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ANNIHILATOR-PRESERVING CONGRUENCE RELATIONS IN DISTRIBUTIVE NEARLATTICES

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Abstract. In this note we give some new characterizations of distributivity of a nearlattice and we study annihilator-preserving congruence relations.

Keywords: distributive nearlattice; ideal; filter; congruence; annihilator

MSC 2010: 06A12, 03G10, 06D50

1. Introduction and preliminaries

There exists a correspondence between the class of implication algebras and join-semilattices with greatest element in which every principal filter is a Boolean lattice, given by Abbott in [1]. The nearlattices are a natural generalization of the implication algebras, i.e., join-semilattices with greatest element in which every principal filter is a bounded lattice. The class of nearlattices has been studied in [14] and [16] by Cornish and Hickman, and in [15], [8], [10], [9] and [11] by Chajda, Kolařík, Halaš and Kühr. An important class of nearlattices are the distributive nearlattices. Recently in [7] and [3], the authors develop a full duality for distributive nearlattices and propose a definition of relative annihilator different from that given in [10].

On the other hand, in [17], Janowitz defines the notion of annihilator-preserving congruence relation in a bounded distributive lattice $\mathbf{A}$, called AP-congruence, as a lattice-congruence θ such that for all $a, b \in A$, if $a \wedge b \equiv_\theta 0$, then there exists $c \in A$ such that $a \wedge c = 0$ and $c \equiv_\theta b$. If $\mathbf{A}$ is pseudocomplemented, then a lattice-congruence θ is an AP-congruence if and only if it preserves pseudocomplements. A new characterization of the AP-congruences for bounded distributive lattices is given in [5] and in [6] this concept was extended to the bounded distributive semilattices.

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This paper has two objectives: to give some new characterizations of distributivity of a nearlattice and to study the notion of annihilator-preserving congruence relations in the class of distributive nearlattices. In the rest of this section we shall give some necessary notation and definitions. We will recall the topological representation for distributive nearlattices developed in [7]. In Section 2 we will show some new characterizations of distributivity of a nearlattice using ideals, filters and relative annihilators. Finally, in Section 3, we study the AP-congruences in distributive nearlattices. We shall see that the AP-congruences in a distributive nearlattice $\mathbf{A}$ are in a bijective correspondence to certain N-subspaces of the dual space of $\mathbf{A}$. This correspondence extends the results developed in [5] for distributive lattices.

Let $\mathbf{A} = \langle A, \vee, 1\rangle$ be a join-semilattice with greatest element. The set complement of a subset X of $\mathbf{A}$ will be denoted by X^{c}. A *filter* is a subset F of $\mathbf{A}$ such that $1 \in F$, if $a \leqslant b$ and $a \in F$ then $b \in F$, and if $a, b \in F$ then $a \wedge b \in F$, whenever $a \wedge b$ exists. If X is a subset of $\mathbf{A}$, the least filter containing X is called the *filter generated by* X and will be denoted by $F(X)$. A filter G is said to be *finitely generated* if $G = F(X)$ for some finite subset X of $\mathbf{A}$. If $X = a$, then $F(\{a\}) = [a)$ is called the *principal filter of* a. We will denote by $\mathrm{Fi}(\mathbf{A})$ and $\mathrm{Fi}_{\mathrm{f}}(\mathbf{A})$ the set of all filters and finitely generated filters of $\mathbf{A}$, respectively. A proper filter F of $\mathbf{A}$ is called *prime* if for all $a, b \in A$, $a \vee b \in F$ implies $a \in F$ or $b \in F$. An *order filter* is a subset F of $\mathbf{A}$ such that $1 \in F$, if $a \leqslant b$ and $a \in F$ then $b \in F$, and for all $a, b \in F$, there exists $c \in F$ such that $c \leqslant a$ and $c \leqslant b$. A subset F of $\mathbf{A}$ is a *Frink filter* if $1 \in F$ and for all $a_1, \ldots, a_n \in F$ and $b \in A$, whenever $(a_1] \cap \ldots \cap (a_n] \subseteq (b]$ we have $b \in F$. Denote the set of all order filters and Frink filters of $\mathbf{A}$ by $\mathrm{Fi}_{\mathrm{Or}}(\mathbf{A})$ and $\mathrm{Fi}_{\mathrm{F}}(\mathbf{A})$, respectively. Note that every order filter and Frink filter is, in particular, a filter.

A subset I of $\mathbf{A}$ is called an *ideal* if $a \leqslant b$ and $b \in I$ imply $a \in I$, and $a, b \in I$ imply $a \vee b \in I$. If X is a nonempty set, the least ideal containing X is called the *ideal generated by* X and will be denoted by $I(X)$. We shall say that a nonempty proper ideal P is *prime* if for all $a, b \in A$, $a \wedge b \in I$ implies $a \in I$ or $b \in I$, whenever $a \wedge b$ exists. We denote by $\mathrm{Id}(\mathbf{A})$ and $X(\mathbf{A})$ the set of all ideals and prime ideals of $\mathbf{A}$, respectively. Let $\max X(\mathbf{A})$ denote the maximal elements of $X(\mathbf{A})$. For each $P \in X(\mathbf{A})$, let $\max [P) = \max X(\mathbf{A}) \cap [P)$. An ideal I of $\mathbf{A}$ is *irreducible* if for all $I_1, I_2 \in \mathrm{Id}(\mathbf{A})$ such that $I = I_1 \cap I_2$ we have $I = I_1$ or $I = I_2$. An ideal I of $\mathbf{A}$ is *optimal* if I^{c} is a Frink filter, i.e., if for all $a_1, \ldots, a_n \notin I$ and $b \in A$, whenever $(a_1] \cap \ldots \cap (a_n] \subseteq (b]$ we have $b \notin I$. Denote by $\mathrm{Irr}(\mathbf{A})$ and $\mathrm{Id}_{\mathrm{Op}}(\mathbf{A})$ the set of all optimal and irreducible ideals of $\mathbf{A}$, respectively. It follows that $\mathrm{Id}_{\mathrm{Op}}(\mathbf{A}) \subseteq X(\mathbf{A})$.

The following results were investigated for join-semilattices and will be useful later.

Theorem 1.1 ([4]). *Let* $\mathbf{A}$ *be a join-semilattice with greatest element. Let* $I \in \mathrm{Id}(\mathbf{A})$ *and* $F \in \mathrm{Fi}_{\mathrm{Or}}(\mathbf{A})$ *such that* $I \cap F = \emptyset$. *Then there exists* $P \in \mathrm{Irr}(\mathbf{A})$ *such that* $I \subseteq P$ *and* $P \cap F = \emptyset$.

Lemma 1.1 ([4]). *Let* $\mathbf{A}$ *be a join-semilattice with greatest element and* $I \in \mathrm{Id}(\mathbf{A})$. *Then* I *is irreducible if and only if for each* $a_1, \ldots, a_n \notin I$ *there exists* $b \notin I$ *and there exists* $c \in I$ *such that* $b \leqslant a_i \vee c$ *for all* $i = 1, \ldots, n$.

We introduce the class of algebras that are the objects of study in this paper.

Definition 1.1. Let $\mathbf{A}$ be a join-semilattice with greatest element. Then $\mathbf{A}$ is called a *nearlattice* if each principal filter is a bounded lattice with respect to the induced order.

The class of nearlattices can be regarded as pure algebras through a ternary operation. This fact was proved by Hickman in [16] and by Chajda and Kolařík in [11]. Araújo and Kinyon in [2] found a smaller equational base.

Proposition 1.1 ([2]). *Let* $\mathbf{A}$ *be a nearlattice. Let* $m\colon A^3 \to A$ *be a ternary operation given by* $m(x,y,z) = (x \vee z) \wedge_z (y \vee z)$. *The following identities are satisfied:*

(1) $m(x,y,x) = x$,
(2) $m(m(x,y,z), m(y, m(u,x,z), z), w) = m(w,w,m(y,m(x,u,z),z))$,
(3) $m(x,x,1) = 1$.

Conversely, let $\mathbf{A} = \langle A, m, 1\rangle$ *be an algebra of type* $(3,0)$ *satisfying the identities* (1)–(3). *If we define* $x \vee y = m(x,x,y)$, *then* $\mathbf{A}$ *is a semilattice. Moreover, for each* $a \in A$, $[a)$ *is a bounded lattice where for* $x, y \in [a)$ *the infimum is* $x \wedge_a y = m(x,y,a)$. *Hence* $\mathbf{A}$ *is a nearlattice.*

Definition 1.2. Let $\mathbf{A}$ be a nearlattice. Then $\mathbf{A}$ is called *distributive* if each principal filter is a bounded distributive lattice.

