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SUMS OF MULTIPLICATIVE FUNCTION IN SPECIAL ARITHMETIC PROGRESSIONS

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Abstract. We find, via the Selberg-Delange method, an asymptotic formula for the mean of arithmetic functions on certain APs. It generalizes a result due to Cui and Wu (2014).

Keywords: Selberg-Delange method; multiplicative function; arithmetic progressions

MSC 2010: 11N37

1. Introduction

Many number-theoretic problems lead to the study of mean values of arithmetic functions. In [1] Cui and Wu obtain mean values of certain arithmetic functions over short intervals by using the Selberg-Delange method and zero density estimates of the Riemann zeta function.

In order to state their result, it is necessary to introduce some notations. From [10], Theorem II.5.1, the function

$$Z(s;z) := \frac{((s-1)\zeta(s))^z}{s} \quad (z \in \mathbb{C})$$

is holomorphic in disc $|s-1| < 1$ and admits the Taylor series expansion

$$Z(s;z) = \sum_{j=0}^{\infty} \frac{\gamma_j(z)}{j!}(s-1)^j, \tag{1.1}$$

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where $\gamma_j(z)$ are entire functions of z that satisfy for all $B > 0$ and $\varepsilon > 0$ the estimate

$$\frac{\gamma_j(z)}{j!} \ll_{B,\varepsilon} (1+\varepsilon)^j \quad (j \geqslant 0,\ |z| \leqslant B).$$

The following is the result due to Cui and Wu.

Theorem 1.1. *Let $f(n) \ll_\varepsilon n^\varepsilon$ be a multiplicative funtion and let $\kappa > 0$, $w \in \mathbb{C}$, $\alpha > 0$, $\delta \geqslant 0$, $A \geqslant 0$, $B > 0$, $M > 0$ be some constants. Suppose that the Dirichlet series $\mathcal{F}(s) := \sum_{n=1}^{\infty} f(n)n^{-s}$ is of type $\mathcal{P}(\kappa, w, \alpha, \delta, A, B, M)$. Then for any $\varepsilon > 0$ we have*

$$\sum_{x<n\leqslant x+y} f(n) = y(\log x)^{\kappa-1}\left(\sum_{l=0}^{N} \frac{\lambda_l(\kappa, w)}{(\log x)^l} + O(R_N(x,y))\right) \tag{1.2}$$

uniformly for

$$x \geqslant y \geqslant x^{\theta(\kappa,\delta)+\varepsilon} \geqslant 2, \quad N \geqslant 0, \quad 0 \leqslant \kappa \leqslant B, \quad |w| \leqslant B,$$

where $\theta(\kappa,\delta) := (5\kappa + 15\delta + 21)/(5\kappa + 15\delta + 36)$, $\lambda_l(\kappa, w) := g_l(\kappa, w)/\Gamma(\kappa - l)$,

$$g_l(\kappa, w) := \frac{1}{l!}\sum_{j=0}^{l}\binom{l}{j}\frac{\partial^{l-j}(G(s;\kappa,w)\zeta(2s)^{-w})}{\partial s^{l-j}}\Big|_{s=1}\gamma_j(\kappa),$$

where $\gamma_j(\kappa)$ is defined by (1.1) and

$$R_N(x,y) := \frac{y}{x}\sum_{l=1}^{N+1}\frac{l|\lambda_{l-1}(\kappa,w)|}{(\log x)^l} + M\left(\left(\frac{c_1 N + 1}{\log x}\right)^{N+1} + \frac{(c_1N+1)^{N+1}}{\mathrm{e}^{c_2(\log x)^{1/3}(\log_2 x)^{-1/3}}}\right)$$

for some constants $c_1 > 0$ and $c_2 > 0$. The implied constant in the O-term depends only on A, B, α, δ and ε.

