Petr Salač; Jan Stebel

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Applications of Mathematics, Vol. 64 (2019), No. 2, 195–224

Persistent URL: `http://dml.cz/dmlcz/147663`

SHAPE OPTIMIZATION FOR A TIME-DEPENDENT MODEL OF A CAROUSEL PRESS IN GLASS PRODUCTION

Petr Salač, Jan Stebel, Liberec

Received October 30, 2018. Published online March 18, 2019.

Abstract. This contribution presents the shape optimization problem of the plunger cooling cavity for the time dependent model of pressing the glass products. The system of the mould, the glass piece, the plunger and the plunger cavity is considered in four consecutive time intervals during which the plunger moves between 6 glass moulds.

The state problem is represented by the steady-state Navier-Stokes equations in the cavity and the doubly periodic energy equation in the whole system, under the assumption of rotational symmetry, supplemented by suitable boundary conditions.

The cost functional is defined as the squared weighted L^2 norm of the difference between a prescribed constant and the temperature of the plunger surface layer at the moment before separation of the plunger and the glass piece.

The existence and uniqueness of the solution to the state problem and the existence of a solution to the optimization problem are proved.

Keywords: shape optimization; Navier-Stokes equations; heat transfer

MSC 2010: 49Q10, 76D55, 93C20

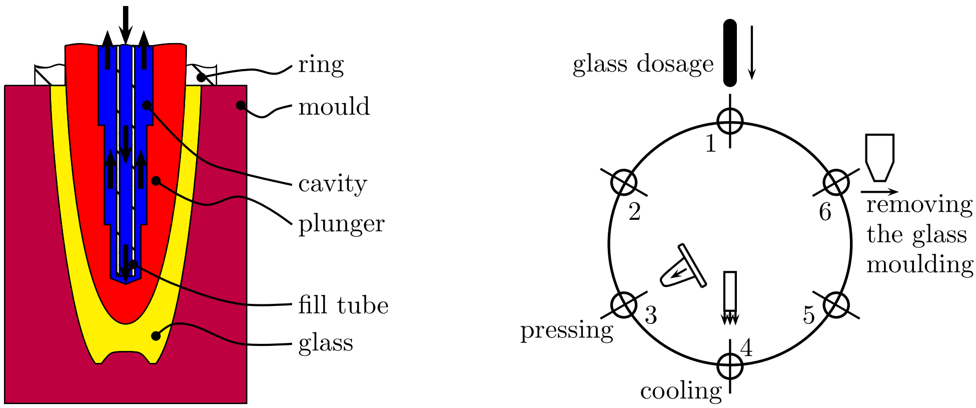
1. Introduction

1.1. Technical motivation. We are motivated by the industrial production of the glass vase of height 250 mm and weight 1.9 kg made from lead crystal glassware. The vase is formed in a three-segment mould which is a part of the carousel press NOVA [9] where the plunger handles successively 6 moulds. The mould is made from austenitic nickel steel, while the ring, the plunger, the crown and the glass mould basket are made from gray cast iron (see Fig. 1 a)). The whole press working cycle takes 162 seconds while the duty cycle of the plunger lasts 27 seconds. The glass is dosed into the mould during the first second (see Fig. 1 b), position 1), then

The research has been supported by TUL.

DOI: 10.21136/AM.2019.0301-18

rotation to position 3 follows. Pressing takes place at the time from 3.5 s to 4.5 s. In the period from 4.5 s to 13.5 s the plunger is in its lower position where it enforces cooling of the vase by conduction. At the time 13.5 s the plunger raises and the vase is cooled by free convection on the inner side and by forced conduction to the outer side until the time 88 s, when the vase is removed from the mould. Then the mould is prepared for pressing another piece until the time 162 s, when the mould is filled again.



a) Scheme of the system. b) Scheme of the carousel press.

Figure 1.

For an ideal surface finish of the moulded piece it is necessary to obtain approximately constant optimal temperature on the outward surface of the plunger at the moment of plunger removal. If some part of the plunger surface is too hot at the moment of separation, then the glass melt adheres to the device and deformation of the moulded piece follows. On the other hand, if the surface of the plunger is too cold, then small fire cracks appear on the surface of the moulded glass piece. Both means less quality of production. The temperature of the plunger is controlled by cooling of the plunger by flowing water, which is forced to the cavity located in the plunger axis. The goal of the optimization is to find an optimal inner shape of the cavity, which allows us to cool down the plunger in such a way to attain prescribed constant temperature on its outward surface.

Glass forming is a complex process involving many challenges from the point of mathematical modelling. The research topics include the nonisothermal flow of glass, fibres drawing, radiative heat transfer or glass moulding. Scientific studies usually focus on detailed analysis of particular component of the system or on a "global" model with simplified description of components. For a nice overview of state-of-the-art and list of relevant references we mention the book [2]. The present paper deals with a mathematical model of fluid flow and heat transfer in several interacting components of the carousel press whose dynamics is described by appropriate partial differential equations.

Compared to the previous work [13], where steady heat transfer and potential flow was considered, here we deal with the time-dependent energy equation describing the heat conduction during the working cycle of the carousel press with different periods in the plunger and in the mould, coupled with the steady-state Navier-Stokes equations for the flow of cooling water in the plunger cavity. The model is studied in rotationally symmetric setting. We formulate the optimization problem under the assumption of nonunique flow field. The target is to reach prescribed optimal temperature on the outward layer of the plunger at the moment of separation from the glass moulding. We focus on the rigorous mathematical formulation of the problem and establish the existence of its solution.

The main challenge in the proof of well-posedness of the flow problem is to obtain solution bounds that are uniform with respect to the shape of the domain. Due to the nonhomogeneous Dirichlet boundary condition, it relies on a special extension of the boundary value that allows an appropriate estimate of the convective term. Since in the rotationally symmetric case the existence of such an extension is not known, we overcome this by transforming the problem to the full 3D setting.

The considered heat transfer problem is nonstandard in two aspects: First, it combines an initial-value problem and time-periodic problems with two different periods. Second, the interface conditions between the four parts of the domain are changing with time. These properties require suitable function spaces (see e.g. [16], [8] for theory of time-periodic problems). The existence and uniqueness of the weak solution is shown using the Rothe method [11] and a fixed-point argument. This approach can also be used for the numerical solution, together with suitable discretization in space (see e.g. [4], [15]).

The existence of a solution of the optimization problem is proved using the compactness of the set of admissible domains, the continuity of the control-to-state mapping in the weak topology of appropriate spaces and the weak lower semicontinuity of the cost functional. We are able to show only weak convergence of sequences of minimizers due to the mere boundedness of the coefficients of heat conduction.

1.2. Structure of the paper. The paper is organized as follows. In the rest of this section we introduce the geometrical setting and definitions of function spaces. Then, in Section 2 we present the fluid flow model and prove the existence of weak solutions that are bounded for a class of admissible domains having the uniform cone property [1]. The main result of this section is the continuous dependence of the velocity field on the shape of the flow domain. Section 3 deals with the heat transfer problem for which we give the weak formulation and establish the existence and uniqueness result, which is then proved in Section 5. In Section 4 we give the precise form of the shape optimization problem, allowing for possible nonuniqueness of the flow field. The existence of an optimal solution is then proved using a continuity result for the solutions of the heat transfer problem.

1.3. Description of the geometry. Due to the rotational symmetry of the system we transform the problem to the cylindrical coordinates and reduce the angle coordinate so that we work with the coordinates $(x, r) \in \mathcal{U} := \mathbb{R} \times [0, \infty)$. Further we rotate the system to the horizontal position. Let us consider the system of four adjacent nonempty domains representing a planar cross section of the mould Ω_{M}, the glass piece Ω_{G}, the plunger Ω_{P} and the cavity of the plunger Ω_{C}, situated in the left part of the xr halfplane (see Fig. 2). The part consisting of the cavity and the plunger will be denoted $\Omega_{\mathrm{CP}} := \operatorname{Int} \overline{\Omega_{\mathrm{C}} \cup \Omega_{\mathrm{P}}}$, the part glass-mould $\Omega_{\mathrm{GM}} := \operatorname{Int} \overline{\Omega_{\mathrm{G}} \cup \Omega_{\mathrm{M}}}$ and the whole system $\Omega := \operatorname{Int} \overline{\Omega_{\mathrm{C}} \cup \Omega_{\mathrm{P}} \cup \Omega_{\mathrm{G}} \cup \Omega_{\mathrm{M}}}$.

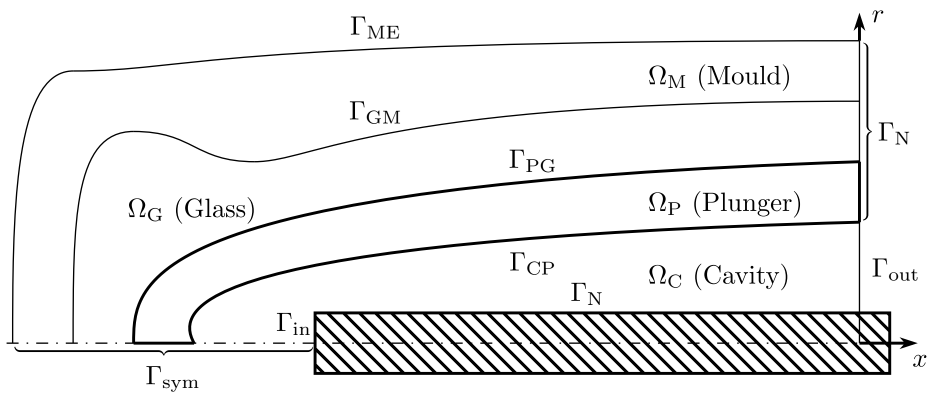


Figure 2. Scheme of the system mould, glass piece, plunger, cavity of plunger and supply tube.

Furthermore, we denote the following boundary segments (see Fig. 2):

▷ Γ_{PG} the boundary between the plunger Ω_{P} and the moulded piece Ω_{G},
▷ Γ_{CP} the boundary between the plunger Ω_{P} and the plunger cavity Ω_{C},
▷ Γ_{N} the thermally isolated part of the boundary connecting the system mould, moulded piece and plunger with the presser, and the part of the boundary formed by the tube,
▷ Γ_{sym} the part of the axis of symmetry,
▷ Γ_{GM} the boundary between the moulded piece Ω_{G} and the mould Ω_{M},
▷ Γ_{ME} the outward boundary of the mould, surrounded by an external environment,

▷ Γ_{in} the part of the boundary, where cooling water comes into the cooling cavity of the plunger,

▷ Γ_{out} the part of the boundary, where water exits.

We denote by G_{C} the 3D region created by rotation of Ω_{C} around the x axis. Analogously, we denote by $\Gamma_{\mathrm{N}}^{3\mathrm{D}}$, $\Gamma_{\mathrm{in}}^{3\mathrm{D}}$ and $\Gamma_{\mathrm{out}}^{3\mathrm{D}}$ the boundaries created by rotation of the appropriate planar boundary segment around the x axis.

1.4. Formulation of the problem of optimal design. We use the polar coordinate system to describe the shape of the inner surface of the plunger. We assume that

$$\Gamma_{\mathrm{CP}} = \Gamma_{\mathrm{CP}}(\varkappa) := \{(x,r) = (\varkappa(\xi)\cos\xi, \varkappa(\xi)\sin\xi);\ \xi \in [\pi/2, \pi]\},$$

where $\varkappa \in U_{\mathrm{ad}}$ is a design function from the admissible set

$$U_{\mathrm{ad}} = \{\varkappa \in C^{(0),1}([\pi/2,\pi]);\quad \forall\,\text{a.a. } \xi \in [\pi/2,\pi]\colon\ \underline{\varkappa}(\xi) \leqslant \varkappa(\xi) \leqslant \overline{\varkappa}(\xi),\ |\varkappa'(\xi)| \leqslant \gamma\}$$

(see Fig. 3).

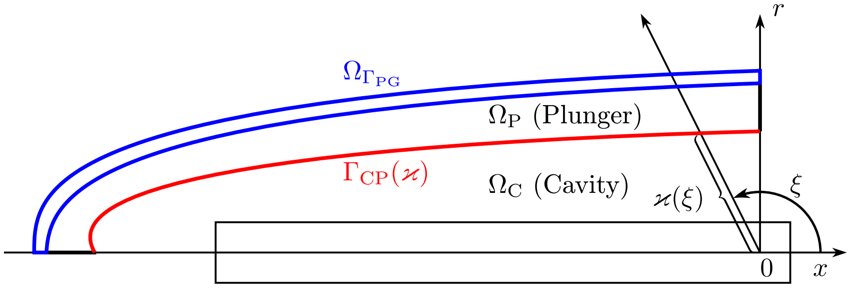


Figure 3. Scheme of the outer surface layer of the plunger and the design function.