E x a m p l e 1.1 ([1]). An implication algebra is defined as a join-semilattice with greatest element such that each principal filter is a Boolean lattice with respect to the induced order. If $\mathbf{A} = \langle A, \to, 1\rangle$ is an implication algebra, then the join of two elements x and y is given by $x \vee y = (x \to y) \to y$ and for each $a \in A$, $[a) = \{x \in A\colon a \leqslant x\}$ is a Boolean lattice where for $x, y \in [a)$ the meet is given by $x \wedge_a y = (x \to (y \to a)) \to a$ and $x \to a$ is the complement of x in $[a)$. So, $\mathbf{A} = \langle A, \vee, 1\rangle$ is a distributive nearlattice.

Note that from the results given in [14] we have the following characterization of the filter generated by a subset X in a distributive nearlattice $\mathbf{A}$:

$$F(X) = \{a \in A\colon \exists x_1, \ldots, x_n \in [X) \ \exists x_1 \wedge \ldots \wedge x_n \ (a = x_1 \wedge \ldots \wedge x_n)\}.$$

Theorem 1.2 ([15])**.** *Let* $\mathbf{A}$ *be a distributive nearlattice. Let* $I \in \mathrm{Id}(\mathbf{A})$ *and let* $F \in \mathrm{Fi}(\mathbf{A})$ *such that* $I \cap F = \emptyset$*. Then there exists* $P \in X(\mathbf{A})$ *such that* $I \subseteq P$ *and* $P \cap F = \emptyset$*.*

Recall some topological notions. A topological space with a base $\mathcal{K}$ will be denoted by $\langle X, \mathcal{K}\rangle$. We consider the set $D_{\mathcal{K}}(X) = \{U\colon U^{\mathrm{c}} \in \mathcal{K}\}$. A subset $Y \subseteq X$ is *basic saturated* if it is an intersection of basic open sets, i.e., $Y = \bigcap\{U_i \in \mathcal{K}\colon Y \subseteq U_i\}$. The *basic saturation* $\mathrm{Bs}(Y)$ of a subset Y is the smallest basic saturated set containing Y. If $Y = \{y\}$, we write $\mathrm{Bs}(\{y\}) = \mathrm{Bs}(y)$. On X, a binary relation $\leqslant$ is defined as $x \leqslant y$ if and only if $y \in \mathrm{Bs}(x)$. It is easy to see that the relation $\leqslant$ is a partial order if and only if $\langle X, \mathcal{K}\rangle$ is T_0. Let Y be a nonempty subset of X. We say that Y is *irreducible* if for every $U, V \in D_{\mathcal{K}}(X)$ such that $U \cap V \in D_{\mathcal{K}}(X)$ and $Y \cap (U \cap Y) = \emptyset$ we have $Y \cap U = \emptyset$ or $Y \cap V = \emptyset$. We say that Y is *dually compact* if for every family $\mathcal{F} = \{U_i\colon i \in I\} \subseteq \mathcal{K}$ such that $\bigcap\{U_i\colon i \in I\} \subseteq Y$ there exists a finite family $\{U_1, \ldots, U_n\} \subseteq \mathcal{K}$ such that $U_1 \cap \ldots \cap U_n \subseteq Y$. Finally, remember that we can define a topology on Y by taking as its base the family $\mathcal{K}_Y = \{U \cap Y\colon U \in \mathcal{K}\}$ such that the pair $\langle Y, \mathcal{K}_Y\rangle$ is a topological space. For more details see [7].

Definition 1.3 ([7])**.** Let $\langle X, \mathcal{K}\rangle$ be a topological space. Then $\langle X, \mathcal{K}\rangle$ is an N-*space* if:

(1) $\mathcal{K}$ is a basis of open, compact and dually compact subsets for the topology $\mathcal{T}_{\mathcal{K}}$ on X.
(2) For every $U, V, W \in \mathcal{K}$, $(U \cap W) \cup (V \cap W) \in \mathcal{K}$.
(3) For every irreducible basic saturated subset Y of X there exists a unique $x \in X$ such that $Y = \mathrm{Bs}(x)$.

Proposition 1.2 ([7])**.** *Let* $\langle X, \mathcal{K}\rangle$ *be a topological space where* $\mathcal{K}$ *is a basis of open and compact subsets for the topology* $\mathcal{T}_{\mathcal{K}}$ *on* X*. Suppose that* $(U \cap W) \cup (V \cap W) \in \mathcal{K}$ *for all* $U, V, W \in \mathcal{K}$*. The following conditions are equivalent:*

(1) $\langle X, \mathcal{K}\rangle$ *is* T_0 *and if* $A = \{U_i\colon i \in I\}$ *and* $B = \{V_j\colon j \in J\}$ *are nonempty families of* $D_{\mathcal{K}}(X)$ *such that* $\bigcap\{U_i\colon i \in I\} \subseteq \bigcup\{V_j\colon j \in J\}$*, then there exist* $U_1, \ldots, U_n \in [A)$ *and* $V_1, \ldots, V_k \in B$ *such that* $U_1 \cap \ldots \cap U_n \in D_{\mathcal{K}}(X)$ *and* $U_1 \cap \ldots \cap U_n \subseteq V_1 \cup \ldots \cup V_k$*.*

(2) $\langle X,\mathcal{K}\rangle$ *is* T_0, *every* $U\in\mathcal{K}$ *is dually compact and the assignment* $H\colon X\to X(D_{\mathcal{K}}(X))$ *defined by*

$$H(x)=\{U\in D_{\mathcal{K}}(X)\colon\ x\notin U\}$$

for each $x\in X$, *is onto.*

(3) *Every* $U\in\mathcal{K}$ *is dually compact and for every irreducible basic saturated subset* Y *of* X *there exists a unique* $x\in X$ *such that* $Y=\mathrm{Bs}(x)$.

If $\langle X,\mathcal{K}\rangle$ is an N-space, then $\langle D_{\mathcal{K}}(X),\cup,X\rangle$ is a distributive nearlattice. Note that $X\in\mathcal{K}$ if and only if $D_{\mathcal{K}}(X)$ is a bounded distributive lattice. Thus, $\mathcal{K}$ is the family of all open and compact subsets of X and we obtain the topological representation for bounded distributive lattices given by Stone in [18].

Let $\mathbf{A}$ be a distributive nearlattice. Let us consider the poset $\langle X(\mathbf{A}),\subseteq\rangle$ and the mapping $\varphi_{\mathbf{A}}\colon A\to\mathcal{P}_{\mathrm{d}}(X(\mathbf{A}))$ defined by $\varphi_{\mathbf{A}}(a)=\{P\in X(\mathbf{A})\colon\ a\notin P\}$. Then $\mathbf{A}$ is isomorphic to the subalgebra $\varphi_{\mathbf{A}}[\mathbf{A}]=\{\varphi_{\mathbf{A}}(a)\colon\ a\in A\}$ of $\mathcal{P}_{\mathrm{d}}(X(\mathbf{A}))$ and the pair $\langle X(\mathbf{A}),\mathcal{K}_{\mathbf{A}}\rangle$ is an N-space, called the *dual space of* $\mathbf{A}$, where the topology $\mathcal{T}_{\mathbf{A}}$ is generated by taking as its base the family $\mathcal{K}_{\mathbf{A}}=\{\varphi_{\mathbf{A}}(a)^{\mathrm{c}}\colon\ a\in A\}$. Let $\mathbf{A}$ and $\mathbf{B}$ be two distributive nearlattices. A mapping $h\colon A\to B$ is a *homomorphism* if $h(1)=1$, $h(a\vee b)=h(a)\vee h(b)$ for all $a,b\in A$, and $h(a\wedge b)=h(a)\wedge h(b)$ whenever $a\wedge b$ exists. For more details see [7].

2. Some equivalences of distributivity

In [3] the authors obtain new equivalences of distributivity of a nearlattice. In this section we present some new characterizations of distributivity using ideals, filters and the theory of relative annihilators.

Lemma 2.1. *Let* $\mathbf{A}$ *be a distributive nearlattice and* $I\in\mathrm{Id}(\mathbf{A})$. *Then* I *is prime if and only if it is optimal.*

Proof. We need only to prove that every prime ideal is optimal. Let $a_1,\ldots,a_n,b\in A$ and $P\in X(\mathbf{A})$ such that $a_1,\ldots,a_n\notin P$ and $(a_1]\cap\ldots\cap(a_n]\subseteq(b]$. Suppose that $b\in P$. As $b\leqslant a_i\vee b$ for all $i=1,\ldots,n$ and $[b)$ is a bounded distributive lattice, $b\leqslant(a_1\vee b)\wedge\ldots\wedge(a_n\vee b)$. Since $(a_1]\cap\ldots\cap(a_n]\subseteq(b]$, it follows that $b=(a_1\vee b)\wedge\ldots\wedge(a_n\vee b)\in P$ and due to primality of P there exists $j\in\{1,\ldots,n\}$ such that $a_j\vee b\in P$. So, $a_j\in P$ which is a contradiction. Thus P is optimal. $\square$

Theorem 1.2, proved by Halaš in [15], generalizes the well-known Prime Ideal Theorem. In fact, this result is equivalent to the distributivity of nearlattices.