Note. Let $\kappa > 0$, $w \in \mathbb{C}$, $\alpha > 0$, $\delta \geqslant 0$, $A \geqslant 0$, $B > 0$, $M > 0$ be some constants. A Dirichlet series $\mathcal{F}(s) := \sum_{n=1}^{\infty} f(n)n^{-s}$ is said to be of type $\mathcal{P}(\kappa, w, \alpha, \delta, A, B, M)$ if the following conditions are verified:

(a) for any $\varepsilon > 0$ we have

$$|f(n)| \ll_\varepsilon n^\varepsilon \quad (n \geqslant 1), \tag{1.3}$$

where the implied constant depends only on ε;

(b) we have

$$\sum_{n=1}^{\infty} |f(n)| n^{-\sigma} \leqslant (\sigma - 1)^{-\alpha} \quad (\sigma > 1);$$

(c) the Dirichlet series

$$G(s;\kappa,w) := \mathcal{F}(s)\zeta(s)^{-z}\zeta(2s)^{w} \tag{1.4}$$

can be analytically continued to a holomorphic function in (an open set containing) $\sigma \geqslant \frac{1}{2}$ and, in this region, $G(s;\kappa,w)$ satisfies the bound

$$|G(s;\kappa,w)| \leqslant M(|\tau|+1)^{\max(\delta(1-\sigma),0)} \log^{A}(|\tau|+1) \tag{1.5}$$

uniformly for $0 < \kappa \leqslant B$ and $|w| \leqslant B$.

It is natural to consider the analogous result of (1.2) for arithmetic progressions $\bmod\, q$. Nevertheless, the problem in arithmetic progressions is more difficult for large moduli q, partly because of the possible Siegel zero, and partly because of a zero-free region for the L-function (excluding the Siegel zero). So we only consider some special arithmetic progressions following the idea of Gallagher [4]. In this note, we prove the following results.

Theorem 1.2. *Let p_0 be a fixed odd prime, $q = p_0^r$ be an integer with r an integer and l be an integer such that $(l,q) = 1$ and let $f(n)$ be a multiplicative function such that $f(p) = \alpha$ $(0 < \alpha < 1)$, $f(p^\nu) \ll p^{\delta\nu}$ $(\nu \geqslant 2)$ for some $\delta < 0$. For any $\varepsilon > 0$, $0 < \theta < 1$ and $q \leqslant x^{15/(2(10\alpha+21))-\varepsilon}$,*

$$\sum_{\substack{n \leqslant x \\ n \equiv l \pmod q}} f(n) = \frac{x}{\varphi(q)} \biggl(\sum_{0 \leqslant k \leqslant N} \frac{a_k}{(\log x)^{k+1-\alpha}} + O(R_N(x)) \biggr) \tag{1.6}$$

uniformly for $x \geqslant 2$, $N \geqslant 0$, where

$$a_k := \frac{1}{\Gamma(\alpha - k)} \sum_{h+j=k} \frac{G^{(h)}(1,\chi_0)\gamma_j(\alpha)}{h!\, j!},$$

$$G(s,\chi_0) := \prod_p \biggl(1 + \frac{\chi_0(p) f(p)}{p^s} + \sum_{\nu=2}^{\infty} \frac{\chi_0(p^\nu) f(p^\nu)}{p^{\nu s}}\biggr)\biggl(1 - \frac{\chi_0(p)}{p^s}\biggr)^{\alpha},$$

$$R_N(x) := O\biggl(\Bigl(\frac{1}{\log x}\Bigr)^{N+2-\alpha} + \mathrm{e}^{-\log^{(1-\theta)/2} x}\biggr)$$

and the implied constant depends only on ε, $\gamma_j(z)$ is defined by (1.1).

Note that for generally large moduli, at present there seems to be little hope for a proof of this, the reason being the Grand Riemman Hypothesis.

2. Notations and preliminaries

Throughout the paper we will use the following notations: $s = \sigma + \mathrm{i}\tau$, ε always denotes a sufficiently small positive number, p denotes a prime number, the parameters T and x are sufficiently large real numbers. When we write $f = O(g)$ or $f \ll g$ we will mean $|f| \leqslant Cg$ for some absolute constant C. When implied constants depend upon some parameters, we sometimes indicate that by a subscript.

Our work is inspired by Selberg-Delange method, which was developed by Selberg [9] and Delange [2], [3]. The method has been applied to some arithmetic problems, see [5], [6] and [7]. For more details, the reader is referred to the book by Tenenbaum [10].