Here $C^{(0),1}([\pi/2,\pi])$ is the space of Lipschitz continuous functions in $[\pi/2,\pi]$, $\underline{\varkappa}, \overline{\varkappa} \in C^{(0),1}([\pi/2,\pi])$ are given functions satisfying $\underline{\varkappa}(\pi/2) = \overline{\varkappa}(\pi/2)$ which realize the constraints for the minimal distance from the supply tube and the minimal thickness of the plunger wall and $\gamma > 0$ is a constant. The functions $\underline{\varkappa}$, $\overline{\varkappa}$ and the constant γ are chosen in such a way that $U_{\mathrm{ad}} \neq \emptyset$. Accordingly, we shall write $\Omega_{\mathrm{C}} = \Omega_{\mathrm{C}}(\varkappa)$, $\Omega_{\mathrm{P}} = \Omega_{\mathrm{P}}(\varkappa)$ etc. to emphasize the dependence on the design function $\varkappa$. The interfaces Γ_{PG}, Γ_{GM} as well as $\partial\Omega$ are fixed, hence Ω_{CP}, Ω_{G}, Ω_{M} and Ω are independent of $\varkappa$.

To represent the plunger outward surface temperature we denote the outer surface layer of the plunger of the thickness $\varepsilon > 0$ as

$$\Omega_{\Gamma_{\mathrm{PG}}} = \{(x,r) \in \Omega_{\mathrm{CP}};\ \operatorname{dist}((x,r);\Gamma_{\mathrm{PG}}) < \varepsilon\},$$

where $\operatorname{dist}((x,r);\Gamma_{\rm PG})$ is the distance of the point (x,r) from the boundary $\Gamma_{\rm PG}$ and ε is sufficiently small so that $\Omega_{\Gamma_{\rm PG}} \subset \Omega_{\rm P}(\varkappa)$ for all $\varkappa \in U_{\rm ad}$.

1.5. Function spaces. For any $q \in [1,\infty)$, open set $D \subset \mathcal{U}$ and measurable function $w\colon D \to [0,\infty)$, the space $L^q_w(D)$ is defined as the set of measurable functions $v\colon D \to \mathbb{R}$ such that

$$\|v\|_{L^q_w(D)} = \left(\int_D |v|^q w\right)^{1/q} < \infty.$$

Following [7], we introduce the weighted Sobolev space $H^1_r(D)$ as the subspace of functions in $L^2_r(D)$ whose first order derivatives are in $L^2_r(D)$. It is a Banach space endowed with the norm

$$\|v\|_{H^1_r(D)} := \sqrt{\|v\|^2_{L^2_r(D)} + \|\nabla v\|^2_{L^2_r(D)}},$$

see [7]. We shall also need the space

$$V^1_r(D) := H^1_r(D) \cap L^2_{1/r}(D).$$

It can be shown [10] that every function from $V^1_r(D)$ has zero trace on the symmetry axis $\{r = 0\}$.

For any $\varkappa \in U_{\rm ad}$ we define the spaces

$$\begin{gathered}\boldsymbol{V}(\varkappa) := H^1_r(\Omega_{\rm C}(\varkappa)) \times V^1_r(\Omega_{\rm C}(\varkappa)) = \{\boldsymbol{v} = (v_x, v_r) \in \boldsymbol{H}^1_r(\Omega_{\rm C}(\varkappa));\ v_r \in L^2_{1/r}(\Omega_{\rm C}(\varkappa))\},\\ \boldsymbol{V}_{0,\rm div}(\varkappa) := \{\boldsymbol{v} \in \boldsymbol{V}(\varkappa);\ \operatorname{div}\boldsymbol{v} = 0 \text{ a.e. in } \Omega_{\rm C}(\varkappa),\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\Omega_{\rm C}(\varkappa) \setminus \Gamma_{\rm sym}\},\end{gathered}$$

equipped by the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{V}(\varkappa)} := \sqrt{\|\nabla \boldsymbol{v}\|^2_{L^2_r(\Omega_{\rm C}(\varkappa))} + \|v_r\|^2_{L^2_{1/r}(\Omega_{\rm C}(\varkappa))}}.$$

For a bounded domain D in $\mathbb{R}^2$ or $\mathbb{R}^3$, the standard Lebesgue and Sobolev spaces will be denoted by $L^q(D)$ and $H^k(D)$, respectively. Further, we introduce the space of functions from $\boldsymbol{H}^1(D)$ with vanishing trace and divergence

$$\boldsymbol{H}^1_{0,\rm div}(D) := \{\boldsymbol{v} \in \boldsymbol{H}^1(D);\ \boldsymbol{v}|_{\partial D} = \boldsymbol{0}, \operatorname{div}\boldsymbol{v} = 0 \text{ in } D\}$$

and its subspace $\boldsymbol{C}^\infty_{0,\rm div}(D)$ of smooth and compactly supported solenoidal functions.

For every $\varkappa \in U_{\rm ad}$ we consider the stationary flow of an incompressible fluid in the cavity $\Omega_{\rm C}(\varkappa)$ described by the axisymmetric Navier-Stokes equations

$$(P_f(\varkappa)) \qquad \begin{cases} -\nu\Delta_r \boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{w} - \nabla\Pi = \boldsymbol{g} & \text{in } \Omega_{\rm C}(\varkappa),\\ \operatorname{div}\boldsymbol{w} = 0 & \text{in } \Omega_{\rm C}(\varkappa),\\ \boldsymbol{w} = \boldsymbol{w}_D & \text{on } \Gamma_{\rm in}\cup\Gamma_{\rm out},\\ \boldsymbol{w} = \boldsymbol{0} & \text{on } \Gamma_{\rm CP}(\varkappa)\cup\Gamma_{\rm N}(\varkappa),\\ w_r = 0 & \text{on } \Gamma_{\rm sym}(\varkappa), \end{cases}$$

where $\boldsymbol{w} = \boldsymbol{w}(\varkappa) = (w_x, w_r)$, $\Pi = \Pi(\varkappa)$ is the velocity and the kinematic pressure, respectively, $\nu > 0$ is the (constant) kinematic viscosity, $\boldsymbol{g} = (-g, 0, 0)$ is the (constant) gravity force and $\boldsymbol{w}_D$ the inlet/outlet velocity. We note that the density does not appear in $(P_f(\varkappa))$ since the kinematic variables are used. By bold symbols we denote vector-valued functions and corresponding function spaces. The symbol Δ_r denotes the Laplace operator for axisymetric functions, i.e.

$$\Delta_r \boldsymbol{w} := \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial \boldsymbol{w}}{\partial r}\Big) + \frac{\partial^2 \boldsymbol{w}}{\partial x^2}.$$

Before we give the weak formulation of $(P_f(\varkappa))$ we shall introduce some notations.

2.1. Weak formulation. In this subsection we shall fix $\varkappa \in U_{\rm ad}$. Therefore, we shall often omit the symbol $\varkappa$ and write e.g. $\boldsymbol{V}$, $\Omega_{\rm C}$ instead of $\boldsymbol{V}(\varkappa)$ and $\Omega_{\rm C}(\varkappa)$, respectively. We impose the following assumptions concerning the function $\boldsymbol{w}_D$:

$$(2.1) \qquad \begin{aligned} &\boldsymbol{w}_D && \text{is a trace of a function from } \boldsymbol{V};\\ &\boldsymbol{w}_D = \boldsymbol{0} && \text{on } \Gamma_{\rm CP}\cup\Gamma_{\rm N};\\ &\boldsymbol{w}_D\cdot\boldsymbol{n} \geqslant 0 && \text{on } \Gamma_{\rm out};\\ &w_{Dr} = 0 && \text{on } \Gamma_{\rm sym};\\ &\int_{\Gamma_{\rm in}\cup\Gamma_{\rm out}} \boldsymbol{w}_D\cdot\boldsymbol{n}\, r = 0. \end{aligned}$$

Here $\boldsymbol{n}$ denotes the unit normal vector, outward to the respective domain. It follows from Lemma 2.3 in [6] that there exists a divergence-free extension of $\boldsymbol{w}_D$ to $\boldsymbol{V}$. We shall denote this extension again by $\boldsymbol{w}_D$.

For any $\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{z} \in \boldsymbol{V}$, $q \in L^2_r(\Omega_{\rm C})$ we introduce the following forms:

$$a^f(\boldsymbol{w}, \boldsymbol{v}) = \nu\int_{\Omega_{\rm C}} \Big[(\nabla\boldsymbol{w} : \nabla\boldsymbol{v})r + w_r v_r\frac{1}{r}\Big], \quad c^f(\boldsymbol{w}, \boldsymbol{z}, \boldsymbol{v}) = \int_{\Omega_{\rm C}} ((\boldsymbol{w}\cdot\nabla)\boldsymbol{z})\cdot\boldsymbol{v}r.$$

Then we have the following weak formulation of $(P_f(\varkappa))$.

Definition 2.1. A function $\boldsymbol{w} \in \boldsymbol{V}$ is said to be a weak solution of $(P_f(\varkappa))$ if

▷ $\boldsymbol{z} := \boldsymbol{w} - \boldsymbol{w}_D \in \boldsymbol{V}_{0,\mathrm{div}}$;

▷ for all $\boldsymbol{v} \in \boldsymbol{V}_{0,\mathrm{div}}$:

$$a^f(\boldsymbol{w}, \boldsymbol{v}) + c^f(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{v}) = \int_{\Omega_\mathrm{C}} \boldsymbol{g} \cdot \boldsymbol{v}. \tag{2.2}$$

We shall reformulate Definition 2.1 in the three-dimensional Cartesian coordinate system. For $\boldsymbol{v} = (v_x, v_r) \in \boldsymbol{V}$ we define

$$\breve{\boldsymbol{v}}(\boldsymbol{x}) := (v_x(x_1, r), v_r(x_1, r)\cos\phi, v_r(x_1, r)\sin\phi), \quad \boldsymbol{x} \in G_\mathrm{C},$$

where $r = \sqrt{x_2^2 + x_3^2}$, $\cos\phi = x_2/r$, and $\sin\phi = x_3/r$. It can be easily verified that $\breve{\boldsymbol{v}} \in \boldsymbol{H}^1(G_\mathrm{C})$ and that the following identities hold:

$$\|\breve{\boldsymbol{v}}\|^2_{L^2(G_\mathrm{C})} = 2\pi\|\boldsymbol{v}\|^2_{L^2_r(\Omega_\mathrm{C})}, \quad \|\nabla\breve{\boldsymbol{v}}\|^2_{L^2(G_\mathrm{C})} = 2\pi\|\boldsymbol{v}\|^2_{\boldsymbol{V}}.$$

We introduce the following problem in the 3D domain G_C:

Definition 2.2 (Problem $(P_f^{3\mathrm{D}}(\varkappa))$)**.** A function $\boldsymbol{u} \in \boldsymbol{H}^1(G_\mathrm{C})$ is said to be the solution of $(P_f^{3\mathrm{D}}(\varkappa))$ if

▷ $\boldsymbol{u} - \breve{\boldsymbol{w}}_D \in \boldsymbol{H}^1_{0,\mathrm{div}}(G_\mathrm{C})$;

▷ for all $\boldsymbol{v} \in \boldsymbol{H}^1_{0,\mathrm{div}}(G_\mathrm{C})$:

$$\int_{G_\mathrm{C}} [\nu\nabla\boldsymbol{u} : \nabla\boldsymbol{v} + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\cdot\boldsymbol{v}] = \int_{G_\mathrm{C}} \boldsymbol{g}\cdot\boldsymbol{v}. \tag{2.3}$$

It is easy to check that $\boldsymbol{w}$ is a weak solution to $(P_f(\varkappa))$ if and only if $\boldsymbol{u} := \breve{\boldsymbol{w}}$ is the solution of $(P_f^{3\mathrm{D}}(\varkappa))$.