Theorem 2.1. *Let* $\mathbf{A}$ *be a nearlattice. The following conditions are equivalent:*

(1) $\mathbf{A}$ *is distributive.*

(2) *Let* $I \in \mathrm{Id}(\mathbf{A})$ *and let* $F \in \mathrm{Fi}(\mathbf{A})$ *such that* $I \cap F = \emptyset$. *Then there exists* $P \in X(\mathbf{A})$ *such that* $I \subseteq P$ *and* $P \cap F = \emptyset$.

(3) *If* $x \nleqslant y$, *then there exists* $P \in X(\mathbf{A})$ *such that* $y \in P$ *and* $x \notin P$.

P r o o f. (1) $\Rightarrow$ (2) It follows from [15].

(2) $\Rightarrow$ (3) It is immediate.

(3) $\Rightarrow$ (1) Let $a \in A$ and $x, y, z \in [a)$. We know that the inequality $x \vee (y \wedge z) \leqslant (x \vee y) \wedge (x \vee z)$ always holds. We prove the other inequality. We suppose the contrary, i.e., $(x \vee y) \wedge (x \vee z) \nleqslant x \vee (y \wedge z)$. Then, by hypothesis, there exists $P \in X(\mathbf{A})$ such that $x \vee (y \wedge z) \in P$ and $(x \vee y) \wedge (x \vee z) \notin P$. So, $x, y \wedge z \in P$ and since P is prime, $y \in P$ or $z \in P$. It follows that $x \vee y \in P$ or $x \vee z \in P$. On the other hand, as $(x \vee y) \wedge (x \vee z) \notin P$, $x \vee y \notin P$ and $x \vee z \notin P$, which is a contradiction. Therefore, $\mathbf{A}$ is distributive. □

Let $\mathbf{A}$ be a semilattice. Let $a_1, \ldots, a_n \in A$ and $I \in \mathrm{Id}(\mathbf{A})$. We consider the following property:

$$(*) \qquad (a_1] \cap \ldots \cap (a_n] \subseteq I \quad \text{implies} \quad a_i \in I$$

for some $i \in \{1, \ldots, n\}$. It follows that every ideal satisfying the property $(*)$ is irreducible. Indeed, let $I, I_1, I_2 \in \mathrm{Id}(\mathbf{A})$ such that $I = I_1 \cap I_2$. Suppose that $I \subset I_1$ and $I \subset I_2$. Then there exist $a, b \in A$ such that $a \in I_1 - I$ and $b \in I_2 - I$. As $(a] \cap (b] \subseteq I_1 \cap I_2 = I$ and I satisfies $(*)$, $a \in I$ or $b \in I$, which is a contradiction.

Theorem 2.2. *Let* $\mathbf{A}$ *be a nearlattice. The following conditions are equivalent:*

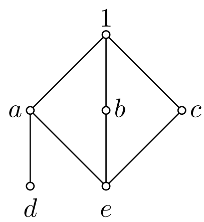
(1) $\mathbf{A}$ *is distributive.*

(2) *Every irreducible ideal satisfies the property* $(*)$.

P r o o f. (1) $\Rightarrow$ (2) Let $a_1, \ldots, a_n \in A$ and $I \in \mathrm{Irr}(\mathbf{A})$ such that $(a_1] \cap \ldots \cap (a_n] \subseteq I$. Suppose that $a_1, \ldots, a_n \notin I$. Since I is irreducible, by Lemma 1.1, there exists $b \notin I$ and there exists $c \in I$ such that $b \leqslant a_i \vee c$, i.e., $a_i \vee c \in [b)$ for all $i = 1, \ldots, n$. As $[b)$ is a bounded distributive lattice, $b \leqslant (a_1 \vee c) \wedge \ldots \wedge (a_n \vee c) = (a_1 \wedge \ldots \wedge a_n) \vee c$. Then $(b] \subseteq (a_1 \wedge \ldots \wedge a_n] \vee (c] \subseteq ((a_1] \cap \ldots \cap (a_n]) \vee (c] \subseteq I$. Thus, $b \in I$, which is a contradiction. Therefore I satisfies the property $(*)$.

(2) $\Rightarrow$ (1) Let $a \in A$ and $x, y, z \in [a)$. We prove that $(x \vee y) \wedge (x \vee z) \leqslant x \vee (y \wedge z)$. Suppose the contrary. Then, by Theorem 1.1, there exists $P \in \mathrm{Irr}(\mathbf{A})$ such that $x \vee (y \wedge z) \in P$ and $(x \vee y) \wedge (x \vee z) \notin P$. So, $x, y \wedge z \in P$ and $x \vee y, x \vee z \notin P$. Since $(y] \cap (z] \subseteq P$ and P satisfies the property $(*)$, $y \in P$ or $z \in P$. So, $x \vee y \in P$ or $x \vee z \in P$, which is a contradiction. Thus, $\mathbf{A}$ is distributive. □

E x a m p l e 2.1. Not every irreducible ideal is optimal in a nearlattice. We consider the following configuration:



It is easy to prove that $I = \{a, d, e\}$ is an irreducible ideal but not optimal, i.e., $I^{\rm c} = \{1, b, c\}$ is not Frink filter because $(b] \cap (c] \subseteq (a]$ and $a \notin I^{\rm c}$.

In the class of distributive nearlattices every irreducible ideal is optimal.

Theorem 2.3. *Let* $\mathbf{A}$ *be a nearlattice. The following conditions are equivalent:*
(1) $\mathbf{A}$ *is distributive.*
(2) $\mathrm{Irr}(\mathbf{A}) \subseteq \mathrm{Id}_{\mathrm{Op}}(\mathbf{A})$.

P r o o f. (1) $\Rightarrow$ (2) Let $I \in \mathrm{Irr}(\mathbf{A})$. Since $\mathbf{A}$ is distributive, it follows by the results developed in [7] that I is prime. Thus, by Lemma 2.1, I is optimal.

(2) $\Rightarrow$ (1) Let $x, y \in A$ such that $x \nleqslant y$. By Theorem 1.1 there exists $P \in \mathrm{Irr}(\mathbf{A})$ such that $y \in P$ and $x \notin P$. As $\mathrm{Irr}(\mathbf{A}) \subseteq \mathrm{Id}_{\mathrm{Op}}(\mathbf{A})$ and $\mathrm{Id}_{\mathrm{Op}}(\mathbf{A}) \subseteq X(\mathbf{A})$ we have that $P \in X(\mathbf{A})$. Then, by Theorem 2.1, $\mathbf{A}$ is distributive. □

The following definition given in [3] is an alternative definition of relative annihilators in distributive nearlattices different from that given in [10].

Definition 2.1. Let $\mathbf{A}$ be a semilattice and $a, b \in A$. The set

$$a \circ b = \{x \in A\colon\ b \leqslant x \vee a\}$$

is called *annihilator of a relative to b*. In particular, the relative annihilator $a^{\top} = a \circ 1 = \{x \in A\colon\ x \vee a = 1\}$ is called the *annihilator of a*.

Let $a \in A$ and $F \in \mathrm{Fi}(\mathbf{A})$. We consider the set

$$a \circ F = \{x \in A\colon\ \exists f \in F\ (f \leqslant x \vee a)\}.$$

By the results developed in [3] we have that a nearlattice $\mathbf{A}$ is distributive if and only if $a \circ b \in \mathrm{Fi}(\mathbf{A})$ if and only if $a \circ F \in \mathrm{Fi}(\mathbf{A})$ for all $a, b \in A$ and all $F \in \mathrm{Fi}(\mathbf{A})$.

Lemma 2.2 ([3]). *Let* $\mathbf{A}$ *be a distributive nearlattice. Let* $a \in A$ *and* $I \in \mathrm{Id}(\mathbf{A})$. *Then* $I \cap a^{\top} = \emptyset$ *if and only if there exists* $Q \in \max X(\mathbf{A})$ *such that* $I \subseteq Q$ *and* $a \in Q$.

Lemma 2.3. *Let* $\mathbf{A}$ *be a semilattice and* $F \in \mathrm{Fi_F}(\mathbf{A})$. *Then* F *is prime if and only if* $a \circ F = F$ *for all* $a \notin F$.