3. Some lemmas

Lemma 3.1. *Let* $F(s) := \sum_{n=1}^{\infty} a_n n^{-s}$ *be a Dirichlet series with finite abscissa of absolute convergence* σ_a. *Suppose that there exists a real number* $\eta > 0$ *such that:*

(i) $\sum_{n=1}^{\infty} |a_n| n^{-\sigma} \ll (\sigma - \sigma_a)^{-\eta} \quad (\sigma > \sigma_a)$,
and that $B(n)$ *is a nondecreasing function satisfying*

(ii) $|a_n| \leqslant B(n) \quad (n \geqslant 1)$.

Then for $x \geqslant 2$, $T \geqslant 2$, $\sigma \leqslant \sigma_a$, $\kappa := \sigma_a - \sigma + 1/\log x$ *we have*

$$\sum_{n \leqslant x} \frac{a_n}{n^s} = \frac{1}{2\pi \mathrm{i}} \int_{\kappa + \mathrm{i}T}^{\kappa - \mathrm{i}T} F(s+w) x^w \frac{\mathrm{d}w}{w} + O\Big(x^{\sigma_a - \sigma} \frac{(\log x)^\eta}{T} + \frac{B(2x)}{x^\sigma}\Big(1 + x \frac{\log T}{T}\Big)\Big).$$

Proof. See [10], Corollary II.2.2.1. □

Lemma 3.2. *Let* $q > 2$ *be an integer and* χ *be Dirichlet character modulo* q. *Then we have*

$$L(\sigma + \mathrm{i}\tau, \chi) \ll q^{1-\sigma} (|\tau| + 1)^{1/6} \ln(|\tau| + 1).$$

Proof. See [8], Theorem 1, page 485. □

Lemma 3.3. *Let* $q > 2$ *be an integer and* χ *be a non principal Dirichlet character modulo* q. *For* $s = \sigma + \mathrm{i}\tau$, $0 < \varepsilon < \frac{1}{2}$, $\varepsilon \leqslant \sigma < 1$, $|\tau| + 2 \leqslant T$ *we have*

$$L(\sigma + \mathrm{i}\tau, \chi) \ll_\varepsilon (q^{1/2} T)^{1-\sigma+\varepsilon}.$$

P r o o f. The lemma is Exercise 241 of [11]. $\square$

Lemma 3.4. *For $q = p_0^r$ (p_0 odd prime), the Dirichlet L-function to modulus q has no zeros in region*

$$\sigma > 1 - \frac{c}{\log^{\gamma}(q(|t|+2))}, \quad \gamma < 1. \tag{3.1}$$

P r o o f. See [4], Theorem 2. $\square$

Lemma 3.5. *Let $f(n)$ be a multiplicative function such that $f(p) = \alpha$ $(0 < \alpha < 1)$, $f(p^\nu) \ll p^{\delta\nu}$ $(\nu \geqslant 2)$ for some $\delta < 0$ and χ_0 be the principal character to the modulus q. For any $\varepsilon > 0$,*

$$\sum_{n \leqslant x} \chi_0(n) f(n) = \frac{x}{(\log x)^{1-\alpha}} \Bigg(\sum_{0 \leqslant k \leqslant N} \frac{a_k}{(\log x)^k} + O_\varepsilon\Big(\frac{1}{(\log x)^{N+1}}\Big)\Bigg) \tag{3.2}$$

holds for $x \geqslant 2$, where

$$a_k := \frac{1}{\Gamma(\alpha - k)} \sum_{h+j=k} \frac{G^{(h)}(1)\gamma_j(\alpha)}{h!\, j!},$$

$$G(s, \chi_0) := \prod_p \Bigg(1 + \frac{\chi_0(p) f(p)}{p^s} + \sum_{\nu=2}^{\infty} \frac{\chi_0(p^\nu) f(p^\nu)}{p^{\nu s}}\Bigg)\Big(1 - \frac{\chi_0(p)}{p^s}\Big)^{\alpha}$$

and $\gamma_j(z)$ is defined by (1.1).