It is known [6] that $(P_f(\varkappa))$ has at least one weak axisymmetric solution provided that (2.1) holds true. In next subsections we shall study the dependence of solutions to $(P_f(\varkappa))$ on the design variable $\varkappa \in U_\mathrm{ad}$.

2.2. Uniform bounds and stability of solutions with respect to the domain. Our next aim is to show that weak solutions of $(P_f(\varkappa))$ are bounded in a suitable norm by a constant that does not depend on $\varkappa \in U_\mathrm{ad}$. For this purpose we shall rewrite (2.3) to the fixed domain $\widehat{G} := G_\mathrm{C}(\overline{\varkappa})$. Let $\boldsymbol{u}(\varkappa) := \breve{\boldsymbol{w}}(\varkappa)$, where $\boldsymbol{w}(\varkappa)$ is a weak solution to $(P_f(\varkappa))$. Then (2.3) is equivalent to

$$\int_{\widehat{G}} [\nu\nabla\widetilde{\boldsymbol{u}}(\varkappa) : \nabla\widetilde{\boldsymbol{v}} + ((\widetilde{\boldsymbol{u}}(\varkappa)\cdot\nabla)\widetilde{\boldsymbol{u}}(\varkappa))\cdot\widetilde{\boldsymbol{v}}] = \int_{\widehat{G}} \boldsymbol{g}\cdot\widetilde{\boldsymbol{v}}. \tag{2.4}$$

The symbol "$\sim$" denotes the zero extension of a function to $\widehat{G}$. We shall construct a specific extension of the nonhomogeneous boundary data $\boldsymbol{u}_D := \breve{\boldsymbol{w}}_D$, which is independent of the design variable $\varkappa \in U_{\mathrm{ad}}$. Let $\breve{G} := G_{\mathrm{C}}(\underline{\varkappa})$. In accordance with (2.1) we shall assume that

$$\boldsymbol{u}_D \in \boldsymbol{H}^{1/2}(\partial\breve{G}), \quad \boldsymbol{u}_D = \boldsymbol{0} \quad \text{on} \quad \partial\breve{G} \setminus (\Gamma^{\mathrm{3D}}_{\mathrm{in}} \cup \Gamma^{\mathrm{3D}}_{\mathrm{out}}), \quad \int_{\partial\breve{G}} \boldsymbol{u}_D \cdot \boldsymbol{n} = 0. \tag{2.5}$$

We emphasize that the sets $\Gamma^{\mathrm{3D}}_{\mathrm{in}}$ and $\Gamma^{\mathrm{3D}}_{\mathrm{out}}$ are independent of $\varkappa \in U_{\mathrm{ad}}$.

Lemma 2.1. *There exists a function $\boldsymbol{U} \in \boldsymbol{H}^1(\widehat{G})$ with the following properties:*
(i) $\boldsymbol{U} = \boldsymbol{u}_D$ *on* $\partial\breve{G}$,
(ii) $\operatorname{div}\boldsymbol{U} = 0$ *a.e. in* $\widehat{G}$,
(iii) $\boldsymbol{U} = \boldsymbol{0}$ *a.e. in* $\widehat{G} \setminus \breve{G}$,
(iv) *for every* $\boldsymbol{v} \in \boldsymbol{H}^1_{0,\mathrm{div}}(\widehat{G})$:

$$\int_{\widehat{G}} ((\boldsymbol{v} \cdot \nabla)\boldsymbol{U}) \cdot \boldsymbol{v} \leqslant \frac{\nu}{2} \|\nabla \boldsymbol{v}\|^2_{L^2(\widehat{G})}. \tag{2.6}$$

P r o o f. It is known that for any $\varepsilon > 0$ there exists a solenoidal extension $\boldsymbol{U}_\varepsilon \in \boldsymbol{H}^1(\breve{G})$ of $\boldsymbol{u}_D$ satisfying $\boldsymbol{U}_\varepsilon = \boldsymbol{u}_D$ on $\partial\breve{G}$ and

$$\int_{\breve{G}} ((\boldsymbol{v} \cdot \nabla)\boldsymbol{U}_\varepsilon) \cdot \boldsymbol{v} \leqslant \varepsilon \|\nabla \boldsymbol{v}\|^2_{L^2(\breve{G})}, \tag{2.7}$$

see e.g. Lemma II.1.8 in [14] or Lemma VIII.4.2 in [3]. Due to $(2.5)_2$, $\boldsymbol{U}_\varepsilon$ can be extended by zero to $\boldsymbol{H}^1(\widehat{G})$. Taking $\boldsymbol{U} := \boldsymbol{U}_\varepsilon$ with $\varepsilon = \nu/2$, one can easily verify properties (i)–(iv). $\square$

Let us note that $\boldsymbol{U}$ is in general not axisymmetric, even though $\boldsymbol{u}_D$ is. In fact, it is not known whether an axisymmetric extension having properties (i)–(iv) of Lemma 2.1 exists.

To prove the estimate of solutions independent of $\varkappa \in U_{\mathrm{ad}}$, we use the Sobolev embedding and the Friedrichs inequality on $\widehat{G}$. For any $q \in [1,6]$ we shall denote by $C_{\mathrm{I}}(q) > 0$ the constant of the Sobolev embedding

$$\|v\|_{L^q(\widehat{G})} \leqslant C_{\mathrm{I}}(q) \|v\|_{H^1(\widehat{G})}, \tag{2.8}$$

and by $C_{\mathrm{F}} > 0$ the constant of the Friedrichs inequality

$$\|v\|_{H^1(\widehat{G})} \leqslant C_{\mathrm{F}} \|\nabla v\|_{L^2(\widehat{G})}, \tag{2.9}$$

which hold in the space $\{v \in H^1(\widehat{G});\ v_{|\Gamma^{\mathrm{3D}}_{\mathrm{N}}} = 0\}$.

Theorem 2.1. *There exists a constant $C_{\mathrm{E}}^{f} > 0$ such that for every $\varkappa \in U_{\mathrm{ad}}$ and every weak solution $\boldsymbol{w}(\varkappa)$ to $(P_f(\varkappa))$ the following estimate holds true:*

$$\|\boldsymbol{w}(\varkappa)\|_{\boldsymbol{V}(\varkappa)} \leqslant C_{\mathrm{E}}^{f}. \tag{2.10}$$

P r o o f. We shall prove the inequality

$$\|\nabla \boldsymbol{u}(\varkappa)\|_{L^2(G_{\mathrm{C}}(\varkappa))} \leqslant C, \tag{2.11}$$

where $C > 0$ is independent of $\varkappa \in U_{\mathrm{ad}}$ and $\boldsymbol{u}(\varkappa) = \breve{\boldsymbol{w}}(\varkappa)$. Then (2.10) holds with $C_{\mathrm{E}}^{f} := C/\sqrt{2\pi}$. Identity (2.4) is equivalent to

$$\begin{aligned}\int_{\widehat{G}} [\nu \nabla \widetilde{\boldsymbol{z}} : \nabla \widetilde{\boldsymbol{v}} + ((\widetilde{\boldsymbol{z}} \cdot \nabla)\widetilde{\boldsymbol{z}}) \cdot \widetilde{\boldsymbol{v}} + ((\widetilde{\boldsymbol{z}} \cdot \nabla)\boldsymbol{U}) \cdot \widetilde{\boldsymbol{v}} + ((\boldsymbol{U} \cdot \nabla)\widetilde{\boldsymbol{z}}) \cdot \widetilde{\boldsymbol{v}}] \\ = \int_{\widehat{G}} [\boldsymbol{g} \cdot \widetilde{\boldsymbol{v}} - \nu \nabla \boldsymbol{U} : \nabla \widetilde{\boldsymbol{v}} - ((\boldsymbol{U} \cdot \nabla)\boldsymbol{U}) \cdot \widetilde{\boldsymbol{v}}],\end{aligned} \tag{2.12}$$

where $\boldsymbol{z} = \boldsymbol{u}(\varkappa) - \boldsymbol{U}_{|G_{\mathrm{C}}(\varkappa)}$. We use $\boldsymbol{v} := \boldsymbol{z}$ as a test function in (2.12). Applying (2.6) and Green's theorem we obtain

$$\begin{aligned}\nu \|\nabla \boldsymbol{z}\|^2_{L^2(G_{\mathrm{C}}(\varkappa))} + \underbrace{\int_{\widehat{G}} ((\widetilde{\boldsymbol{z}} \cdot \nabla)\widetilde{\boldsymbol{z}}) \cdot \widetilde{\boldsymbol{z}}}_{=0} + \underbrace{\int_{\widehat{G}} ((\widetilde{\boldsymbol{z}} \cdot \nabla)\boldsymbol{U}) \cdot \widetilde{\boldsymbol{z}}}_{\leqslant \frac{1}{2}\nu \|\nabla \boldsymbol{z}\|^2_{L^2(G_{\mathrm{C}}(\varkappa))}} + \underbrace{\int_{\widehat{G}} ((\boldsymbol{U} \cdot \nabla)\widetilde{\boldsymbol{z}}) \cdot \widetilde{\boldsymbol{z}}}_{=0} \\ = \int_{\widehat{G}} \boldsymbol{g} \cdot \widetilde{\boldsymbol{z}} - \nu \int_{\widehat{G}} \nabla \boldsymbol{U} : \nabla \widetilde{\boldsymbol{z}} - \int_{\widehat{G}} ((\boldsymbol{U} \cdot \nabla)\boldsymbol{U}) \cdot \widetilde{\boldsymbol{z}}.\end{aligned} \tag{2.13}$$

With the help of Hölder's inequality, (2.8) and (2.9), we estimate the right-hand side of (2.13):

$$\begin{aligned}\int_{\widehat{G}} \boldsymbol{g} \cdot \widetilde{\boldsymbol{z}} &\leqslant C_{\mathrm{F}} \|\boldsymbol{g}\|_{L^2(\widehat{G})} \|\nabla \boldsymbol{z}\|_{L^2(G_{\mathrm{C}}(\varkappa))}, \\ \left| \nu \int_{\widehat{G}} (\nabla \boldsymbol{U} : \nabla \widetilde{\boldsymbol{z}}) \right| &\leqslant \nu \|\nabla \boldsymbol{U}\|_{L^2(\widehat{G})} \|\nabla \boldsymbol{z}\|_{L^2(G_{\mathrm{C}}(\varkappa))}, \\ \left| \int_{\widehat{G}} ((\boldsymbol{U} \cdot \nabla)\boldsymbol{U}) \cdot \widetilde{\boldsymbol{z}} \right| &\leqslant \|\boldsymbol{U}\|_{L^4(\widehat{G})} \|\nabla \boldsymbol{U}\|_{L^2(\widehat{G})} \|\boldsymbol{z}\|_{L^4(\widehat{G})} \\ &\leqslant C_{\mathrm{I}}(4)^2 C_{\mathrm{F}}^2 \|\nabla \boldsymbol{U}\|^2_{L^2(\widehat{G})} \|\nabla \boldsymbol{z}\|_{L^2(G_{\mathrm{C}}(\varkappa))}.\end{aligned}$$

Putting these estimates back to (2.13) and dividing it by $\|\nabla \boldsymbol{z}\|_{L^2(G_{\mathrm{C}}(\varkappa))}$, we obtain:

$$\frac{\nu}{2} \|\nabla \boldsymbol{z}\|_{L^2(G_{\mathrm{C}}(\varkappa))} \leqslant C_{\mathrm{F}} g |\widehat{G}|^{1/2} + \nu \|\nabla \boldsymbol{U}\|_{L^2(\widehat{G})} + C_{\mathrm{I}}(4)^2 C_{\mathrm{F}}^2 \|\nabla \boldsymbol{U}\|^2_{L^2(\widehat{G})},$$

which implies (2.11) with

$$C := \frac{2}{\nu} C_{\mathrm{F}} g |\widehat{G}|^{1/2} + 2\|\nabla \boldsymbol{U}\|_{L^2(\widehat{G})} + \frac{2}{\nu} C_{\mathrm{I}}(4)^2 C_{\mathrm{F}}^2 \|\nabla \boldsymbol{U}\|^2_{L^2(\widehat{G})}.$$

□

Having the uniform estimate of solutions, we can prove the following stability result.