P r o o f. Let $a \in A$ such that $a \notin F$. It is easy to see that $F \subseteq a \circ F$. Let $x \in a \circ F$. Then there exists $f \in F$ such that $f \leqslant x \vee a$. So, $x \vee a \in F$. Since F is prime, $x \in F$ or $a \in F$. Therefore, it should be $x \in F$ and $a \circ F = F$ for all $a \notin F$. Conversely, let $x, y \in A$ such that $x \vee y \in F$ and suppose that $x \notin F$ and $y \notin F$. As $a \circ F = F$ for all $a \notin F$, in particular, $x \circ F = F$. Thus, $y \notin x \circ F$. On the other hand, if we take $f = x \vee y \in F$, $f \leqslant y \vee x$ and $y \in x \circ F$, which is a contradiction. □

Theorem 2.4. *Let* $\mathbf{A}$ *be a nearlattice. The following conditions are equivalent:*

(1) $\mathbf{A}$ *is distributive.*
(2) *If* $x \circ F \subseteq y \circ F$ *for every prime Frink filter* F, *then* $x \leqslant y$.

P r o o f. (1) ⇒ (2) Let $x, y \in A$ such that $x \circ F \subseteq y \circ F$ for every prime Frink filter F. Suppose that $x \nleqslant y$. Then, by Theorem 1.2, there exists $P \in X(\mathbf{A})$ such that $y \in P$ and $x \notin P$. By Lemma 2.1, $X(\mathbf{A}) = \mathrm{Id_{Op}}(\mathbf{A})$ and P is optimal. So, $F = P^{\mathrm{c}}$ is a prime Frink filter. It follows that $x \circ F = A$ and therefore $y \circ F = A$. On the other hand, by Lemma 2.3, $y \in y \circ F = F$ and $y \notin F$, which is a contradiction.

(2) ⇒ (1) Let $x, y \in A$ such that $x \nleqslant y$. Then there exists $F \in \mathrm{Fi_F}(\mathbf{A})$ such that F is prime and $x \circ F \nsubseteq y \circ F$. So, there exists $z \in x \circ F$ such that $z \notin y \circ F$. Note that $z \notin F$. Since F is a Frink filter and $z \notin F$, by Lemma 2.3 we have $z \circ F = F$. It follows that $x \in F$ and $y \notin F$. As $P = F^{\mathrm{c}}$ is an optimal ideal, and therefore prime, $x \notin P$ and $y \in P$. Then, by Theorem 2.1, $\mathbf{A}$ is distributive. □

3. Annihilator-preserving congruence relations

In the present section, following the results developed in [3] and [7], we study the concept of annihilator-preserving congruence relations in the class of distributive nearlattices. We shall prove that there exists a dual isomorphism between annihilator-preserving congruence relations of a distributive nearlattice $\mathbf{A}$ and certain N-subspaces of the dual space of $\mathbf{A}$ satisfying an aditional condition.

Let $\mathbf{A}$ be a distributive nearlattice. We denote by $\mathrm{Con}(\mathbf{A})$ the set of all congruences of $\mathbf{A}$. If $\theta \in \mathrm{Con}(\mathbf{A})$, then we will write $(a, b) \in \theta$ or $a \equiv_\theta b$. The equivalence class of an element $a \in A$ is denoted by $|a|_\theta = \{b \in A\colon a \equiv_\theta b\}$. Recall that $\mathrm{Con}(\mathbf{A})$ is a distributive lattice where for any $\theta_1, \theta_2 \in \mathrm{Con}(\mathbf{A})$, $\theta_1 \wedge \theta_2 = \theta_1 \cap \theta_2$ and

$$(a, b) \in \theta_1 \vee \theta_2 \text{ if and only if there exist } c_0 = a, c_1, \ldots, c_n = b \in A \text{ such that } (c_i, c_{i+1}) \in \theta_1 \cup \theta_2 \text{ for all } i = 0, \ldots, n-1.$$

The canonical or natural map with respect to θ is the function $q_\theta\colon A \to A/\theta$ defined by $q_\theta(a) = |a|_\theta$. For a subset $S \subseteq A$ we will write $|S|_\theta = \{|a|_\theta\colon a \in S\}$.

Definition 3.1. Let $\mathbf{A}$ be a distributive nearlattice and $\theta \in \mathrm{Con}(\mathbf{A})$. We say that θ is an *annihilator-preserving congruence*, or AP-*congruence*, if for each $a, b \in A$, $a \equiv_\theta b$ implies that for each $x \in a^\top$ there exists $y \in b^\top$ such that $x \equiv_\theta y$.

If $\mathbf{A}$ is a distributive nearlattice and θ is an AP-congruence, we will use the notation $a^\top \equiv_{\tilde\theta} b^\top$ to indicate that θ satisfies the condition of Definition 3.1. So, a congruence is an AP-congruence if for each $a, b \in A$, $a^\top \equiv_{\tilde\theta} b^\top$ whenever $a \equiv_\theta b$. We denote by $\mathrm{Con}_{\mathrm{AP}}(\mathbf{A})$ the set of all AP-congruences of $\mathbf{A}$.

Example 3.1. Let $h\colon A \to B$ be a homomorphism such that $h(a) = 1$ implies $a = 1$. Then $\mathrm{Ker}\, h = \{(a, b)\colon h(a) = h(b)\}$ is an AP-congruence. Let $(a, b) \in \mathrm{Ker}\, h$. If $x \in a^\top$, then $x \vee a = 1$ and $h(x \vee a) = h(x) \vee h(a) = h(x) \vee h(b) = h(x \vee b) = 1$. Thus, by the assumption, $x \vee b = 1$ and $x \in b^\top$. It follows that $\mathrm{Ker}\, h \in \mathrm{Con}_{\mathrm{AP}}(\mathbf{A})$.

Remark 3.1. If $\mathbf{A} = \langle A, \to, 1\rangle$ is an implication algebra, then every congruence is an AP-congruence, that is, both concepts coincide. Let $\theta \in \mathrm{Con}(\mathbf{A})$. Let $a, b \in A$ such that $a \equiv_\theta b$ and $x \in a^\top$. Then $a \vee x = 1$, or equivalently, $(a \to x) \to x = 1$ and $a \to x \leqslant x$. On the other hand, it always holds that $x \leqslant a \to x$ in $\mathbf{A}$. So, $x = a \to x$. Let $y = b \to x$. Since $a \equiv_\theta b$,

$$x = a \to x \equiv_\theta b \to x = y,$$

i.e., $x \equiv_\theta y$. We prove that $y \in b^\top$. If $1 \nleq b \vee y$, then there exists a maximal deductive system P such that $b \vee y = (b \to y) \to y \notin P$. Since P is maximal, $b \to y \in P$ and $y = b \to x \notin P$. Again, since P is maximal, $b \in P$ and $x \notin P$. Then $b, b \to y \in P$ and $y \in P$, which is a contradiction. Therefore, $y \in b^\top$ and $\theta \in \mathrm{Con}_{\mathrm{AP}}(\mathbf{A})$.

Lemma 3.1. *Let* $\mathbf{A}$ *be a distributive nearlattice. Then* $\mathrm{Con}_{\mathrm{AP}}(\mathbf{A})$ *is a sublattice of* $\mathrm{Con}(\mathbf{A})$.

Proof. Let $\theta_1, \theta_2 \in \mathrm{Con}_{\mathrm{AP}}(\mathbf{A})$. We prove that $\theta_1 \wedge \theta_2, \theta_1 \vee \theta_2 \in \mathrm{Con}_{\mathrm{AP}}(\mathbf{A})$. Let $(a, b) \in \theta_1 \wedge \theta_2$ and $x \in a^\top$. Since θ_1 is an AP-congruence, there exists $y \in b^\top$ such that $x \equiv_{\theta_1} y$. Similarly, as $(a, b) \in \theta_2$, there exists $\overline{y} \in b^\top$ such that $x \equiv_{\theta_2} \overline{y}$. Then $x \equiv_{\theta_1} y \vee x$ and $x \equiv_{\theta_2} \overline{y} \vee x$. Since $[x)$ is a distributive lattice, there exist $(y \vee x) \wedge (\overline{y} \vee x)$, $x \wedge (\overline{y} \vee x)$ and $x \wedge (y \vee x)$. It follows that

$$x = x \wedge (\overline{y} \vee x) \equiv_{\theta_1} (y \vee x) \wedge (\overline{y} \vee x)$$

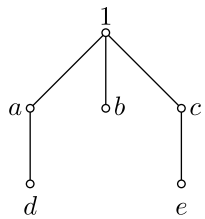
and

$$x = x \wedge (y \vee x) \equiv_{\theta_2} (y \vee x) \wedge (\overline{y} \vee x).$$

Then $x \equiv_{\theta_1 \wedge \theta_2} (y \vee x) \wedge (\overline{y} \vee x)$. On the other hand, $y \vee x, \overline{y} \vee x \in b^{\top}$ and since $b^{\top}$ is a filter, we have $(y \vee x) \wedge (\overline{y} \vee x) \in b^{\top}$. Thus, $\theta_1 \wedge \theta_2$ is an AP-congruence.

