$. Thus there is a homomorphism $\varphi\colon F \to M'$ such that $\varphi i = f$ by [4], Lemma 4.1. Set $t = \varphi h\colon M \to M'$, then $t\varepsilon(x) = \varphi h\varepsilon(x) = \varphi i h\varepsilon(x) = fh\varepsilon(x) = x$. Therefore ε is a locally split monomorphism. $\square$

Note that a submodule N of an R-module M is called locally split, if the inclusion map $i\colon N \hookrightarrow M$ is locally split. Therefore, the above theorem shows that any pure submodule of a strict Mittag-Leffler module is a locally split submodule. Now, as a corollary of the above theorem and Proposition 2.6 (2), we can give the following result, which can be viewed as an extension of the relations between split monomorphisms and split epimorphisms to the case of locally split morphisms.

Corollary 2.8. *Let $0 \to M' \xrightarrow{\varepsilon} M \xrightarrow{\pi} M'' \to 0$ be an exact sequence of modules with M strict Mittag-Leffler. Then π is a locally split epimorphism if and only if ε is a locally split monomorphism and M'' is a strict Mittag-Leffler module.*

Moreover, as another corollary, we have the following result, which has also been proven by Herbera and Trlifaj in [14], Lemma 3.3.

Corollary 2.9. *Every finitely generated pure submodule of a strict Mittag-Leffler module is a direct summand.*

3. An application

During the past several years, Gorenstein modules were deeply studied by many authors. It is well known that these modules have many similar properties as projective, injective and flat modules. But there are significant differences. For example, projective modules are always flat modules, but it is not clear whether Gorenstein projective modules are Gorenstein flat. Recently, Ding et al. in [8] considered a special class of the Gorenstein projective modules, which they called strongly Gorenstein flat modules. Gillespie renamed strongly Gorenstein flat modules as Ding projective modules (see [12] for details).

We now recall the definitions of relevant notions.

Definition 3.1.

(1) ([11], Definition 10.2.1) An R-module M is called Gorenstein projective if there exists an exact sequence $\ldots \to P_1 \to P_0 \to P^0 \to P^1 \to \ldots$ of projective modules with $M = \mathrm{Ker}(P^0 \to P^1)$ such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence exact whenever Q is projective.

(2) ([6], Definition 2.1) An R-module M is called strongly Gorenstein projective if there exists an exact sequence $\ldots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \ldots$ of projective modules with $M = \mathrm{Ker}(f)$ such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence exact whenever Q is projective.

(3) ([12], Definition 3.7) An R-module M is called Ding projective if there exists an exact sequence $\ldots \to P_1 \to P_0 \to P^0 \to P^1 \to \ldots$ of projective modules with $M = \mathrm{Ker}(P^0 \to P^1)$ such that $\mathrm{Hom}_R(-, F)$ leaves the sequence exact whenever F is flat.

(4) ([11], Definition 10.3.1) An R-module M is called Gorenstein flat if there exists an exact sequence $\ldots \to F_1 \to F_0 \to F^0 \to F^1 \to \ldots$ of flat modules with $M = \mathrm{Ker}(F^0 \to F^1)$ such that $I \otimes_R -$ leaves the sequence exact whenever I is an injective right R-module.

We denote by $\mathcal{GP}(R)$, $\mathcal{SGP}(R)$, $\mathcal{DP}(R)$ and $\mathcal{GF}(R)$ the classes of Gorenstein projective, strongly Gorenstein projective, Ding projective and Gorenstein flat R-modules, respectively. By Definition 3.1, it is easy to get that $\mathcal{SGP}(R) \subseteq \mathcal{GP}(R)$ and $\mathcal{DP}(R) \subseteq \mathcal{GP}(R)$. Bennis and Mahdou in [6] gave an example of a Gorenstein projective module which is not strongly Gorenstein projective, and so $\mathcal{GP}(R) \nsubseteq \mathcal{SGP}(R)$.

However, we have not been able to find examples of Gorenstein projective modules which are not Ding projective, so it remains open whether the inclusion of $\mathcal{GP}(R) \subseteq \mathcal{DP}(R)$ is true. Furthermore, it is not clear whether all Gorenstein projective modules are Gorenstein flat. Ding et al. proved in [8], Proposition 2.3, that over coherent rings, Ding projective (i.e., strongly Gorenstein flat in the proposition) modules are Gorenstein flat. Therefore, if the class of Ding projective modules (strongly Gorenstein flat modules) happens to be the class of Gorenstein projective modules, then, over coherent rings, all Gorenstein projective modules are Gorenstein flat. In the rest of this section, we first focus on the question when strongly Gorenstein projective modules are Ding projective, then we apply our result to show when the class of Gorenstein projective modules coincides with the class of Ding projective modules.