Theorem 2.2. *Let $\{\varkappa_n\} \subset U_{\rm ad}$ and $\{\boldsymbol{w}_n\}$ be a sequence of weak solutions to $(P_f(\varkappa_n))$, $n = 1, 2, \ldots$ Then there exists a subsequence denoted $\{\varkappa_{n_k}\}$ and functions $\varkappa \in U_{\rm ad}$, $\widehat{\boldsymbol{w}} \in \boldsymbol{V}(\overline{\varkappa})$ such that*

$$\varkappa_{n_k} \rightrightarrows \varkappa \quad \text{in } [\pi/2, \pi], \tag{2.14}$$

$$\widetilde{\boldsymbol{w}}_{n_k} \rightharpoonup \widehat{\boldsymbol{w}} \quad \text{weakly in } \boldsymbol{V}(\overline{\varkappa}). \tag{2.15}$$

In addition, the function $\widehat{\boldsymbol{w}}_{|\Omega_{\rm C}(\varkappa)}$ is a weak solution of $(P_f(\varkappa))$.

P r o o f. The existence of a subsequence $\{n_k\}$ and $\varkappa \in U_{\rm ad}$ satisfying (2.14) follows from the compactness of $U_{\rm ad}$.

Due to estimate (2.10), the sequence $\{\widetilde{\boldsymbol{w}}_n\}$ is bounded in $\boldsymbol{V}(\overline{\varkappa})$. Thus, one can choose another subsequence (denoted by the same symbol) and a function $\widehat{\boldsymbol{w}}$ such that (2.15) is satisfied.

It remains to show that $\boldsymbol{w} := \widehat{\boldsymbol{w}}_{|\Omega_{\rm C}(\varkappa)}$ is a weak solution of $(P_f(\varkappa))$. Clearly $\widehat{\boldsymbol{w}}_{|\Omega_{\rm C}(\overline{\varkappa})\setminus\Omega_{\rm C}(\varkappa)} = \boldsymbol{0}$ and thus $\boldsymbol{w} - \boldsymbol{w}_D \in \boldsymbol{V}_{0,\rm div}(\Omega_{\rm C}(\varkappa))$. We shall prove that $\boldsymbol{u} := \breve{\boldsymbol{w}}$ satisfies (2.4). Let $\boldsymbol{u}_k := \breve{\boldsymbol{w}}_{n_k}$. Then we have

$$\int_{\widehat{G}} [\nu \nabla \widetilde{\boldsymbol{u}}_k : \nabla \widetilde{\boldsymbol{v}} + ((\widetilde{\boldsymbol{u}}_k \cdot \nabla)\widetilde{\boldsymbol{u}}_k) \cdot \widetilde{\boldsymbol{v}}] = \int_{\widehat{G}} \boldsymbol{g} \cdot \widetilde{\boldsymbol{v}} \tag{2.16}$$

for any $\boldsymbol{v} \in \boldsymbol{H}^1_{0,\rm div}(G_{\rm C}(\varkappa_{n_k}))$.

Let $\boldsymbol{v} \in \boldsymbol{C}^\infty_{0,\rm div}(G_{\rm C}(\varkappa))$. Since $\varkappa_n \rightrightarrows \varkappa$, there exists an index k_0 such that $\operatorname{supp} \boldsymbol{v} \subset G_{\rm C}(\varkappa_{n_k})$ for $k \geqslant k_0$ and hence $\boldsymbol{v}$ is a suitable test function in (2.16). The weak convergence (2.15) is sufficient to show that

$$\int_{\widehat{G}} \nu \nabla \widetilde{\boldsymbol{u}}_k : \nabla \widetilde{\boldsymbol{v}} \to \int_{\widehat{G}} \nu \nabla \widetilde{\boldsymbol{u}} : \nabla \widetilde{\boldsymbol{v}}. \tag{2.17}$$

From the compact embedding $H^1(\widehat{G}) \hookrightarrow L^4(\widehat{G})$ we have that

$$\widetilde{\boldsymbol{u}}_k \to \widetilde{\boldsymbol{u}} \quad \text{strongly in } \boldsymbol{L}^4(\widehat{G}), \quad k \to \infty,$$

passing eventually to a subsequence, which yields:

$$\int_{\widehat{G}} ((\widetilde{\boldsymbol{u}}_k \cdot \nabla)\widetilde{\boldsymbol{u}}_k) \cdot \widetilde{\boldsymbol{v}} \to \int_{\widehat{G}} ((\widetilde{\boldsymbol{u}} \cdot \nabla)\widetilde{\boldsymbol{u}}) \cdot \widetilde{\boldsymbol{v}}, \quad k \to \infty. \tag{2.18}$$

The density of $\boldsymbol{C}^\infty_{0,\rm div}(G_{\rm C}(\varkappa))$ in $\boldsymbol{H}^1_{0,\rm div}(G_{\rm C}(\varkappa))$, (2.16), (2.17) and (2.18) imply that $\boldsymbol{u}$ satisfies (2.4) for any $\boldsymbol{v} \in \boldsymbol{H}^1_{0,\rm div}(G_{\rm C}(\varkappa))$. □

The pressing cycle has four important time points $0 < T_1 < T_2 < T_3 < T_4$. In the interval $(0, T_1)$ the system is in the process of pressing with the glass in the mould and the plunger in the lower position. The separation of the plunger from the glass, which remains in the mould, follows in the time $T_1 = T_4/12$ $(= 13.5)$. In the time $T_2 = T_4/6$ $(= 27)$ the plunger goes down to the lower position into another mould (on a carousel press, the plunger presses sequentially in six moulds). The separation of the mould and the glass piece follows in the time $T_3 = (13/24)T_4$ $(= 87.75)$ and the mould is refilled by the molten glass and the plunger descends at the time $T_4 = 6T_2$ $(= 162)$.
We consider doubly periodic process with the period T_2 in the region Ω_{CP}, and with the period T_4 in the region Ω_{M}.

We consider the mixed problem for the energy equation in the form
$(P_h(\varkappa, \boldsymbol{w}))$

$$\left\{\begin{aligned}
\frac{\partial\vartheta}{\partial t} + \nabla\vartheta\cdot\boldsymbol{w} - a(\varkappa)\Delta_r\vartheta &= 0 && \text{in } (0,T_4)\times(\Omega_{\mathrm{CP}}\cup\Omega_{\mathrm{G}}\cup\Omega_{\mathrm{M}}),\\
\vartheta(0,\cdot) &= \vartheta_0 && \text{in } \Omega_{\mathrm{G}},\\
\forall\, t\in[T_2,T_4]\colon\ \vartheta(t-T_2,\cdot) &= \vartheta(t,\cdot) && \text{in } \Omega_{\mathrm{CP}},\\
\vartheta(0,\cdot) &= \vartheta(T_4,\cdot) && \text{in } \Omega_{\mathrm{M}},\\
\vartheta &= \vartheta_{\mathrm{in}} && \text{in } (0,T_4)\times\Gamma_{\mathrm{in}},\\
\frac{\partial\vartheta}{\partial n} &= 0 && \text{in } (0,T_4)\times(\Gamma_{\mathrm{out}}\cup\Gamma_{\mathrm{N}}\cup\Gamma_{\mathrm{sym}}),\\
\left[a_{\mathrm{P}}\frac{\partial\vartheta}{\partial n}\right]_{|\Omega_{\mathrm{CP}}} + \left[a_{\mathrm{G}}\frac{\partial\vartheta}{\partial n}\right]_{|\Omega_{\mathrm{G}}} &= \beta && \text{in } (0,T_1)\times\Gamma_{\mathrm{PG}},\\
\vartheta_{|\Omega_{\mathrm{CP}}} &= \vartheta_{|\Omega_{\mathrm{G}}} && \text{in } (0,T_1)\times\Gamma_{\mathrm{PG}},\\
\left[a_{\mathrm{P}}\frac{\partial\vartheta}{\partial n} + \alpha\vartheta\right]_{|\Omega_{\mathrm{CP}}} &= \alpha\vartheta_{\mathrm{ext}} && \text{in } (T_1,T_2)\times\Gamma_{\mathrm{PG}},\\
\left[a_{\mathrm{G}}\frac{\partial\vartheta}{\partial n} + \alpha\vartheta\right]_{|\Omega_{\mathrm{G}}} &= \alpha\vartheta_{\mathrm{ext}} && \text{in } (T_1,T_4)\times\Gamma_{\mathrm{PG}},\\
\left[a_{\mathrm{M}}\frac{\partial\vartheta}{\partial n}\right]_{|\Omega_{\mathrm{M}}} + \left[a_{\mathrm{G}}\frac{\partial\vartheta}{\partial n}\right]_{|\Omega_{\mathrm{G}}} &= \beta && \text{in } (0,T_3)\times\Gamma_{\mathrm{GM}},\\
\vartheta_{|\Omega_{\mathrm{M}}} &= \vartheta_{|\Omega_{\mathrm{G}}} && \text{in } (0,T_3)\times\Gamma_{\mathrm{GM}},\\
\left[a_{\mathrm{G}}\frac{\partial\vartheta}{\partial n} + \alpha\vartheta\right]_{|\Omega_{\mathrm{G}}} &= \alpha\vartheta_{\mathrm{ext}} && \text{in } (T_3,T_4)\times\Gamma_{\mathrm{GM}},\\
\left[a_{\mathrm{M}}\frac{\partial\vartheta}{\partial n} + \alpha\vartheta\right]_{|\Omega_{\mathrm{M}}} &= \alpha\vartheta_{\mathrm{ext}} && \text{in } (T_3,T_4)\times\Gamma_{\mathrm{GM}},\\
\left[a_{\mathrm{M}}\frac{\partial\vartheta}{\partial n} + \alpha\vartheta\right]_{|\Omega_{\mathrm{M}}} &= \alpha\vartheta_{\mathrm{ext}} && \text{in } (0,T_4)\times\Gamma_{\mathrm{ME}},
\end{aligned}\right.$$

where $\boldsymbol{w} := \boldsymbol{w}(\varkappa)$ is the solution of $(P_f(\varkappa))$ and $\varkappa \in U_{\rm ad}$. We assume that $\boldsymbol{w}(\varkappa) = \boldsymbol{0}$ in $\Omega_{\rm P}(\varkappa) \cup \Omega_{\rm G} \cup \Omega_{\rm M}$. Further,

$$a(\varkappa) = \begin{cases} a_{\rm C} = \dfrac{k_{\rm C}}{\varrho_{\rm C} c_{\rm C}} & \text{in } \Omega_{\rm C}(\varkappa), \\ a_{\rm P} = \dfrac{k_{\rm P}}{\varrho_{\rm P} c_{\rm P}} & \text{in } \Omega_{\rm P}(\varkappa), \\ a_{\rm G} = \dfrac{k_{\rm G}}{\varrho_{\rm G} c_{\rm G}} & \text{in } \Omega_{\rm G}, \\ a_{\rm M} = \dfrac{k_{\rm M}}{\varrho_{\rm M} c_{\rm M}} & \text{in } \Omega_{\rm M}, \end{cases} \tag{3.1}$$

is the material constant, where $k_i > 0$ represents the coefficient of thermal conductivity, $\varrho_i > 0$ the density, $c_i > 0$ the specific heat in Ω_i, $i = {\rm C, P, G, M}$, ϑ_0 is the initial distribution of temperature in $\Omega_{\rm G}$, $\vartheta_{\rm in}$ the absolute temperature of water at the inlet, $\alpha > 0$ the coefficient of heat-transfer, $\beta > 0$ the flux density of the modified mass of the body and $\vartheta_{\rm ext} > 0$ the temperature of environment. The symbol $[\partial\vartheta/\partial n]_{|\Omega_i}$ denotes the derivative with respect to the outward unit normal with respect to the region Ω_i, $i = {\rm C, P, G, M}$. In this section we shall omit the symbol $\varkappa$ and write $\Omega_{\rm C}$, $\Omega_{\rm P}$ instead of $\Omega_{\rm C}(\varkappa)$, $\Omega_{\rm P}(\varkappa)$, respectively.