Let $(a,b) \in \theta_1 \vee \theta_2$ and $x \in a^{\top}$. Then there exist $c_0 = a, c_1, \ldots, c_n = b \in A$ such that $(c_i, c_{i+1}) \in \theta_1 \cup \theta_2$ for all $i = 0, \ldots, n-1$. So, $x \in a^{\top} = c_0^{\top}$ and $(c_0, c_1) \in \theta_1 \cup \theta_2$. Since θ_1 and θ_2 are AP-congruences, there exists $y_1 \in c_1^{\top}$ such that $(x, y_1) \in \theta_1 \cup \theta_2$. By induction, there exist $y_1, \ldots, y_n \in A$ such that $y_j \in c_j^{\top}$ and $(x, y_1), (y_j, y_{j+1}) \in \theta_1 \cup \theta_2$ for all $j = 1, \ldots, n-1$. Therefore, there exists $y_n \in c_n^{\top} = b^{\top}$ and $x \equiv_{\theta_1 \vee \theta_2} y_n$. So, $\theta_1 \vee \theta_2$ is an AP-congruence. $\square$

R e m a r k 3.2. Note that not every congruence is an AP-congruence. We consider the following distributive nearlattice $\mathbf{A}$:



Let $\theta(c,1)$ be the congruence generated by the pair $(c,1)$. Let $e \in 1^{\top} = A$. Thus, $1 \equiv_{\theta} c$ but there does not exist $y \in c^{\top}$ such that $e \equiv_{\theta} y$. So, $\theta(c,1) \notin \mathrm{Con}_{\mathrm{AP}}(\mathbf{A})$.

It was proved in [7] that the congruences lattice of a distributive nearlattice $\mathbf{A}$ is dually isomorphic to the lattice of certain subspaces, called N-subspaces, of the dual space $\langle X(\mathbf{A}), \mathcal{K}_{\mathbf{A}} \rangle$.

Definition 3.2 ([7]). Let $\langle X, \mathcal{K} \rangle$ be an N-space and let Y be a subset of X. We say that Y is an N-*subspace* if $\langle Y, \mathcal{K}_Y \rangle$ is an N-space.

The set of all N-subspaces of an N-space $\langle X, \mathcal{K} \rangle$ will be denoted by $\mathcal{S}(X)$.

Let $\mathbf{A}$ be a distributive nearlattice and $\theta \in \mathrm{Con}(\mathbf{A})$. We consider

$$Y_{\theta} = \{q_{\theta}^{-1}(P) \colon \ P \in X(\mathbf{A}/\theta)\}.$$

Since q_{θ} is a homomorphism, $q_{\theta}^{-1}(P) \in X(\mathbf{A})$ and $Y_{\theta} \subseteq X(\mathbf{A})$. It follows that the pair $\langle Y_{\theta}, \mathcal{K}_{Y_{\theta}} \rangle$ is an N-space and therefore Y_{θ} is an N-subspace. Reciprocally, if $Y \subseteq X(\mathbf{A})$, then the binary relation $\theta(Y) \subseteq A \times A$ given by

$$\theta(Y) = \{(a,b) \in A \times A \colon \ \varphi_{\mathbf{A}}(a)^{\mathrm{c}} \cap Y = \varphi_{\mathbf{A}}(b)^{\mathrm{c}} \cap Y\}$$

is a congruence relation of $\mathbf{A}$ such that $\theta = \theta(Y_{\theta})$. For more details see [7].

For each $F \in \mathrm{Fi}(\mathbf{A})$ we consider the set $\gamma(F) = \{P \in X(\mathbf{A})\colon\ P \cap F \neq \emptyset\}$. We have that $\gamma(F) = \bigcup\{\varphi_{\mathbf{A}}(a)^{\mathrm{c}}\colon\ a \in F\}$ and in particular $\gamma([a)) = \varphi_{\mathbf{A}}(a)^{\mathrm{c}}$.

Corollary 3.1. *Let $\mathbf{A}$ be a distributive nearlattice and $Y \in \mathcal{S}(X(\mathbf{A}))$. Let $a, b \in A$ such that $a \equiv_{\theta(Y)} b$. The following conditions are equivalent:*

(1) $a^{\top} \equiv_{\tilde{\theta}(Y)} b^{\top}$.
(2) $\gamma(a^{\top}) \cap Y = \gamma(b^{\top}) \cap Y$.

P r o o f. (1) $\Rightarrow$ (2) We prove that $\gamma(a^{\top}) \cap Y \subseteq \gamma(b^{\top}) \cap Y$. Let $P \in \gamma(a^{\top}) \cap Y$. Then $P \in \bigcup\{\varphi_{\mathbf{A}}(x)^{\mathrm{c}}\colon\ x \in a^{\top}\} \cap Y$. So, there exists $e \in a^{\top}$ such that $P \in \varphi_{\mathbf{A}}(e)^{\mathrm{c}}$ and as $a^{\top} \equiv_{\tilde{\theta}(Y)} b^{\top}$, there exists $f \in b^{\top}$ such that $e \equiv_{\theta(Y)} f$, i.e., $\varphi_{\mathbf{A}}(e)^{\mathrm{c}} \cap Y = \varphi_{\mathbf{A}}(f)^{\mathrm{c}} \cap Y$. It follows that

$$P \in \varphi_{\mathbf{A}}(e)^{\mathrm{c}} \cap Y = \varphi_{\mathbf{A}}(f)^{\mathrm{c}} \cap Y \subseteq \bigcup\{\varphi_{\mathbf{A}}(y)^{\mathrm{c}}\colon\ y \in b^{\top}\} \cap Y = \gamma(b^{\top}) \cap Y.$$

Thus, $P \in \gamma(b^{\top}) \cap Y$. The other inclusion can be shown similarly.

(2) $\Rightarrow$ (1) By hypothesis, $\gamma(a^{\top}) \cap Y = \gamma(b^{\top}) \cap Y$, i.e.,

$$\bigcup\{\varphi_{\mathbf{A}}(x)^{\mathrm{c}}\colon\ x \in a^{\top}\} \cap Y = \bigcup\{\varphi_{\mathbf{A}}(y)^{\mathrm{c}}\colon\ y \in b^{\top}\} \cap Y.$$

Let $e \in a^{\top}$. We prove that there exists $f \in b^{\top}$ such that $(e, f) \in \theta(Y)$. Note that $\varphi_{\mathbf{A}}(e)^{\mathrm{c}} \cap Y \subseteq \bigcup\{\varphi_{\mathbf{A}}(x)^{\mathrm{c}}\colon\ x \in a^{\top}\} \cap Y = \bigcup\{\varphi_{\mathbf{A}}(y)^{\mathrm{c}}\colon\ y \in b^{\top}\} \cap Y$. Thus,

$$\varphi_{\mathbf{A}}(e)^{\mathrm{c}} \cap Y \subseteq \bigcup\{\varphi_{\mathbf{A}}(y)^{\mathrm{c}} \cap Y\colon\ y \in b^{\top}\}$$

or equivalently,

$$\bigcap\{\varphi_{\mathbf{A}}(y) \cap Y\colon\ y \in b^{\top}\} \subseteq \varphi_{\mathbf{A}}(e) \cap Y.$$

Since Y is an N-subspace, by Proposition 1.2, there exist $y_1, \ldots, y_n \in [b^{\top}) = b^{\top}$ such that $y_1 \wedge \ldots \wedge y_n$ exists and $[\varphi_{\mathbf{A}}(y_1) \cap Y] \cap \ldots \cap [\varphi_{\mathbf{A}}(y_n) \cap Y] \subseteq \varphi_{\mathbf{A}}(e) \cap Y$. If $y = y_1 \wedge \ldots \wedge y_n$, then $\varphi_{\mathbf{A}}(y) \cap Y \subseteq \varphi_{\mathbf{A}}(e) \cap Y$ and $\varphi_{\mathbf{A}}(e \vee y)^{\mathrm{c}} \cap Y = \varphi_{\mathbf{A}}(e)^{\mathrm{c}} \cap Y$. Since $b^{\top} \in \mathrm{Fi}(\mathbf{A})$ and $y \in b^{\top}$, it follows that $f = e \vee y \in b^{\top}$. So, $\varphi_{\mathbf{A}}(e)^{\mathrm{c}} \cap Y = \varphi_{\mathbf{A}}(f)^{\mathrm{c}} \cap Y$ and $(e, f) \in \theta(Y)$. Therefore, $a^{\top} \equiv_{\tilde{\theta}(Y)} b^{\top}$. □

Theorem 3.1. *Let $\mathbf{A}$ be a distributive nearlattice and $\theta \in \mathrm{Con}(\mathbf{A})$. Let $Y_{\theta} \in \mathcal{S}(X(\mathbf{A}))$ such that $\theta = \theta(Y_{\theta})$. The following conditions are equivalent:*

(1) *If $a \equiv_{\theta} 1$, then $a^{\top} \equiv_{\tilde{\theta}} A$.*
(2) *θ is an* AP*-congruence.*
(3) *$|a^{\top}|_{\theta} = |a|_{\theta}^{\top}$ for all $a \in A$.*
(4) *If $a \vee b \equiv_{\theta} 1$, then there exists $c \in a^{\top}$ such that $c \equiv_{\theta} b$.*