P r o o f. We write for $\operatorname{Re} s > 1$,

(3.3)

$$\begin{aligned}
\mathcal{F}(s, \chi_0) &:= \sum_{n=1}^{\infty} \frac{\chi_0(n) f(n)}{n^s} = \prod_p \Bigg(1 + \frac{\chi_0(p) f(p)}{p^s} + \sum_{\nu=2}^{\infty} \frac{\chi_0(p^\nu) f(p^\nu)}{p^{\nu s}}\Bigg) \\
&= \prod_p \Bigg(1 + \frac{\chi_0(p) f(p)}{p^s} + \sum_{\nu=2}^{\infty} \frac{\chi_0(p^\nu) f(p^\nu)}{p^{\nu s}}\Bigg)\Big(1 - \frac{\chi_0(p)}{p^s}\Big)^{\alpha}\Big(1 - \frac{\chi_0(p)}{p^s}\Big)^{-\alpha} \\
&= L(s, \chi_0)^{\alpha} G(s, \chi_0) = (s-1)^{-\alpha} \cdot \prod_{p|q} \Big(1 - \frac{1}{p^s}\Big)^{\alpha} ((s-1)\zeta(s))^{\alpha} \cdot G(s, \chi_0) \\
&=: (s-1)^{-\alpha} \cdot Z(s; \alpha) \cdot G(s, \chi_0),
\end{aligned}$$

where $Z(s; \alpha)$ is holomorphic and $O_q(M)$ in the disc $|s-1| \leqslant c$ $(0 < c < 1/10)$ and $G(s, \chi_0)$ is expandable as a Dirichlet series

$$G(s, \chi_0) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s},$$

where $g(n)$ is a multiplicative function for which the values on primes' powers are determined by the identity

$$1+\sum_{\nu\geqslant 1} g(p^\nu)\xi^\nu = (1-\xi)^\alpha \sum_{\nu\geqslant 0} f(p^\nu)\xi^\nu \quad (|\xi|\leqslant 1).$$

In particular, we have $g(p)=0$ and Cauchy inequality implies that $|g(p^\nu)| \ll_\varepsilon p^{\varepsilon\nu}$ ($\varepsilon>0$). So we have shown that for $\sigma > \frac{1}{2}+\varepsilon$

$$\sum_p \sum_{\nu\geqslant 1} |g(p^\nu)| p^{-\nu\sigma} \leqslant \sum_p \frac{1}{p^{\sigma-\varepsilon}(p^{\sigma-\varepsilon}-1)} \leqslant \frac{c}{(\sigma-\varepsilon)-1/2},$$

where c is an absolute constant. Then we deduce that $G(s,\chi_0)$ is absolutely convergent for $\sigma>\frac{1}{2}+\varepsilon$ and $G(s,\chi_0)\ll_\varepsilon 1$. By using Selberg-Delange theorem [10], Theorem III. 5.3, we obtain (3.2). □

Lemma 3.6. *Let $f(n)$ be a multiplicative function such that $f(p)=\alpha$ $(0<\alpha<1)$, $f(p^\nu)\ll p^{\delta\nu}$ $(\nu\geqslant 2)$ for some $\delta<0$ and χ be a nonprincipal Dirichlet character modulo q, where $q=p_0^r$ (p_0 odd prime). For any $0<\varepsilon<\frac{1}{2}$, $0<\theta<1$ and $q\leqslant x^{15/(2(10\alpha+21))-\varepsilon}$,*

$$\sum_{\chi\neq\chi_0} \max_{y\leqslant x}\Big|\sum_{n\leqslant y}\chi(n)f(n)\Big| = O\big(x\exp(-\log^{(1-\theta)/2} x)\big) \tag{3.4}$$

holds for $x\geqslant 2$, where the implied constant depends only on ε.