3.1. Weak formulation. We define the space

$$H := \{v \in L^2_r(\Omega);\ v_{|\Omega_{\rm CP}} \in H^1_r(\Omega_{\rm CP}),\ v_{|\Omega_{\rm G}} \in H^1_r(\Omega_{\rm G}),\ v_{|\Omega_{\rm M}} \in H^1_r(\Omega_{\rm M})\},$$

endowed by the norm

$$\|v\|_H := \sqrt{\|v_{|\Omega_{\rm CP}}\|^2_{H^1_r(\Omega_{\rm CP})} + \|v_{|\Omega_{\rm G}}\|^2_{H^1_r(\Omega_{\rm G})} + \|v_{|\Omega_{\rm M}}\|^2_{H^1_r(\Omega_{\rm M})}}$$

and its subspace

$$H_0 := \{v \in H;\ v_{|\Omega_{\rm CP}} \in H^1_{r,0}(\Omega_{\rm CP})\},$$

where $H^1_{r,0}(\Omega_{\rm CP})$ is the closure of the set $\mathcal{H} := \{v \in C^\infty(\overline{\Omega}_{\rm CP});\ v_{|\Gamma_{\rm in}} = 0\}$ in the norm of $H^1_r(\Omega_{\rm CP})$. Further we define the space

$$H_{\rm GM} := \{v \in L^2_r(\Omega_{\rm GM});\ v_{|\Omega_{\rm G}} \in H^1_r(\Omega_{\rm G}), v_{|\Omega_{\rm M}} \in H^1_r(\Omega_{\rm M})\}$$

with the norm

$$\|v\|_{H_{\rm GM}} := \sqrt{\|v_{|\Omega_{\rm G}}\|^2_{H^1_r(\Omega_{\rm G})} + \|v_{|\Omega_{\rm M}}\|^2_{H^1_r(\Omega_{\rm M})}}.$$

In the weak formulation we shall use the following space of test functions:

$$\begin{aligned} S_0 := \{v = v(t,\cdot) \in L^2(0,T_4;H_0);\ & v_{|\Omega_{\rm CP}} = v_{|\Omega_{\rm G}} \text{ in } (0,T_1) \times \Gamma_{\rm PG}, \\ & v_{|\Omega_{\rm G}} = v_{|\Omega_{\rm M}} \text{ in } (0,T_3) \times \Gamma_{\rm GM}\} \end{aligned}$$

with the norm

$$\|v\|_{S_0} := \|v\|_{L^2(0,T_4;H_0)}$$

and the space

$$W := \Big\{v \in S_0\,;\ \frac{\partial v}{\partial t} \in S_0^*\Big\},$$

where S_0^* denotes the dual space to the Banach space S_0 with the norm

$$\|v\|_W := \sqrt{\|v\|_{S_0}^2 + \Big\|\frac{\partial v}{\partial t}\Big\|_{S_0^*}^2}.$$

By standard arguments (see e.g. [12]) one can show that $W \hookrightarrow C(0,T_4;L_r^2(\Omega))$.

Hence, we can define the subspace of time-periodic functions:

$$\begin{aligned} W_{\mathrm{per}} := \{v \in W;\ & v_{|\Omega_{\mathrm{M}}}(T_4,\cdot) = v_{|\Omega_{\mathrm{M}}}(0,\cdot),\\ & \forall\,\text{a.a. } t \in (T_2,T_4)\colon\ v_{|\Omega_{\mathrm{CP}}}(t,\cdot) = v_{|\Omega_{\mathrm{CP}}}(t-T_2,\cdot)\}, \end{aligned}$$

in which the following by-parts formula holds:

$$\forall\, u,v \in W_{\mathrm{per}}\colon\ \Big\langle\frac{\partial u}{\partial t},v\Big\rangle_{S_0^*,S_0} + \Big\langle\frac{\partial v}{\partial t},u\Big\rangle_{S_0^*,S_0} = \int_{\Omega_{\mathrm{G}}} (u(T_4,\cdot)v(T_4,\cdot) - u(0,\cdot)v(0,\cdot))r.$$

In particular,

$$\forall\, u \in W_{\mathrm{per}}\colon\ \Big\langle\frac{\partial u}{\partial t},u\Big\rangle_{S_0^*,S_0} = \frac{1}{2}(\|u(T_4,\cdot)\|^2_{L_r^2(\Omega_{\mathrm{G}})} - \|u(0,\cdot)\|^2_{L_r^2(\Omega_{\mathrm{G}})}). \tag{3.2}$$

We assume the existence of a function $\widetilde{\vartheta}_{\mathrm{in}} \in H_r^1(\Omega)$ such that $\widetilde{\vartheta}_{\mathrm{in}|\Gamma_{\mathrm{in}}} = \vartheta_{\mathrm{in}}$. In addition, we shall assume that $\vartheta_0 \in H_r^1(\Omega_{\mathrm{G}})$.

Definition 3.1. We say that $\vartheta \in L^2(0,T_4;H)$ is a weak solution to the heat conduction problem $(P_h(\varkappa,\boldsymbol{w}))$ if

- ▷ $(\vartheta - \widetilde{\vartheta}_{\mathrm{in}}) \in W_{\mathrm{per}}$;
- ▷ $\vartheta_{|\Omega_{\mathrm{G}}}(0,\cdot) = \vartheta_0$;
- ▷ for all $\psi \in S_0$:

$$M\Big(\frac{\partial\vartheta}{\partial t},\psi\Big) + \int_0^{T_4} A(\tau,\vartheta(\tau,\cdot),\psi(\tau,\cdot)) = \int_0^{T_4} F(\tau,\psi(\tau,\cdot)), \tag{3.3}$$

where the forms $M\colon\ S_0^*\times S_0\to\mathbb{R}$, $A\colon\ (0,T_4)\times H\times H\to\mathbb{R}$ and $F\colon\ (0,T_4)\times H\to\mathbb{R}$ are defined as

$$\begin{aligned}
M\Big(\frac{\partial\vartheta}{\partial t},\psi\Big) &:= \int_0^{T_2}\Big\langle\frac{\partial\vartheta}{\partial t},\psi\Big\rangle_{H^*,H}+\int_{T_2}^{T_4}\Big\langle\frac{\partial\vartheta}{\partial t},\psi\Big\rangle_{H^*_{\rm GM},H_{\rm GM}},\\
A(\tau,\vartheta,\psi) &:= \chi_{(0,T_2)}(\tau)\int_{\Omega_{\rm CP}}(\boldsymbol{w}\cdot\nabla\vartheta)\psi r\\
&\quad+\chi_{(0,T_2)}(\tau)\int_{\Omega_{\rm CP}}a(\varkappa)(\nabla\vartheta\cdot\nabla\psi)r+\int_{\Omega_{\rm GM}}a(\varkappa)(\nabla\vartheta\cdot\nabla\psi)r\\
&\quad+\int_{\Gamma_{\rm PG}}\alpha[\chi_{(T_1,T_2)}(\tau)(\vartheta\psi)_{|\Omega_{\rm CP}}+\chi_{(T_1,T_4)}(\tau)(\vartheta\psi)_{|\Omega_{\rm G}}]r\\
&\quad+\chi_{(T_3,T_4)}(\tau)\int_{\Gamma_{\rm GM}}\alpha[(\vartheta\psi)_{|\Omega_{\rm G}}+(\vartheta\psi)_{|\Omega_{\rm M}}]r+\int_{\Gamma_{\rm ME}}\alpha\vartheta\psi r,\\
F(\tau,\psi) &:= \int_{\Gamma_{\rm PG}}[\chi_{(0,T_1)}(\tau)\beta\psi_{|\Omega_{\rm CP}}+\chi_{(T_1,T_2)}(\tau)\alpha\vartheta_{\rm ext}\psi_{|\Omega_{\rm CP}}\\
&\qquad\qquad+\chi_{(T_1,T_4)}(\tau)\alpha\vartheta_{\rm ext}\psi_{|\Omega_{\rm G}}]r\\
&\quad+\int_{\Gamma_{\rm GM}}[\chi_{(0,T_3)}(\tau)\beta\psi_{|\Omega_{\rm G}}+\chi_{(T_3,T_4)}(\tau)\alpha\vartheta_{\rm ext}(\psi_{|\Omega_{\rm G}}+\psi_{|\Omega_{\rm M}})]r\\
&\quad+\int_{\Gamma_{\rm ME}}\alpha\vartheta_{\rm ext}\psi r.
\end{aligned}$$

Here χ_E denotes the characteristic function of a set E.

The following variant of the Friedrichs inequality will be used in the proof of ellipticity of A.

Lemma 3.1. *Let D be a domain in $\mathbb{R}^2$ and Γ be a nonempty measurable part of ∂D. Then there exists a constant $c(D,\Gamma)>0$ such that*

$$\forall\psi\in H^1_r(D)\colon\ \|\nabla\psi\|^2_{L^2_r(D)}+\|\psi\|^2_{L^2_r(\Gamma)}\geqslant c(D,\Gamma)\|\psi\|^2_{H^1_r(D)}. \tag{3.4}$$

Lemma 3.2. *Let $\varkappa\in U_{\rm ad}$, and $\boldsymbol{w}\in\boldsymbol{V}(\varkappa)$ be the velocity field from Definition* 2.1. *Then the bilinear form A is bounded and elliptic in the following sense: There exist constants C_A, $K>0$ independent of $\varkappa\in U_{\rm ad}$ such that*

(i) *for a.a. $\tau\in(0,T_4)$ and for all φ, $\psi\in H$:*

$$A(\tau,\varphi,\psi)\leqslant C_A(\chi_{(0,T_2)}(\tau)\|\varphi\|_H\|\psi\|_H+\chi_{(T_2,T_4)}(\tau)\|\varphi\|_{H_{\rm GM}}\|\psi\|_{H_{\rm GM}}), \tag{3.5}$$

(ii) *for a.a. $\tau\in(0,T_4)$ and for all $\psi\in S_0$:*

$$A(\tau,\psi(\tau,\cdot),\psi(\tau,\cdot))\geqslant K(\chi_{(0,T_2)}(\tau)\|\psi(\tau,\cdot)\|^2_H+\chi_{(T_2,T_4)}(\tau)\|\psi(\tau,\cdot)\|^2_{H_{\rm GM}}). \tag{3.6}$$

P r o o f. (i) Let $C_1, C_2 > 0$ be the constants of the embedding $H^1_r(\Omega_{\rm CP}) \times V^1_r(\Omega_{\rm CP}) \hookrightarrow \boldsymbol{L}^4_r(\Omega_{\rm CP})$, $H^1_r(\Omega_{\rm CP}) \hookrightarrow L^4_r(\Omega_{\rm CP})$, respectively. Then we can estimate the first term in $A(\tau, \varphi, \psi)$ as

$$\begin{aligned} \int_{\Omega_{\rm CP}} (\boldsymbol{w} \cdot \nabla\varphi)\psi r &\leqslant \|\boldsymbol{w}\|_{L^4_r(\Omega_{\rm CP})} \|\nabla\varphi\|_{L^2_r(\Omega_{\rm CP})} \|\psi\|_{L^4_r(\Omega_{\rm CP})} \\ &\leqslant C_1 C_2 \|\boldsymbol{w}\|_{\boldsymbol{V}(\varkappa)} \|\nabla\varphi\|_{L^2_r(\Omega_{\rm CP})} \|\psi\|_{H^1_r(\Omega_{\rm CP})} \\ &\leqslant C_1 C_2 C^f_{\rm E} \|\varphi\|_H \|\psi\|_H, \end{aligned} \tag{3.7}$$

using also estimate (2.10).

Next two terms in $A(\tau, \varphi, \psi)$ are estimated as follows:

$$\int_{\Omega_{\rm CP}} a(\varkappa)(\nabla\varphi \cdot \nabla\psi) r \leqslant \max_{\Omega_{\rm CP}} a(\varkappa) \|\varphi\|_H \|\psi\|_H, \tag{3.8}$$

$$\int_{\Omega_{\rm GM}} a(\varkappa)(\nabla\varphi \cdot \nabla\psi) r \leqslant \max_{\Omega_{\rm GM}} a(\varkappa) \|\varphi\|_{H_{\rm GM}} \|\psi\|_{H_{\rm GM}}. \tag{3.9}$$

Denoting by $C_T(D)$ the norm of the trace operator in $H^1_r(D)$, where D is a domain in $\mathbb{R}^2$, we can estimate the remaining terms in $A(\tau, \varphi, \psi)$:

$$\int_{\Gamma_{\rm PG}} \alpha(\varphi\psi)_{|\Omega_{\rm CP}} r \leqslant \alpha \|\varphi\|_{L^2_r(\partial\Omega_{\rm CP})} \|\psi\|_{L^2_r(\partial\Omega_{\rm CP})} \leqslant \alpha C^2_T(\Omega_{\rm CP}) \|\varphi\|_H \|\psi\|_H, \tag{3.10}$$

$$\int_{\Gamma_{\rm PG}} \alpha(\varphi\psi)_{|\Omega_{\rm G}} r \leqslant \alpha C^2_T(\Omega_{\rm G}) \|\varphi\|_{H_{\rm GM}} \|\psi\|_{H_{\rm GM}}, \tag{3.11}$$

$$\int_{\Gamma_{\rm GM}} \alpha(\varphi\psi)_{|\Omega_{\rm G}} r \leqslant \alpha C^2_T(\Omega_{\rm G}) \|\varphi\|_{H_{\rm GM}} \|\psi\|_{H_{\rm GM}}, \tag{3.12}$$

$$\int_{\Gamma_{\rm GM}} \alpha(\varphi\psi)_{|\Omega_{\rm M}} r \leqslant \alpha C^2_T(\Omega_{\rm M}) \|\varphi\|_{H_{\rm GM}} \|\psi\|_{H_{\rm GM}}. \tag{3.13}$$

Combining (3.7)–(3.13) we obtain (3.5).