P r o o f. (1) $\Rightarrow$ (2) Let $a, b \in A$ such that $a \equiv_\theta b$. By Corollary 3.1, we only need to prove that $\gamma(a^\top) \cap Y_\theta = \gamma(b^\top) \cap Y_\theta$. If $P \in \gamma(a^\top) \cap Y_\theta$, then $P \cap a^\top \neq \emptyset$ and $P \in Y_\theta$. So, there exists $c \in P$ such that $a \vee c = 1$. As $a \equiv_\theta b$, we have $b \vee c \equiv_\theta 1$ and by hypothesis $(b \vee c)^\top \equiv_{\tilde{\theta}} A$, i.e.,

$$\gamma((b \vee c)^\top) \cap Y_\theta = \gamma(A) \cap Y_\theta = X(\mathbf{A}) \cap Y_\theta = Y_\theta.$$

On the other hand, $\varphi_{\mathbf{A}}(c)^{\mathrm{c}} \cap \gamma(b^\top) = \varphi_{\mathbf{A}}(c)^{\mathrm{c}} \cap \gamma((b \vee c)^\top)$. Indeed, if $P \in \varphi_{\mathbf{A}}(c)^{\mathrm{c}} \cap \gamma(b^\top)$, then $c \in P$ and $P \cap b^\top \neq \emptyset$. Since $P \cap b^\top \neq \emptyset$, there exists $z \in P$ such that $b \vee z = 1$. It follows that $z \in (b \vee c)^\top$ and $P \cap (b \vee c)^\top \neq \emptyset$, i.e., $P \in \gamma((b \vee c)^\top)$. So, $P \in \varphi_{\mathbf{A}}(c)^{\mathrm{c}} \cap \gamma((b \vee c)^\top)$. The other inclusion is similar. Therefore, $\varphi_{\mathbf{A}}(c)^{\mathrm{c}} \cap \gamma(b^\top) = \varphi_{\mathbf{A}}(c)^{\mathrm{c}} \cap \gamma((b \vee c)^\top)$ and

$$\varphi_{\mathbf{A}}(c)^{\mathrm{c}} \cap \gamma(b^\top) \cap Y_\theta = \varphi_{\mathbf{A}}(c)^{\mathrm{c}} \cap \gamma((b \vee c)^\top) \cap Y_\theta = \varphi_{\mathbf{A}}(c)^{\mathrm{c}} \cap Y_\theta.$$

As $c \in P$, $P \in \varphi_{\mathbf{A}}(c)^{\mathrm{c}} \cap Y_\theta$. Thus, $P \in \gamma(b^\top) \cap Y_\theta$ and $\gamma(a^\top) \cap Y_\theta \subseteq \gamma(b^\top) \cap Y_\theta$. The inclusion $\gamma(b^\top) \cap Y_\theta \subseteq \gamma(a^\top) \cap Y_\theta$ is analogous.

(2) $\Rightarrow$ (3) Let $a \in A$. We see that $|a^\top|_\theta = |a|_\theta^\top$. If $|x| \in |a^\top|_\theta$, then there exists $\bar{x} \in a^\top$ such that $|x|_\theta = |\bar{x}|_\theta$. Thus, $|x|_\theta \vee |a|_\theta = |\bar{x}|_\theta \vee |a|_\theta = |\bar{x} \vee a|_\theta = |1|_\theta$ and $|x|_\theta \in |a|_\theta^\top$. Therefore, $|a^\top|_\theta \subseteq |a|_\theta^\top$.

For the other inclusion, suppose there exists $|x|_\theta \in |a|_\theta^\top$ such that $|x|_\theta \notin |a^\top|_\theta$. Then $x \vee a \equiv_\theta 1$ and we consider the filter $F(|a^\top|_\theta)$. Note that $|x|_\theta \notin F(|a^\top|_\theta)$. Indeed, if $|x|_\theta \in F(|a^\top|_\theta)$, then there exist $|x_1|_\theta, \ldots, |x_n|_\theta \in [|a^\top|_\theta)$ such that $|x_1|_\theta \wedge \ldots \wedge |x_n|_\theta$ exists and $|x|_\theta = |x_1|_\theta \wedge \ldots \wedge |x_n|_\theta$. It is easy to see that $|a^\top|_\theta$ is increasing. So, $|x_1|_\theta, \ldots, |x_n|_\theta \in |a^\top|_\theta$ and there exist $\bar{x}_1, \ldots, \bar{x}_n \in a^\top$ such that $|\bar{x}_i|_\theta = |x_i|_\theta$ for all $i = 1, \ldots, n$. Then $|\bar{x}_1|_\theta, \ldots, |\bar{x}_n|_\theta \in |a^\top|_\theta$ and since $a^\top \in \mathrm{Fi}(\mathbf{A})$,

$$|x|_\theta = |x_1|_\theta \wedge \ldots \wedge |x_n|_\theta = |\bar{x}_1|_\theta \wedge \ldots \wedge |\bar{x}_n|_\theta = |\bar{x}_1 \wedge \ldots \wedge \bar{x}_n|_\theta \in |a^\top|_\theta,$$

which is a contradiction. Then $|x|_\theta \notin F(|a^\top|_\theta)$ and by Theorem 1.2 there exists $P_\theta \in X(\mathbf{A}/\theta)$ such that $|x|_\theta \in P_\theta$ and $P_\theta \cap |a^\top|_\theta = \emptyset$. Since $q_\theta \colon A \to A/\theta$ is a homomorphism onto, $q_\theta^{-1}(P_\theta) = P \in X(\mathbf{A})$ and $P \in Y_\theta$. Then $P \cap a^\top = \emptyset$ and by Lemma 2.2 there exists $Q \in \max X(\mathbf{A})$ such that $P \subseteq Q$ and $a \in Q$.

On the other hand, $x \vee a \equiv_\theta 1$ and as θ is an AP-congruence, $(x \vee a)^\top \equiv_{\tilde{\theta}} 1^\top$ and $\gamma((x \vee a)^\top) \cap Y_\theta = \gamma(1^\top) \cap Y_\theta = X(\mathbf{A}) \cap Y_\theta = Y_\theta$, i.e., $Y_\theta \subseteq \gamma((x \vee a)^\top)$. As $P \in Y_\theta$, $P \in \gamma((x \vee a)^\top)$ and $P \cap (x \vee a)^\top \neq \emptyset$. Then there exists $w \in P$ such that $(x \vee a) \vee w = 1$. Since $|x|_\theta \in P_\theta$, $x \in q_\theta^{-1}(P_\theta) = P$. So, $w, x \in P \subseteq Q$. Also, $a \in Q$ and $(x \vee a) \vee w = 1 \in Q$, which is a contradiction because Q is maximal. Therefore, $|a^\top|_\theta = |a|_\theta^\top$.

(3) $\Rightarrow$ (4) Let $a, b \in A$ such that $a \vee b \equiv_\theta 1$. Then $|a \vee b|_\theta = |a|_\theta \vee |b|_\theta = |1|_\theta$. It follows that $|b|_\theta \in |a|_\theta^\top = |a^\top|_\theta$. Then there exists $c \in a^\top$ such that $c \equiv_\theta b$.

(4) $\Rightarrow$ (1) Let $a \in A$ such that $a \equiv_\theta 1$. If $b \in A$, then $a \vee b \equiv_\theta 1$ and by hypothesis there exists $c \in a^\top$ such that $c \equiv_\theta b$, i.e., $a^\top \equiv_{\tilde{\theta}} A$. $\square$

In [3] we introduce and study a particular class of homomorphisms, called $\top$-*homomorphisms*. We say that a homomorphism $h\colon A \to B$ between distributive nearlattices is a $\top$-homomorphism if $F(h(a^\top)) = h(a)^\top$ for all $a \in A$. The following result will be useful later.

Lemma 3.2 ([3]). *Let* $\mathbf{A}, \mathbf{B}$ *be two distributive nearlattices and* $h\colon A \to B$ *be a* $\top$*-homomorphism. Then* $h^{-1}(P) \in \max X(\mathbf{A})$ *for all* $P \in \max X(\mathbf{B})$.

Theorem 3.2. *Let* $\mathbf{A}$ *be a distributive nearlattice and* $\theta \in \mathrm{Con}(\mathbf{A})$. *Let* $Y_\theta \in \mathcal{S}(X(\mathbf{A}))$ *such that* $\theta = \theta(Y_\theta)$. *The following conditions are equivalent:*

(1) θ *is an* AP*-congruence.*
(2) q_θ *is a* $\top$*-homomorphism.*

Proof. (1) $\Rightarrow$ (2) By Theorem 3.1 we have that $F(|a^\top|_\theta) = F(|a|_\theta^\top) = |a|_\theta^\top$. It follows that $F(|q_\theta(a)^\top|) = q_\theta(a)^\top$ and q_θ is a $\top$-homomorphism.