P r o o f. As in Lemma 3.5, we have

$$\begin{aligned}\mathcal{F}(s,\chi) &:= \sum_{n=1}^{\infty}\frac{\chi(n)f(n)}{n^s} = \prod_p\Big(1+\frac{\chi(p)f(p)}{p^s}+\sum_{\nu=2}^{\infty}\frac{\chi(p^\nu)f(p^\nu)}{p^{\nu s}}\Big) \\ &= \prod_p\Big(1+\frac{\chi(p)f(p)}{p^s}+\sum_{\nu=2}^{\infty}\frac{\chi(p^\nu)f(p^\nu)}{p^{\nu s}}\Big)\Big(1-\frac{\chi(p)}{p^s}\Big)^\alpha\Big(1-\frac{\chi(p)}{p^s}\Big)^{-\alpha} \\ &=: L(s,\chi)^\alpha G(s,\chi),\end{aligned} \tag{3.5}$$

where

$$G(s,\chi)=\prod_p\Big(1+\frac{\chi(p)f(p)}{p^s}+\sum_{\nu=2}^{\infty}\frac{\chi(p^\nu)f(p^\nu)}{p^{\nu s}}\Big)\Big(1-\frac{\chi(p)}{p^s}\Big)^\alpha.$$

As similarly shown in Lemma 3.5, we easily see that $G(s,\chi)\ll_\varepsilon 1$ for $\operatorname{Re} s>\frac{1}{2}+\varepsilon$.

We can apply Lemma 3.1 with the choice of parameters $\sigma_a=1$, $B(n)=n^\varepsilon$, $\eta=\eta$, $\sigma=0$ and $T=x^{15/(2(10\alpha+21))+\sqrt{\varepsilon}}$ to write

$$\sum_{n\leqslant y}\chi(n)f(n)=\frac{1}{2\pi\mathrm{i}}\int_{b-\mathrm{i}T}^{b+\mathrm{i}T}\mathcal{F}(s,\chi)\frac{y^s}{s}\,\mathrm{d}s+O\Big(\frac{y^{1+\varepsilon}}{T}\Big),$$

where $b=1+2/\log x$ and $100\leqslant T\leqslant x$ such that $L(\sigma+\mathrm{i}T,\chi)\neq 0$ for $0<\sigma<1$.

Let $\mathcal{L}$ be the boundary of the modified rectangle with vertices $(\frac{1}{2}+\varepsilon)\pm \mathrm{i}T$ and $b\pm \mathrm{i}T$, where

- ▷ $\varepsilon>0$ is a small constant chosen so that $L(\frac{1}{2}+\varepsilon+\mathrm{i}\gamma,\chi)\neq 0$ for $|\gamma|<T$, and
- ▷ the zeros of $L(s,\chi)$ of the form $\varrho=\beta+\mathrm{i}\gamma$ with $\beta>\frac{1}{2}$ and $|\gamma|<T$ are avoided by the horizontal cut drawn from the critical line inside this rectangle to $\varrho=\beta+\mathrm{i}\gamma$.
- ▷ $\mathcal{L}_1$ and $\mathcal{L}_2$ denote horizontal segment $[(\frac{1}{2}+\varepsilon)\pm \mathrm{i}T,(1+2/\log x)\pm \mathrm{i}T]$, $\mathcal{L}_3$ and $\mathcal{L}_4$ denote vertical segment $[(\frac{1}{2}+\varepsilon),(\frac{1}{2}+\varepsilon)\pm \mathrm{i}T]\setminus\{\varrho\colon\ \varrho=(\frac{1}{2}+\varepsilon)\pm \mathrm{i}\gamma\}$, Γ_ϱ denotes horizontal segment $[(\frac{1}{2}+\varepsilon)\pm \mathrm{i}\gamma,\beta\pm \mathrm{i}\gamma]$.