(ii) Let $\psi \in S_0$. Integrating by parts we obtain:

$$\int_{\Omega_{\rm C}} (\boldsymbol{w} \cdot \nabla\psi)\psi r = \int_{\Omega_{\rm C}} \boldsymbol{w} \cdot \nabla\frac{|\psi|^2}{2} r = \int_{\partial\Omega_{\rm C}} \boldsymbol{w} \cdot \boldsymbol{n} \frac{|\psi|^2}{2} r - \int_{\Omega_{\rm C}} \operatorname{div} \boldsymbol{w} \frac{|\psi|^2}{2} r \tag{3.14}$$

for any $\tau \in (0, T_2)$, where $\psi := \psi(\tau, \cdot)$. On each part of $\partial\Omega_{\rm C}$ at least one of the following is satisfied: $\boldsymbol{w} \cdot \boldsymbol{n} \geqslant 0$, $\psi = 0$, or $r = 0$, which means that

$$\int_{\partial\Omega_{\rm C}} \boldsymbol{w} \cdot \boldsymbol{n} \frac{|\psi|^2}{2} r \geqslant 0. \tag{3.15}$$

The remaining term on the right of (3.14) is zero since $\operatorname{div} \boldsymbol{w} = 0$ in $\Omega_{\rm C}$. Hence, the term on the left of (3.14) is nonnegative. Let $\underline{a} := \min\{a_{\rm C}, a_{\rm P}, a_{\rm G}, a_{\rm M}\}$. We estimate the term $A(\tau, \psi, \psi)$ in particular time intervals using (3.4) as follows:

▷ For $\tau \in (0, T_1)$: Since $\psi(\tau, \cdot) \in H^1_r(\Omega) \cap H_0$, we obtain

$$A(\tau, \psi, \psi) \geqslant \underline{a} \|\nabla \psi\|^2_{L^2_r(\Omega)} \geqslant \underline{a} c(\Omega, \Gamma_{\mathrm{in}}) \|\psi\|^2_{H^1_r(\Omega)}.$$

▷ For $\tau \in (T_1, T_2)$:

$$\begin{aligned} A(\tau, \psi, \psi) &\geqslant \underline{a} \|\nabla \psi\|^2_{L^2_r(\Omega)} + \alpha (\|\psi_{|\Omega_{\mathrm{G}}}\|^2_{L^2_r(\Gamma_{\mathrm{PG}})} + \|\psi\|^2_{L^2_r(\Gamma_{\mathrm{ME}})}) \\ &\geqslant \min\{\underline{a}, \alpha\} \min\{c(\Omega_{\mathrm{CP}}, \Gamma_{\mathrm{in}}), c(\Omega_{\mathrm{G}}, \Gamma_{\mathrm{PG}}), c(\Omega_{\mathrm{M}}, \Gamma_{\mathrm{ME}})\} \|\psi\|^2_H. \end{aligned}$$

▷ For $\tau \in (T_2, T_4)$:

$$\begin{aligned} A(\tau, \psi, \psi) &\geqslant \underline{a} \|\nabla \psi\|^2_{L^2_r(\Omega_{\mathrm{GM}})} + \alpha (\|\psi_{|\Omega_{\mathrm{G}}}\|^2_{L^2_r(\Gamma_{\mathrm{PG}})} + \|\psi\|^2_{L^2_r(\Gamma_{\mathrm{ME}})}) \\ &\geqslant \min\{\underline{a}, \alpha\} \min\{c(\Omega_{\mathrm{G}}, \Gamma_{\mathrm{PG}}), c(\Omega_{\mathrm{M}}, \Gamma_{\mathrm{ME}})\} \|\psi\|^2_{H_{\mathrm{GM}}}. \end{aligned}$$

The above inequalities together prove (3.6) with the constant

$$K := \min\{\underline{a}, \alpha\} \min\{c(\Omega, \Gamma_{\mathrm{in}}), c(\Omega_{\mathrm{CP}}, \Gamma_{\mathrm{in}}), c(\Omega_{\mathrm{G}}, \Gamma_{\mathrm{PG}}), c(\Omega_{\mathrm{M}}, \Gamma_{\mathrm{ME}})\}.$$

Since the domains Ω, Ω_{CP}, Ω_{G} and Ω_{M} as well as the sets Γ_{in}, Γ_{PG} and Γ_{ME} are independent of the design variable $\varkappa \in U_{\mathrm{ad}}$, so is the constant K. □

Lemma 3.3. *The linear form F is bounded in the following sense. There exists a constant $C_F > 0$ independent of the design variable $\varkappa \in U_{\mathrm{ad}}$ such that for a.a. $\tau \in (0, T_4)$ and for all $\psi \in H$:*

$$F(\tau, \psi) \leqslant C_F [\chi_{(0,T_2)}(\tau) \|\psi\|_H + \chi_{(T_2,T_4)}(\tau) \|\psi\|_{H_{\mathrm{GM}}}]. \tag{3.16}$$

P r o o f. Lemma 3.3 can be proved using similar arguments as Lemma 3.2. □

Lemma 3.4. *There exists a constant $C^h_E > 0$ such that for every weak solution ϑ of $(P_h(\varkappa, \boldsymbol{w}))$, where $\varkappa \in U_{\mathrm{ad}}$ and $\boldsymbol{w}$ is a weak solution of $(P_f(\varkappa))$, the following estimate holds:*

$$\|\vartheta(T_4, \cdot)\|^2_{L^2_r(\Omega_{\mathrm{G}})} + \int_0^{T_4} \|\vartheta\|^2_H \leqslant C^h_E. \tag{3.17}$$

P r o o f. We use (3.3) with the test function $\delta := \vartheta - \widetilde{\vartheta}_{\rm in}$ and the fact that $\widetilde{\vartheta}_{\rm in}$ is constant in time to obtain the identity

$$M\Big(\frac{\partial\delta}{\partial t},\delta\Big)+\int_0^{T_4}A(\delta,\delta)=\int_0^{T_4}(F(\delta)-A(\widetilde{\vartheta}_{\rm in},\delta)). \tag{3.18}$$

The first term in (3.18) can be estimated with help of (3.2), Hölder's and Young's inequality:

$$\begin{aligned}M\Big(\frac{\partial\delta}{\partial t},\delta\Big)&=\|\delta(T_4,\cdot)\|^2_{L^2_r(\Omega_{\rm G})}-\|\delta(0,\cdot)\|^2_{L^2_r(\Omega_{\rm G})}\\&\geqslant\frac12\|\vartheta(T_4,\cdot)\|^2_{L^2_r(\Omega_{\rm G})}-\|\widetilde{\vartheta}_{\rm in}\|^2_{L^2_r(\Omega_{\rm G})}-\|\vartheta_0-\widetilde{\vartheta}_{\rm in}\|^2_{L^2_r(\Omega_{\rm G})}.\end{aligned}$$

The second term in (3.18) is estimated using ellipticity of A and the fact that $\int_0^{T_2}\|\delta\|^2_{H^1_r(\Omega_{\rm CP})}=\frac16\int_0^{T_4}\|\delta\|^2_{H^1_r(\Omega_{\rm CP})}$ as

$$\int_0^{T_4}A(\delta,\delta)\geqslant\frac{K}{6}\int_0^{T_4}\|\delta\|^2_H. \tag{3.19}$$

The right-hand side of (3.18) can be bounded using (3.5), (3.16), Hölder's and Young's inequality:

$$\Big|\int_0^{T_4}A(\widetilde{\vartheta}_{\rm in},\delta)\Big|\leqslant C_A\|\widetilde{\vartheta}_{\rm in}\|_H\int_0^{T_4}\|\delta\|_H\leqslant\frac{K}{24}\int_0^{T_4}\|\delta\|^2_H+\frac{6C_A^2T_4}{K}\|\widetilde{\vartheta}_{\rm in}\|^2_H, \tag{3.20}$$

$$\int_0^{T_4}F(\delta)\leqslant C_{\rm F}\int_0^{T_4}\|\delta\|_H\leqslant\frac{K}{24}\int_0^{T_4}\|\delta\|^2_H+\frac{6C_{\rm F}^2T_4}{K}. \tag{3.21}$$

Altogether, (3.18)–(3.21) yields the desired estimate with

$$C_E^h:=\frac{1}{\min\{\frac12,\frac1{12}K\}}\Big(\|\widetilde{\vartheta}_{\rm in}\|^2_{L^2_r(\Omega_{\rm G})}+\|\vartheta_0-\widetilde{\vartheta}_{\rm in}\|^2_{L^2_r(\Omega_{\rm G})}+\frac{6C_A^2T_4}{K}\|\widetilde{\vartheta}_{\rm in}\|^2_H+\frac{6C_{\rm F}^2T_4}{K}\Big).$$

$\square$

Theorem 3.1 (Existence and uniqueness for $(P_h(\varkappa,\boldsymbol{w}))$). *Let $\boldsymbol{w}\in\boldsymbol{V}(\varkappa)$ be a weak solution of $(P_f(\varkappa))$, where $\varkappa\in U_{\rm ad}$. Then there exists a unique weak solution of the problem $(P_h(\varkappa,\boldsymbol{w}))$, which satisfies* (3.17).

We shall prove Theorem 3.1 in Section 5.

4. Optimization problem

For every $\varkappa \in U_{\rm ad}$ and $\boldsymbol{w}$ a weak solution of $(P_f(\varkappa))$, we denote by $\vartheta(\varkappa, \boldsymbol{w})$ the unique weak solution to the heat conduction problem $(P_h(\varkappa, \boldsymbol{w}))$.

Let

$$\mathcal{A} := \{(\varkappa, \boldsymbol{w}, \vartheta);\ \varkappa \in U_{\rm ad},\ \boldsymbol{w} \text{ is a weak solution of } (P_f(\varkappa)),\ \vartheta = \vartheta(\varkappa, \boldsymbol{w})\}.$$

Given a constant target surface temperature $\vartheta_{\rm opt} > 0$, we introduce the cost functional $\mathcal{J}\colon\ L^2(0, T_4; H) \mapsto \mathbb{R}$ as follows:

$$\mathcal{J}(\vartheta) := \int_{\Omega_{\Gamma_{\rm PG}}} |\vartheta(T_1, \cdot) - \vartheta_{\rm opt}|^2 r. \tag{4.1}$$

The optimization problem reads:

$$\begin{cases} \text{Find } (\varkappa^*, \boldsymbol{w}^*, \vartheta^*) \in \mathcal{A} \text{ such that} \\ \mathcal{J}(\vartheta^*) \leqslant \mathcal{J}(\vartheta) \quad \forall\, (\varkappa, \boldsymbol{w}, \vartheta) \in \mathcal{A}. \end{cases} \tag{4.2}$$

In what follows we show that the map $(\varkappa, \boldsymbol{w}) \mapsto \vartheta(\varkappa, \boldsymbol{w})$ is in certain sense continuous. Combining this with the continuity of $\mathcal{J}$ and the closedness of $\mathcal{A}$ we will be able to prove the existence of a solution to (4.2).