(2) $\Rightarrow$ (1) Let $a, b \in A$ such that $a \equiv_\theta b$. We see that $a^\top \equiv_{\tilde{\theta}(Y_\theta)} b^\top$, or equivalently by Corollary 3.1, that $\gamma(a^\top) \cap Y_\theta = \gamma(b^\top) \cap Y_\theta$. Let $P \in \gamma(a^\top) \cap Y_\theta$. Then $P \in \bigcup\{\varphi_{\mathbf{A}}(x)^{\mathrm{c}}\colon\ x \in a^\top\}$ and $P \in Y_\theta$, i.e., there exists $e \in a^\top$ such that $P \in \varphi_{\mathbf{A}}(e)^{\mathrm{c}}$ and $P = q_\theta^{-1}(P_\theta)$ for some $P_\theta \in X(\mathbf{A}/\theta)$. So, $e \in P$ and $|e|_\theta \in P_\theta$. Since $e \in a^\top$, $|e|_\theta \in |a^\top|_\theta \subseteq F(|a^\top|_\theta) = |a|_\theta^\top = |b|_\theta^\top$. By hypothesis, q_θ is a $\top$-homomorphism and $|b|_\theta^\top = F(|b^\top|_\theta)$. Then $|e|_\theta \in F(|b^\top|_\theta)$ and there exist $|y_1|_\theta, \ldots, |y_n|_\theta \in [|b^\top|_\theta) = |b^\top|_\theta$ such that $|y_1|_\theta \wedge \ldots \wedge |y_n|_\theta$ exists and $|e|_\theta = |y_1|_\theta \wedge \ldots \wedge |y_n|_\theta$. Thus, there exist $t_1, \ldots, t_n \in b^\top$ such that $|t_i|_\theta = |y_i|_\theta$ for all $i = 1, \ldots, n$. Since $|e|_\theta = |y_1|_\theta \wedge \ldots \wedge |y_n|_\theta \in P_\theta$ and $P_\theta \in X(\mathbf{A}/\theta)$, there exists $j \in \{1, \ldots, n\}$ such that $|y_j|_\theta \in P_\theta$. So, $|t_j|_\theta \in P_\theta$ and $t_j \in q_\theta^{-1}(P_\theta) = P$, i.e., $P \in \varphi_{\mathbf{A}}(t_j)^{\mathrm{c}}$ and as $t_j \in b^\top$, $P \in \bigcup\{\varphi_{\mathbf{A}}(y)^{\mathrm{c}}\colon\ y \in b^\top\}$. Therefore, $P \in \gamma(b^\top) \cap Y_\theta$ and $\gamma(a^\top) \cap Y_\theta \subseteq \gamma(b^\top) \cap Y_\theta$. The other inclusion can be shown similarly. $\square$

Now, we study the structure of the quotient algebra $\mathbf{A}/\theta$ of a distributive nearlattice $\mathbf{A}$ when θ is an AP-congruence. In the following definition we generalize the normal and quasicomplemented lattices studied by Cornish in [12] and [13].

Definition 3.3. Let $\mathbf{A}$ be a distributive nearlattice.

(1) We say that $\mathbf{A}$ is *normal* if each prime ideal is contained in a unique maximal ideal.

(2) We say that $\mathbf{A}$ is *quasicomplemented* if for each $a \in A$ there exists $b \in A$ such that $a^{\top\top} = b^{\top}$, where

$$a^{\top\top} = \{c \in A\colon\ (\forall\, e \in a^{\top})\ (c \vee e = 1)\}.$$

Theorem 3.3. *Let* $\mathbf{A}$ *be a distributive nearlattice and* $\theta \in \mathrm{Con}_{\mathrm{AP}}(\mathbf{A})$.

(1) *If* $\mathbf{A}$ *is normal, then* $\mathbf{A}/\theta$ *is normal.*

(2) *If* $\mathbf{A}$ *is quasicomplemented, then* $\mathbf{A}/\theta$ *is quasicomplemented.*

P r o o f. (1) Let $P \in X(\mathbf{A}/\theta)$ and $U_1, U_2 \in \max X(\mathbf{A}/\theta)$ such that $P \subseteq U_1$ and $P \subseteq U_2$. By Lemma 3.2, $q_\theta^{-1}(U_1), q_\theta^{-1}(U_2) \in \max X(\mathbf{A})$. As $q_\theta^{-1}(P) \subseteq q_\theta^{-1}(U_1) \cap q_\theta^{-1}(U_2)$ and $\mathbf{A}$ is normal, $q_\theta^{-1}(U_1) = q_\theta^{-1}(U_2)$. Thus, $U_1 = q_\theta(q_\theta^{-1}(U_1)) = q_\theta(q_\theta^{-1}(U_2)) = U_2$ and $\mathbf{A}/\theta$ is normal.

(2) Let $|a|_\theta \in X(\mathbf{A}/\theta)$. Then $a \in A$ and as $\mathbf{A}$ is quasicomplemented, there exists $b \in A$ such that $a^{\top\top} = b^{\top}$. We prove that $|a|_\theta^{\top\top} = |b|_\theta^{\top}$, or equivalently by Theorem 3.1, $|a|_\theta^{\top\top} = |a^{\top\top}|_\theta$. If $|x|_\theta \in |a^{\top\top}|_\theta$, then there exists $\bar{x} \in a^{\top\top}$ such that $|\bar{x}|_\theta = |x|_\theta$. Let $|y|_\theta \in |a|_\theta^{\top}$. By Theorem 3.1, $|a|_\theta^{\top} = |a^{\top}|_\theta$. So, there exists $\overline{y} \in a^{\top}$ such that $|\overline{y}|_\theta = |y|_\theta$. Since $\overline{y} \in a^{\top}$ and $\bar{x} \in a^{\top\top}$, $\bar{x} \vee \overline{y} = 1$. So,

$$|x|_\theta \vee |y|_\theta = |\bar{x}|_\theta \vee |\overline{y}|_\theta = |\bar{x} \vee \overline{y}|_\theta = |1|_\theta,$$

i.e., $|x|_\theta \in |a|_\theta^{\top\top}$ and $|a^{\top\top}|_\theta \subseteq |a|_\theta^{\top\top}$.

Let us prove the other inclusion. Let $|x|_\theta \in |a|_\theta^{\top\top}$. As $a^{\top\top} = b^{\top}$, it follows that $a^{\top} = a^{\top\top\top} = b^{\top\top}$. Since $b \in b^{\top\top}$, $b \in a^{\top}$ and $b \vee a = 1$. Then $|b \vee a|_\theta = |b|_\theta \vee |a|_\theta = |1|_\theta$ and $|b|_\theta \in |a|_\theta^{\top}$. So, as $|x|_\theta \in |a|_\theta^{\top\top}$ and $|b|_\theta \in |a|_\theta^{\top}$, $|x|_\theta \vee |b|_\theta = |1|_\theta$, which implies that $|x|_\theta \in |b|_\theta^{\top}$. By hypothesis, θ is an AP-congruence and $|b|_\theta^{\top} = |b^{\top}|_\theta = |a^{\top\top}|_\theta$. Thus, $|x|_\theta \in |a^{\top\top}|_\theta$ and $|a|_\theta^{\top\top} \subseteq |a^{\top\top}|_\theta$. Therefore $|a|_\theta^{\top\top} = |b|_\theta^{\top}$. $\square$

R e m a r k 3.3. Note that for every $Q \in X(\mathbf{A})$ the set $Q^{\top} = \{a \in A\colon\ Q \cap a^{\top} \neq \emptyset\}$ is a filter of $\mathbf{A}$. Since $1^{\top} = A$, $1 \in Q^{\top}$. Let $x \in Q^{\top}$ and $x \leqslant y$. Then $Q \cap x^{\top} \neq \emptyset$ and $x^{\top} \subseteq y^{\top}$. So, $Q \cap y^{\top} \neq \emptyset$ and $y \in Q^{\top}$. Finally, let $x, y \in Q^{\top}$ such that $x \wedge y$ exists. Then $Q \cap x^{\top} \neq \emptyset$ and $Q \cap y^{\top} \neq \emptyset$. It follows that there exists $q_1, q_2 \in Q$ such that $x \vee q_1 = 1$ and $y \vee q_2 = 1$. Let $q = q_1 \vee q_2 \in Q$. So, $x \vee q = 1$, $y \vee q = 1$ and as $[q)$ is a bounded distributive lattice, $(x \wedge y) \vee q = (x \vee q) \wedge (y \vee q) = 1$, i.e., $q \in Q \cap (x \wedge y)^{\top}$ and $x \wedge y \in Q^{\top}$. Therefore, $Q^{\top} \in \mathrm{Fi}(\mathbf{A})$.