Clearly the function $\mathcal{F}(s,\chi)$ is analytic inside $\mathcal{L}$. By Cauchy residue theorem, we can write

$$\sum_{n\leqslant y}\chi(n)f(n)=I_1+\ldots+I_4+\sum_{\substack{\beta>\frac{1}{2}+\varepsilon\\ |\gamma|<T}} I_\varrho+O_\varepsilon\Big(\frac{y^{1+\varepsilon}}{T}\Big), \tag{3.6}$$

where

$$I_j:=\frac{1}{2\pi \mathrm{i}}\int_{\mathcal{L}_j}\mathcal{F}(s,\chi)\frac{y^s}{s}\,\mathrm{d}s$$

and

$$I_\varrho:=\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_\varrho}\mathcal{F}(s,\chi)\frac{y^s}{s}\,\mathrm{d}s.$$

A. Estimation of I_1 and I_2. In view of (3.5) and Lemma 3.3, we have

$$\mathcal{F}(s,\chi)\ll (q^{1/2}T)^{\alpha(1-\sigma)+\varepsilon}. \tag{3.7}$$

Thus

$$\begin{aligned}|I_1|+|I_2| &\ll \int_{1/2+\varepsilon}^{1+2/\log x}(q^{1/2}T)^{\alpha(1-\sigma)+\varepsilon}\cdot\frac{y^\sigma}{T}\,\mathrm{d}\sigma\\ &\ll \frac{y}{T}\int_{1/2+\varepsilon}^{1+2/\log x}\Big(\frac{(q^{1/2}T)^\alpha}{y}\Big)^{1-\sigma}\mathrm{d}\sigma \ll \frac{y}{T}\cdot \mathrm{e}^{(1/2-\varepsilon)\log((q^{1/2}T)^\alpha/y)}.\end{aligned} \tag{3.8}$$

B. Estimation of I_3 and I_4. For $s=(\frac{1}{2}+\varepsilon)+\mathrm{i}\tau$ with $0\leqslant|\tau|\leqslant T$, in view of (3.5) and Lemma 3.2, we have

$$\mathcal{F}(s,\chi)\ll q^{\alpha/2}(|\tau|+1)^{\alpha/6+\varepsilon}. \tag{3.9}$$

Thus

$$|I_3|+|I_4|\ll\int_0^T(q^{\alpha/2}(|\tau|+1)^{\alpha/6+\varepsilon})\frac{y^{1/2+\varepsilon}}{|(1/2+\varepsilon)+\mathrm{i}\tau)|}\,\mathrm{d}\tau\ll y^{1/2+\varepsilon}q^{\alpha/2}T^{\alpha/6}. \tag{3.10}$$

C. Estimation of I_ϱ. For $s=\sigma+\mathrm{i}\gamma$ with $\frac{1}{2}+\varepsilon<\sigma\leqslant\beta\leqslant 1-\sigma_0(\gamma)$, in view of (3.5) and Lemma 3.2, we have

$$\mathcal{F}(s,\chi)\ll q^{\alpha(1-\sigma)}|\gamma|^{\alpha/6+\varepsilon}.$$

Then we deduce that

$$I_\varrho \ll \int_{1/2+\varepsilon}^{\beta}(q^{\alpha(1-\sigma)}|\gamma|^{\alpha/6+\varepsilon})\frac{y^\sigma}{|\sigma+\mathrm{i}\gamma|}\,\mathrm{d}\sigma. \tag{3.11}$$

Denote by $N(\sigma,T,\chi)$ the number of zeros of $L(s,\chi)$ in the region $\operatorname{Re}s\geqslant\sigma$ and $|\operatorname{Im}s|\leqslant T$. Summing (3.11) over $|\gamma|<T$ and interchanging the summations, we have

$$\sum_{\substack{\beta>1/2+\varepsilon\\|\gamma|<T}}|I_\varrho|\ll \log T\max_{T_0<T}\int_{1/2+\varepsilon}^{1-\sigma_0(T_0)}q^{\alpha(1-\sigma)}T_0^{\alpha/6+\varepsilon}\cdot\frac{y^\sigma}{T_0}\cdot N(\sigma,T_0,\chi)\,\mathrm{d}\sigma.$$