Lemma 4.1. *Let $\varkappa^n \rightrightarrows \varkappa$ in $[\pi/2, \pi]$ and $\{\boldsymbol{w}^n\}$ be weak solutions of $(P_f(\varkappa^n))$, $n \in \mathbb{N}$. Then there exist a subsequence $\{n_k\}$ and a weak solution $\boldsymbol{w}$ of $(P_f(\varkappa))$ such that*

$$\widetilde{\boldsymbol{w}}^{n_k} \rightharpoonup \widetilde{\boldsymbol{w}} \quad \text{weakly in } \boldsymbol{V}(\overline{\varkappa}), \tag{4.3}$$

$$\vartheta^{n_k} \rightharpoonup \vartheta \quad \text{weakly in } L^2(0, T_4; H), \tag{4.4}$$

$$\frac{\partial \vartheta^{n_k}}{\partial t} \rightharpoonup \frac{\partial \vartheta}{\partial t} \quad \text{weakly in } S_0^*,\ k \to \infty, \tag{4.5}$$

where ϑ^{n_k} is the weak solution of $(P_h(\varkappa^{n_k}, \boldsymbol{w}^{n_k}))$ and ϑ is the weak solution of $(P_h(\varkappa, \boldsymbol{w}))$.

Proof. The existence of $\boldsymbol{w}$ satisfying (4.3) for an appropriate subsequence follows from Theorem 2.2. Variational formulation of $(P_h(\varkappa^{n_k}, \boldsymbol{w}^{n_k}))$ has the form

$$M\Big(\frac{\partial \vartheta^{n_k}}{\partial t}, \psi\Big) + \int_0^{T_4} A_{n_k}(\vartheta^{n_k}, \psi) = \int_0^{T_4} F(\psi) \quad \forall\, \psi \in S_0, \tag{4.6}$$

where A_{n_k} denotes the bilinear form A from Definition 3.1 with $\varkappa^{n_k}$ and $\boldsymbol{w}^{n_k}$ in the place of $\varkappa$ and $\boldsymbol{w}$, respectively.

From (3.17) it follows that $\{\vartheta^{n_k}\}$ is bounded in $L^2(0,T_4;H)$. Consequently, (4.6) implies that also $\{\partial\vartheta^{n_k}/\partial t\}$ is bounded in S_0^*. Hence, there exists a subsequence (denoted by the same symbol) and a function $\widehat{\vartheta} \in L^2(0,T_4;H)$ such that $\widehat{\vartheta} - \widetilde{\vartheta}_{\text{in}} \in W_{\text{per}}$,

$$\vartheta^{n_k} \rightharpoonup \widehat{\vartheta} \qquad \text{weakly in } L^2(0,T_4;H), \tag{4.7}$$

$$\frac{\partial\vartheta^{n_k}}{\partial t} \rightharpoonup \frac{\partial\widehat{\vartheta}}{\partial t} \qquad \text{weakly in } S_0^*,\ k\to\infty. \tag{4.8}$$

To show that $\widehat{\vartheta}$ solves $(P_h(\varkappa,\boldsymbol{w}))$, we need to pass to the limit in (4.6).

Clearly, (4.8) implies that

$$M\Big(\frac{\partial\vartheta^{n_k}}{\partial t},\psi\Big) \to M\Big(\frac{\partial\widehat{\vartheta}}{\partial t},\psi\Big) \quad \forall\,\psi\in S_0,\ k\to\infty.$$

It remains to show that

$$\int_0^{T_4} A_{n_k}(\vartheta^{n_k},\psi) \to \int_0^{T_4} A(\widehat{\vartheta},\psi) \quad \forall\,\psi\in S_0. \tag{4.9}$$

To prove it, we write:

$$\begin{aligned}
\int_0^{T_4} A_{n_k}(\vartheta^{n_k},\psi) - \int_0^{T_4} A(\widehat{\vartheta},\psi) &= \int_0^{T_2}\int_{\Omega_{\text{CP}}} (\boldsymbol{w}^{n_k}\cdot\nabla\vartheta^{n_k} - \boldsymbol{w}\cdot\nabla\widehat{\vartheta})\psi r\\
&+ \int_0^{T_2}\int_{\Omega_{\text{CP}}} (a(\varkappa^{n_k})\nabla\vartheta^{n_k} - a(\varkappa)\nabla\widehat{\vartheta})\cdot\nabla\psi r\\
&+ \int_0^{T_4}\int_{\Omega_{\text{GM}}} a(\varkappa)\nabla(\vartheta^{n_k}-\widehat{\vartheta})\cdot\nabla\psi r\\
&+ \int_{T_1}^{T_2}\int_{\Gamma_{\text{PG}}} \alpha((\vartheta^{n_k}-\widehat{\vartheta})\psi)_{|\Omega_{\text{CP}}} r + \int_{T_1}^{T_4}\int_{\Gamma_{\text{PG}}} \alpha((\vartheta^{n_k}-\widehat{\vartheta})\psi)_{|\Omega_{\text{G}}} r\\
&+ \int_{T_3}^{T_4}\int_{\Gamma_{\text{GM}}} \alpha((\vartheta^{n_k}-\widehat{\vartheta})\psi)_{|\Omega_{\text{G}}} r + \int_{T_3}^{T_4}\int_{\Gamma_{\text{GM}}} \alpha((\vartheta^{n_k}-\widehat{\vartheta})\psi)_{|\Omega_{\text{M}}} r\\
&+ \int_0^{T_4}\int_{\Gamma_{\text{ME}}} \alpha(\vartheta^{n_k}-\widehat{\vartheta})\psi r =: I_1+\ldots+I_8.
\end{aligned}$$

From (4.3) it follows that

$$\boldsymbol{w}^{n_k} \to \boldsymbol{w} \quad \text{(strongly) in } L_r^4(\Omega_{\text{CP}}), \tag{4.10}$$

hence,

$$I_1 = \int_0^{T_2}\int_{\Omega_{\text{CP}}} (\boldsymbol{w}^{n_k}-\boldsymbol{w})\cdot\nabla\vartheta^{n_k}\psi r + \int_0^{T_2}\int_{\Omega_{\text{CP}}} \boldsymbol{w}\cdot\nabla(\vartheta^{n_k}-\widehat{\vartheta})\psi r \to 0, \quad k\to\infty,$$

where the first term vanishes due to (4.10) and the second one due to (4.7).

Further, we decompose

$$
\begin{aligned}
I_2 = &\int_0^{T_2}\int_{\Omega_{\rm CP}} (a(\varkappa^{n_k}) - a(\varkappa))\nabla\vartheta^{n_k}\cdot\nabla\psi r \\
&+ \int_0^{T_2}\int_{\Omega_{\rm CP}} a(\varkappa)\nabla(\vartheta^{n_k} - \widehat{\vartheta})\cdot\nabla\psi r =: I_{21} + I_{22}.
\end{aligned}
$$

Assume for a while that $\psi \in L^\infty(0,T_2;\mathcal{H})$. Since $a(\varkappa^{n_k}) \to a(\varkappa)$ strongly in $L^q(\Omega_{\rm CP})$ for any $q\in[1,\infty)$, we have

$$
\begin{aligned}
(4.11)\quad |I_{21}| \leqslant\; &\|a(\varkappa^{n_k}) - a(\varkappa)\|_{L^4_r(\Omega_{\rm CP})} \\
&\times\Bigl(\int_0^{T_2}\|\nabla\vartheta^{n_k}\|^2_{L^2_r(\Omega_{\rm CP})}\Bigr)^{1/2}\Bigl(\int_0^{T_2}\|\nabla\psi\|^2_{L^4_r(\Omega_{\rm CP})}\Bigr)^{1/2} \to 0,\quad k\to\infty.
\end{aligned}
$$

If $\psi \in S_0$, then clearly $\psi|_{(0,T_2)\times\Omega_{\rm CP}} \in L^2(0,T_2;H^1_{r,0}(\Omega_{\rm CP}))$. From the density of simple functions in this space and the density of $\mathcal{H}$ in $H^1_{r,0}(\Omega_{\rm CP})$ we infer that ψ can be approximated by a sequence of functions in $L^\infty(0,T_2;\mathcal{H})$ and thus (4.11) holds true. Terms $I_{22}, I_3,\ldots,I_8$ vanish for $k\to\infty$ due to (4.7).

This completes the proof of (4.9) and thus $\widehat{\vartheta}$ is the weak solution of $(P_h(\varkappa,\boldsymbol{w}))$.

□

R e m a r k 4.1. Let X be a Banach space. We call $\psi\colon [0,T_2]\to X$ simple if it has the form $\psi(t) = \sum\limits_{i=1}^n \chi_{E_i}(t)x_i$, where $E_i\subset[0,T_2]$ is measurable and $x_i\in X$, $i=1,\ldots,n$. See e.g. [12].

Theorem 4.1 (Existence of a solution to (4.2))**.** *The optimal design problem* (4.2) *has at least one solution.*

P r o o f. We use classical arguments of calculus of variations (see e.g. [5], Theorem 2.1). The set $U_{\rm ad}$ is bounded and closed in $C([\pi/2,\pi])$ and, moreover, consists of uniformly continuous functions. The theorem of Arzelà-Ascoli implies the compactness of $U_{\rm ad}$ in $C([\pi/2,\pi])$. Let $(\varkappa^n,\boldsymbol{w}^n,\vartheta^n)$ be a sequence of functions minimizing $J\colon(\varkappa,\boldsymbol{w},\vartheta)\mapsto\mathcal{J}(\vartheta)$. Then there exists a subsequence $\varkappa^{n_k}$ and $\varkappa\in U_{\rm ad}$ such that $\varkappa^{n_k}\rightrightarrows\varkappa$ uniformly in $[\pi/2,\pi]$ and due to Theorem 2.2 also $\widetilde{\boldsymbol{w}}^{n_k}\rightharpoonup\widetilde{\boldsymbol{w}}$ in $\boldsymbol{V}(\overline{\varkappa})$, where $\boldsymbol{w}$ solves $(P_f(\varkappa))$.

From Lemma 4.1 we get that $\vartheta(\varkappa^{n_k},\boldsymbol{w}^{n_k})\rightharpoonup\vartheta(\varkappa,\boldsymbol{w})$ weakly in $L^2(0,T_4;H)$. Since the cost functional $\mathcal{J}$ is weak lower semicontinuous with respect to the above convergence, we have that

$$
J(\varkappa,\boldsymbol{w},\vartheta(\varkappa,\boldsymbol{w})) \leqslant \liminf_{n\to\infty} J(\varkappa^{n_k},\boldsymbol{w}^{n_k},\vartheta(\varkappa^{n_k},\boldsymbol{w}^{n_k})) = \inf J.
$$

Thus $(\varkappa,\boldsymbol{w},\vartheta(\varkappa,\boldsymbol{w}))$ is an optimal triplet for (4.2). □

5. Time discretization

The state problem $(P_h(\varkappa, \boldsymbol{w}))$ is solved by the Rothe method with the following time discretization.

For any $n \in \mathbb{N}$ we set the time step $\Delta t_n := T_4/(24n)$ and the times $t_k^n := k\Delta t_n$, $k = 0, \dots, 24n$, so that the four important time moments become: $T_1 = 2n\Delta t_n$, $T_2 = 4n\Delta t_n$, $T_3 = 13n\Delta t_n$, $T_4 = 24n\Delta t_n$.