We characterize the N-subspaces of $X(\mathbf{A})$ corresponding to AP-congruences.

Theorem 3.4. *Let* $\mathbf{A}$ *be a distributive nearlattice and* $Y \in \mathcal{S}(X(\mathbf{A}))$. *The following conditions are equivalent:*

(1) $\theta(Y)$ *is an* AP-*congruence.*

(2) $\max[Q) \subseteq Y$ *for all* $Q \in Y$.

P r o o f. (1) $\Rightarrow$ (2) Let $Q \in Y$ and $P \in \max[Q)$. Suppose that $P \notin Y$. We consider the family

$$\mathcal{F} = \bigcap\{\varphi_{\mathbf{A}}(b) \cap Y\colon\ \varphi_{\mathbf{A}}(b) \notin H(P)\} \cap \bigcap\{\varphi^{\mathrm{c}}_{\mathbf{A}}(c) \cap Y\colon\ \varphi_{\mathbf{A}}(c) \in H(P)\}.$$

If $\mathcal{F} \neq \emptyset$, then there exists $R \in \mathcal{F}$ such that $H(P) = H(R)$. Since H is 1-1, we have $P = R$ and $P \in Y$, which is a contradiction. So, $\mathcal{F} = \emptyset$ and

$$\bigcap\{\varphi_{\mathbf{A}}(b) \cap Y\colon\ \varphi_{\mathbf{A}}(b) \notin H(P)\} \subseteq \bigcup\{\varphi_{\mathbf{A}}(c) \cap Y\colon\ \varphi_{\mathbf{A}}(c) \in H(P)\}.$$

Let $B = \{b\colon\ \varphi_{\mathbf{A}}(b) \notin H(P)\}$ and $C = \{c\colon\ \varphi_{\mathbf{A}}(c) \in H(P)\}$. As Y is an N-subspace, there exist $b_1, \ldots, b_n \in [B)$ and $c_1, \ldots, c_m \in C$ such that $b_1 \wedge \ldots \wedge b_n$ exists and

$$[\varphi_{\mathbf{A}}(b_1) \cap Y] \cap \ldots \cap [\varphi_{\mathbf{A}}(b_n) \cap Y] \subseteq [\varphi_{\mathbf{A}}(c_1) \cap Y] \cup \ldots \cup [\varphi_{\mathbf{A}}(c_m) \cap Y].$$

Since $b_1, \ldots, b_n \in [B)$, it follows that there exist $\bar{b}_1, \ldots, \bar{b}_n \in B$ such that $\bar{b}_i \leqslant b_i$ for all $i = 1, \ldots, n$. Let $b = b_1 \wedge \ldots \wedge b_n$ and $c = c_1 \vee \ldots \vee c_m$. Then $\varphi_{\mathbf{A}}(b) \cap Y \subseteq \varphi_{\mathbf{A}}(c) \cap Y$ and $\varphi_{\mathbf{A}}(b \vee c)^{\mathrm{c}} \cap Y = \varphi_{\mathbf{A}}(c)^{\mathrm{c}} \cap Y$. So, $(b \vee c, c) \in \theta(Y)$. As $c_1, \ldots, c_m \in C$, then $\varphi_{\mathbf{A}}(c_j) \in H(P)$, i.e., $P \notin \varphi_{\mathbf{A}}(c_j)$ and $c_j \in P$ for all $j = 1, \ldots, m$. Thus, $c \in P$. On the other hand, if $b \in P$, then $b_1 \wedge \ldots \wedge b_n \in P$ and since P is prime, there exists $k \in \{1, \ldots, n\}$ such that $b_k \in P$. As $\bar{b}_k \leqslant b_k$, then $\bar{b}_k \in P$. But $\bar{b}_k \in B$ and $\varphi_{\mathbf{A}}(\bar{b}_k) \notin H(P)$, i.e., $P \in \varphi_{\mathbf{A}}(\bar{b}_k)$ and $\bar{b}_k \notin P$, which is a contradiction. Then $b \notin P$.

We consider the set $Q^{\top} = \{a \in A\colon\ Q \cap a^{\top} \neq \emptyset\}$. By Remark 3.3, $Q^{\top} \in \mathrm{Fi}(\mathbf{A})$. Since P is maximal, $I(P \cup \{b\}) \cap Q^{\top} \neq \emptyset$. Otherwise, if $I(P \cup \{b\}) \cap Q^{\top} = \emptyset$ then there exists $R \in X(\mathbf{A})$ such that $P \subseteq R$, $b \in R$ and $R \cap Q^{\top} = \emptyset$. So, $b \in R$ and $b \notin P$, which is a contradiction because P is maximal. Then $I(P \cup \{b\}) \cap Q^{\top} \neq \emptyset$ and there exists $p \in P$ such that $p \vee b \in Q^{\top}$, i.e., $Q \cap (p \vee b)^{\top} \neq \emptyset$. Thus, there exists $d \in A$ such that $d \in Q \cap (p \vee b)^{\top}$. On the other hand, since $\theta(Y)$ is a congruence and $(b \vee c, c) \in \theta(Y)$, then $(b \vee c \vee p, c \vee p) \in \theta(Y)$. By hypothesis, $\theta(Y)$ is an AP-congruence and $(b \vee c \vee p)^{\top} \equiv_{\tilde{\theta}(Y)} (c \vee p)^{\top}$. As $p \vee b \leqslant b \vee c \vee p$, $(p \vee b)^{\top} \subseteq (b \vee c \vee p)^{\top}$ and $d \in (b \vee c \vee p)^{\top}$. Thus, there exists $f \in (c \vee p)^{\top}$ such that $d \equiv_{\theta(Y)} f$, i.e., $\varphi_{\mathbf{A}}(d)^{\mathrm{c}} \cap Y = \varphi_{\mathbf{A}}(f)^{\mathrm{c}} \cap Y$. Moreover, $Q \in \varphi_{\mathbf{A}}(d)^{\mathrm{c}} \cap Y$. Consequently, $Q \in \varphi_{\mathbf{A}}(f)^{\mathrm{c}} \cap Y$ and $f \in Q$. Since $P \in \max[Q)$, it follows that $Q \subseteq P$ and $f \in P$. Also, $c, p \in P$ and $f \vee c \vee p = 1 \in P$, which is a contradiction. Therefore $P \in Y$ and $\max[Q) \subseteq Y$ for all $Q \in Y$.

(2) $\Rightarrow$ (1) Let $a, b \in A$ such that $a \equiv_{\theta(Y)} b$. Then $\varphi_{\mathbf{A}}(a)^{\mathrm{c}} \cap Y = \varphi_{\mathbf{A}}(b)^{\mathrm{c}} \cap Y$. We prove that $a^{\top} \equiv_{\tilde{\theta}(Y)} b^{\top}$, or equivalently by Corollary 3.1, $\gamma(a^{\top}) \cap Y = \gamma(b^{\top}) \cap Y$. Let $Q \in \gamma(a^{\top}) \cap Y = \bigcup\{\varphi_{\mathbf{A}}(x)^{\mathrm{c}}\colon\ x \in a^{\top}\} \cap Y$. Then there exists $c \in a^{\top}$ such that $Q \in \varphi_{\mathbf{A}}(c)^{\mathrm{c}}$. So, $c \vee a = 1$ and $c \in Q$. Suppose that $Q \notin \gamma(b^{\top}) \cap Y$, i.e., $Q \cap b^{\top} = \emptyset$. Thus, by Lemma 2.2, there exists $P \in \max X(\mathbf{A})$ such that $Q \subseteq P$ and $b \in P$. So,

$P \in \varphi_{\mathbf{A}}(b)^{\mathrm{c}}$. Since $P \in \max[Q] \subseteq Y$, $P \in \varphi_{\mathbf{A}}(b)^{\mathrm{c}} \cap Y = \varphi_{\mathbf{A}}(a)^{\mathrm{c}} \cap Y$ and $a \in P$. Moreover, $c \in Q \subseteq P$ and $c \vee a = 1 \in P$, which is a contradiction. Therefore, $a^{\top} \equiv_{\tilde{\theta}(Y)} b^{\top}$ and $\theta(Y)$ is an AP-congruence. □

Definition 3.4. Let $\mathbf{A}$ be a distributive nearlattice and $Y \in \mathcal{S}(X(\mathbf{A}))$. We say that Y is an APN-*subspace* if $\max[Q] \subseteq Y$ for all $Q \in Y$.

Theorem 3.5. *Let $\mathbf{A}$ be a distributive nearlattice and $\langle X(\mathbf{A}), \mathcal{K}_{\mathbf{A}} \rangle$ be the dual space of $\mathbf{A}$. Then there exists a dual isomorphism between the lattice of* APN-*subspaces of $X(\mathbf{A})$ and the lattice of* AP-*congruences of $\mathbf{A}$.*

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