It is well-known that

$$N(\sigma,T,q):=\sum_{\chi \bmod q}N(\sigma,T,\chi)\ll (qT)^{(12/5)(1-\sigma)}(\ln qT)^9 \tag{3.12}$$

and in view of Lemma 3.4,

$$\sigma_0(t)=\frac{c}{\log^\gamma(q(|t|+2))},\quad \gamma<1,$$

for $\frac{1}{2}+\varepsilon\leqslant\sigma\leqslant 1$ and $T\geqslant 2$. Thus

$$\begin{aligned}
\sum_{\chi\neq\chi_0}\sum_{\substack{\beta>1/2+\varepsilon\\|\gamma|<T}}|I_\varrho| &\\
&\ll \log T\max_{T_0<T}\int_{1/2+\varepsilon}^{1-\sigma_0(T_0)}q^{\alpha(1-\sigma)}T_0^{\alpha/6+\varepsilon}\cdot\frac{y^\sigma}{T_0}\cdot(qT_0)^{(12/5)(1-\sigma)}\log(qT_0)^9\,\mathrm{d}\sigma\\
&\ll y\log T\max_{T_0<T}\log(qT_0)^9\int_{1/2+\varepsilon}^{1-\sigma_0(T_0)}\left(\frac{q^{\alpha+12/5}T_0^{12/5}}{y}\right)^{1-\sigma}\cdot\frac{1}{T_0^{1-\alpha/6}}\,\mathrm{d}\sigma\\
&\ll y\log T\max_{T_0<T}\log(qT_0)^9\int_{1/2+\varepsilon}^{1-\sigma_0(T_0)}\left(\frac{q^{\alpha+12/5}T_0^{2/5+\alpha/3}}{y}\right)^{1-\sigma}\mathrm{d}\sigma\\
&\ll y\log T\max_{T_0<T}\log(qT_0)^9\left(\frac{q^{\alpha+12/5}T_0^{2/5+\alpha/3}}{y}\right)^{\sigma_0(T_0)}\\
&\ll y\log T\log(qT)^9\left(\frac{q^{\alpha+12/5}T^{2/5+\alpha/3}}{y}\right)^{\sigma_0(T)}.
\end{aligned}\tag{3.13}$$

Inserting (3.8), (3.10) and (3.13) into (3.6), we find that

$$\sum_{\chi\neq\chi_0}\max_{y\leqslant x}\Big|\sum_{n\leqslant y}\chi(n)f(n)\Big| = \Big(\frac{qx}{T}\cdot \mathrm{e}^{(1/2-\varepsilon)\log((q^{1/2}T)^{\alpha}/x)} + q^{1+\alpha/2}x^{1/2}T^{\alpha/6+\varepsilon}$$
$$+ x\log T\log(qT)^9\Big(\frac{q^{\alpha+12/5}T^{2/5+\alpha/3}}{x}\Big)^{\sigma_0(T)}\Big) + O_\varepsilon\Big(\frac{qx^{1+\varepsilon}}{T}\Big).$$

For $q\leqslant x^{15/(2(10\alpha+21))-\varepsilon}$, noting $T = x^{15/(2(10\alpha+21))+\sqrt{\varepsilon}}$ and $0<\alpha<1$, we obtain (3.4). □

4. Proof of Theorem 1.2.

We are now ready to prove Theorem 1.2.

Proof. The starting point of the proof is the following observation.

$$\sum_{\substack{n\leqslant x\\ n\equiv l \ (\mathrm{mod}\ q)}} f(n) = \frac{1}{\varphi(q)}\sum_{\chi \ (\mathrm{mod}\ q)}\bar{\chi}(l)\sum_{n\leqslant x}\chi(n)f(n)$$
$$= \frac{1}{\varphi(q)}\Big(\sum_{n\leqslant x}\chi_0(n)f(n) + \sum_{\chi\neq\chi_0}\bar{\chi}(l)\sum_{n\leqslant x}\chi(n)f(n)\Big).$$

According to Lemma 3.5 and Lemma 3.6, we have the following result:

$$\sum_{\substack{n\leqslant x\\ n\equiv l \ (\mathrm{mod}\ q)}} f(n) = \frac{x}{\varphi(q)}\Big(\sum_{0\leqslant k\leqslant N}\frac{a_k}{(\log x)^{k+1-\alpha}}$$
$$+ O_\varepsilon\Big(\frac{1}{(\log x)^{N+2-\alpha}}\Big) + O\big(\mathrm{e}^{-\log^{(1-\theta)/2}x}\big)\Big).$$

This completes the proof of Theorem 1.2. □

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