Recall that an R-module is said to be countably presented if it is the cokernel of a homomorphism between two countably generated free modules. It is well-known that any countably presented module can be expressed as the direct limit of a countable direct system of finitely presented modules. Let M be a countably presented module and P an R-module. Then $\operatorname{Ext}_R^1(M, P^{(\mathbb{N})}) = 0$ implies that $M \in \mathrm{SML}(P)$ by [1], Example 2.4 (4), and [2], Remark 8.3 (3), or by [9], Proposition 2.3. The following result was obtained by Grothendieck in [13], Section 13.1.2.

Lemma 3.2. *Let $0 \to A_i \to B_i \to C_i \to 0$, $i \in \mathbb{N}$, be a countable inverse system of short exact sequences. If the inverse system $(A_i)_{i\in\mathbb{N}}$ satisfies the strict Mittag-Leffler condition, then*

$$0 \to \varprojlim A_i \to \varprojlim \to B_i \to \varprojlim C_i \to 0$$

is exact.

Theorem 3.3. *Every countably presented strongly Gorenstein projective module is Ding projective.*

P r o o f. Let N be a countably presented strongly Gorenstein projective module. Then there is a countable direct system

$$F_1 \overset{f_1}{\to} F_2 \overset{f_2}{\to} F_3 \to \ldots \to F_n \overset{f_n}{\to} F_{n+1} \to \ldots$$

of finitely presented modules such that $N = \varinjlim F_n$. Note that $\operatorname{Ext}_R^1(N, R^{(\mathbb{N})}) = 0$, so $N \in \mathrm{SML}(R)$. We conclude that $N \in \mathrm{SML}(\mathcal{P}(R))$ by [2], Corollary 8.5 (iii).

For any $F \in \mathcal{F}(R)$, we consider a pure exact sequence $0 \to K \to P \to F \to 0$ with P projective. Applying the functor $\operatorname{Hom}_R(F_n, -)$ to the pure exact sequence $0 \to K \to P \to F \to 0$, we obtain an inverse system of exact sequences of the form

$$0 \to \operatorname{Hom}_R(F_n, K) \to \operatorname{Hom}_R(F_n, P) \to \operatorname{Hom}_R(F_n, F) \to 0.$$

Since every projective module is strict Mittag-Leffler, we have that K is a locally split submodule of P by Theorem 2.7. So $N \in \mathrm{SML}(\mathcal{P}(R))$ implies that $N \in \mathrm{SML}(K)$ by Proposition 2.6 (1). So, the inverse system of abelian groups $(\mathrm{Hom}_R(F_n, K), \mathrm{Hom}_R(f_n, K))$ satisfies the strict Mittag-Leffler condition. Then we have an exact sequence

$$0 \to \varprojlim \mathrm{Hom}_R(F_n, K) \to \varprojlim \mathrm{Hom}_R(F_n, P) \to \varprojlim \mathrm{Hom}_R(F_n, F) \to 0$$

by Lemma 3.2. This shows that the sequence

$$0 \to \mathrm{Hom}_R(\varinjlim F_n, K) \to \mathrm{Hom}_R(\varinjlim F_n, P) \to \mathrm{Hom}_R(\varinjlim F_n, F) \to 0$$

is exact. Thus we have that the sequence

$$0 \to \mathrm{Hom}_R(N, K) \to \mathrm{Hom}_R(N, P) \to \mathrm{Hom}_R(N, F) \to 0$$

is exact. The exactness of $0 \to \mathrm{Ext}^1_R(N, K) \to \mathrm{Ext}^1_R(N, P) = 0$ implies that $\mathrm{Ext}^1_R(N, K) = 0$. The definition of a strongly Gorenstein projective module gives that $\mathrm{Ext}^i_R(N, K) = 0$ for any $i \geqslant 1$. It follows that $\mathrm{Ext}^i_R(N, F) = 0$ for any $i \geqslant 1$ by dimension shifting. This proves that N is Ding projective. $\square$

Note that Bennis and Mahdou proved in [6], Theorem 2.7, that an R-module M is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective module. This result shows that strongly Gorenstein projective modules in Gorenstein homological algebra behave just like free modules in homological algebra. Note that every projective module is a direct summand of a free module, and Kaplansky showed that every projective module is a direct sum of countably generated projective R-modules, therefore, it is natural to ask whether every Gorenstein projective module is a direct summand of the direct sum of countably generated (countably presented) strongly Gorenstein projective modules. If this is the case, as a corollary of Theorem 3.3, we have the following result:

Corollary 3.4. *Assume that R is a right coherent ring such that every Gorenstein projective module is a direct summand of the direct sum of countably presented strongly Gorenstein projective modules. Then every Gorenstein projective module is Ding projective, and in particular, it is Gorenstein flat.*

P r o o f. Let M be a Gorenstein projective left R-module such that M is a direct summand of the direct sum of countably presented strongly Gorenstein projective modules which are also Ding projective modules by Theorem 3.3. Since the base ring is right coherent, we get that the class of Ding projective modules is closed under direct sums and direct summands by [8], Proposition 2.10. Therefore M is Ding projective. $\square$

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