We define discretization of the function spaces:

$$\begin{aligned} H_k^n &:= \{\psi \in H;\ \text{if } k \in \{1, \dots, 2n\},\ \text{then } \psi_{|\Omega_{\mathrm{P}}} = \psi_{|\Omega_{\mathrm{G}}} \text{ on } \Gamma_{\mathrm{PG}}, \\ &\qquad\qquad\quad \text{if } k \in \{1, \dots, 13n\},\ \text{then } \psi_{|\Omega_{\mathrm{G}}} = \psi_{|\Omega_{\mathrm{M}}} \text{ on } \Gamma_{\mathrm{GM}}\}, \\ H_{0,k}^n &:= H_k^n \cap H_0, \\ \boldsymbol{S}^n &:= \prod_{k=0}^{24n} H_k^n,\ \boldsymbol{S}_0^n := \prod_{k=1}^{24n} H_{0,k}^n. \end{aligned}$$

Definition 5.1 (Time discretized problem $(\mathbb{T}_n)$)**.** For $n \in \mathbb{N}$, find $(\vartheta_0^n, \vartheta_1^n, \dots, \vartheta_{24n}^n) \in \boldsymbol{S}^n$ such that:

- ▷ $(\vartheta_1^n - \widetilde{\vartheta}_{\mathrm{in}}, \dots, \vartheta_{24n}^n - \widetilde{\vartheta}_{\mathrm{in}}) \in \boldsymbol{S}_0^n$;
- ▷ $\vartheta_0^n = \vartheta_0$ in Ω_{G};
- ▷ $\vartheta_0^n = \vartheta_{24n}^n$ in Ω_{M};
- ▷ for all $k = 0, \dots, 20n$: $\vartheta_k^n = \vartheta_{k+4n}^n$ in Ω_{CP};
- ▷ for all $(\psi_1, \dots, \psi_{24n}) \in \boldsymbol{S}_0^n$:

$$\begin{aligned} &\int_\Omega \vartheta_k^n \psi_k r + \Delta t_n A(t_k^n, \vartheta_k^n, \psi_k) \\ &\qquad = \Delta t_n F(t_k^n, \psi_k) + \int_\Omega \vartheta_{k-1}^n \psi_k r, \quad k = 1, \dots, 4n, \\ &\int_{\Omega_{\mathrm{GM}}} \vartheta_k^n \psi_k r + \Delta t_n A(t_k^n, \vartheta_k^n, \psi_k) \\ &\qquad = \Delta t_n F(t_k^n, \psi_k) + \int_{\Omega_{\mathrm{GM}}} \vartheta_{k-1}^n \psi_k r, \quad k = 4n+1, \dots, 24n. \end{aligned}$$

This problem will be solved by a fixed-point argument. Let us define an auxiliary problem:

Definition 5.2 (Auxiliary problem $(\mathbb{A}_{n,\varphi})$)**.** Given $n \in \mathbb{N}$, $\varphi \in L_r^2(\Omega_{\mathrm{CP}} \cup \Omega_{\mathrm{M}})$. Set

$$z_0^n(\varphi) := \begin{cases} \varphi & \text{in } \Omega_{\mathrm{CP}} \cup \Omega_{\mathrm{M}}, \\ \vartheta_0 - \widetilde{\vartheta}_{\mathrm{in}} & \text{in } \Omega_{\mathrm{G}}. \end{cases}$$

Find $(z_1^n(\varphi),\dots,z_{24n}^n(\varphi))\in \boldsymbol{S}_0^n$ such that:

▷ for all $k=1,\dots,20n$: $z_k^n(\varphi)=z_{k+4n}^n(\varphi)$ in $\Omega_{\rm CP}$;

▷ for all $(\psi_1,\dots,\psi_{24n})\in \boldsymbol{S}_0^n$:

$$\begin{aligned}\int_\Omega z_k^n(\varphi)\psi_k r+\Delta t_n A(t_k^n,z_k^n(\varphi),\psi_k)\\ =\Delta t_n F(t_k^n,\psi_k)-\Delta t_n A(t_k^n,\widetilde{\vartheta}_{\rm in},\psi_k)\\ +\int_\Omega z_{k-1}^n(\varphi)\psi_k r,\quad k=1,\dots,4n,\end{aligned}\tag{5.1}$$

$$\begin{aligned}\int_{\Omega_{\rm GM}} z_k^n(\varphi)\psi_k r+\Delta t_n A(t_k^n,z_k^n(\varphi),\psi_k)\\ =\Delta t_n F(t_k^n,\psi_k)-\Delta t_n A(t_k^n,\widetilde{\vartheta}_{\rm in},\psi_k)\\ +\int_{\Omega_{\rm GM}} z_{k-1}^n(\varphi)\psi_k r,\quad k=4n+1,\dots,24n.\end{aligned}\tag{5.2}$$

Lemma 5.1. *For any $n\in\mathbb{N}$ and $\varphi\in L_r^2(\Omega_{\rm CP}\cup\Omega_{\rm M})$, problem $(\mathbb{A}_{n,\varphi})$ has a unique solution $(z_0^n(\varphi),\dots,z_{24n}^n(\varphi))$. In addition, there exists $q\in(0,1)$ such that for all $n\in\mathbb{N}$ and $\varphi_1,\varphi_2\in L_r^2(\Omega_{\rm CP}\cup\Omega_{\rm M})$ it holds:*

$$\|z_{24n}^n(\varphi_1)-z_{24n}^n(\varphi_2)\|_{L_r^2(\Omega)}\leqslant q\|\varphi_1-\varphi_2\|_{L_r^2(\Omega_{\rm CP}\cup\Omega_{\rm M})}.\tag{5.3}$$

P r o o f. Existence and uniqueness is a consequence of the Lax-Milgram theorem, boundedness and ellipticity of forms in (5.1) and (5.2).

To show (5.3) we subtract (5.1) for φ_1 and φ_2, and take $\psi_k:=d_k^n$, where $d_k^n:=z_k^n(\varphi_1)-z_k^n(\varphi_2)$, $k=1,\dots,4n$:

$$\|d_k^n\|^2_{L_r^2(\Omega)}+\Delta t_n A(t_k^n,d_k^n,d_k^n)=\int_\Omega d_{k-1}^n d_k^n r\leqslant\frac12\|d_{k-1}^n\|^2_{L_r^2(\Omega)}+\frac12\|d_k^n\|^2_{L_r^2(\Omega)}.$$

We use the fact that A is H_0-elliptic with the constant K, subtract $\frac12\|d_k^n\|^2_{L_r^2(\Omega)}$ and multiply it by two to get

$$\|d_k^n\|^2_{L_r^2(\Omega)}+2\Delta t_n K\|d_k^n\|^2_H\leqslant\|d_{k-1}^n\|^2_{L_r^2(\Omega)}.$$

Further we use the estimate $\|d_k^n\|^2_{L_r^2(\Omega)}\leqslant\|d_k^n\|^2_H$ to get

$$\|d_k^n\|^2_{L_r^2(\Omega)}\leqslant\frac{1}{1+2\Delta t_n K}\|d_{k-1}^n\|^2_{L_r^2(\Omega)}.$$

We substitute $\Delta t_n = T_4/(24n)$ and apply the above inequality recurrently to get the estimate on the last time layer $4n$:

$$\begin{aligned}\|d^n_{4n}\|^2_{L^2_r(\Omega)} &\leqslant \frac{1}{1+\frac{1}{12n}T_4K}\|d^n_{4n-1}\|^2_{L^2_r(\Omega)} \leqslant \dots \\ &\leqslant \frac{1}{(1+\frac{1}{12n}T_4K)^{4n}}\|\varphi_1-\varphi_2\|^2_{L^2_r(\Omega_{\mathrm{CP}}\cup\Omega_{\mathrm{M}})}.\end{aligned}$$

Thus, for all $n \in \mathbb{N}$ it holds:

$$\|d^n_{4n}\|^2_{L^2_r(\Omega)} \leqslant \frac{1}{(1+\frac{1}{12}T_4K)^4}\|\varphi_1-\varphi_2\|^2_{L^2_r(\Omega_{\mathrm{CP}}\cup\Omega_{\mathrm{M}})}. \tag{5.4}$$

Similarly for $k = 4n+1,\dots,24n$ we get from (5.2):

$$\begin{aligned}\|d^n_k\|^2_{L^2_r(\Omega_{\mathrm{GM}})} + \Delta t_n A(t^n_k, d^n_k, d^n_k) &= \int_{\Omega_{\mathrm{GM}}} d^n_{k-1} d^n_k r \\ &\leqslant \frac{1}{2}\|d^n_{k-1}\|^2_{L^2_r(\Omega_{\mathrm{GM}})} + \frac{1}{2}\|d^n_k\|^2_{L^2_r(\Omega_{\mathrm{GM}})}.\end{aligned}$$

Analogously to the first part we get

$$\|d^n_k\|^2_{L^2_r(\Omega_{\mathrm{GM}})} \leqslant \frac{1}{1+2\Delta t_n K}\|d^n_{k-1}\|^2_{L^2_r(\Omega_{\mathrm{GM}})}, \quad k = 4n+1,\dots,24n$$

and then

$$\|d^n_{24n}\|^2_{L^2_r(\Omega_{\mathrm{GM}})} \leqslant \frac{1}{1+\frac{1}{12n}T_4K}\|d^n_{24n-1}\|^2_{L^2_r(\Omega_{\mathrm{GM}})} \leqslant \frac{1}{(1+\frac{1}{12n}T_4K)^{20n}}\|d^n_{4n}\|^2_{L^2_r(\Omega_{\mathrm{GM}})}.$$

Thus, for all $n \in \mathbb{N}$ it holds that

$$\|d^n_{24n}\|^2_{L^2_r(\Omega_{\mathrm{GM}})} \leqslant \frac{1}{(1+\frac{1}{12}T_4K)^{20}}\|d^n_{4n}\|^2_{L^2_r(\Omega_{\mathrm{GM}})}. \tag{5.5}$$

Together we have

$$\begin{aligned}\|d^n_{24n}\|^2_{L^2_r(\Omega)} &= \|d^n_{24n}\|^2_{L^2_r(\Omega_{\mathrm{GM}})} + \|d^n_{4n}\|^2_{L^2_r(\Omega_{\mathrm{CP}})} \\ &\leqslant \frac{1}{(1+\frac{1}{12}T_4K)^{20}}\|d^n_{4n}\|^2_{L^2_r(\Omega_{\mathrm{GM}})} + \|d^n_{4n}\|^2_{L^2_r(\Omega_{\mathrm{CP}})} \\ &\leqslant \|d^n_{4n}\|^2_{L^2_r(\Omega)} \leqslant \frac{1}{(1+\frac{1}{12}T_4K)^4}\|\varphi_1-\varphi_2\|^2_{L^2_r(\Omega_{\mathrm{CP}}\cup\Omega_{\mathrm{M}})}.\end{aligned} \tag{5.6}$$

Since

$$q := \frac{1}{(1+\frac{1}{12}T_4K)^2} < 1,$$

(5.3) is proved. □

Lemma 5.2. *For any $n \in \mathbb{N}$, problem $(\mathbb{T}_n)$ has a unique solution.*

P r o o f. Let us define the map $\Psi^n\colon L^2_r(\Omega_{\rm CP} \cup \Omega_{\rm M}) \to L^2_r(\Omega_{\rm CP} \cup \Omega_{\rm M})$:

$$\Psi^n(\varphi) := z^n_{24n}(\varphi).$$

From Lemma 5.1 we have

$$\begin{aligned}\|\Psi^n(\varphi^1) - \Psi^n(\varphi^2)\|_{L^2_r(\Omega_{\rm CP}\cup\Omega_{\rm M})} &= \|z^n_{24n}(\varphi_1) - z^n_{24n}(\varphi_2)\|_{L^2_r(\Omega_{\rm CP}\cup\Omega_{\rm M})} \\ &\leqslant q\|\varphi_1 - \varphi_2\|^2_{L^2_r(\Omega_{\rm CP}\cup\Omega_{\rm M})}.\end{aligned}$$

Banach fixed-point theorem implies that Ψ^n has a unique fixed point $\overline{\varphi} \in L^2_r(\Omega_{\rm CP} \cup \Omega_{\rm M})$. In fact, $z^n_{24n}(\overline{\varphi}) \in H^n_{0,24n}$ and thus $\overline{\varphi}_{|\Omega_{\rm CP}} \in H^1_r(\Omega_{\rm CP})$ and $\overline{\varphi}_{|\Omega_{\rm M}} \in H^1_r(\Omega_{\rm M})$. Setting $\vartheta^n_k := z^n_k(\overline{\varphi}) + \widetilde{\vartheta}_{\rm in}$, $k = 0, \dots, 24n$, we can easily verify that $(\vartheta^n_0, \dots, \vartheta^n_{24n})$ solves $(\mathbb{T}_n)$. □

Lemma 5.3. *There exists a constant $C = C(\|\vartheta_0\|_{L^2_r(\Omega_{\rm G})}, \|\widetilde{\vartheta}_{\rm in}\|_H, T_4)$ (independent of $\varkappa$) such that for all $n \in \mathbb{N}$:*

$$\|\vartheta^n_{24n}\|^2_{L^2_r(\Omega_{\rm G})} + \sum_{k=1}^{24n} \|\vartheta^n_k - \vartheta^n_{k-1}\|^2_{L^2_r(\Omega)} + \Delta t_n \sum_{k=1}^{24n} \|\vartheta^n_k\|^2_H \leqslant C. \tag{5.7}$$

P r o o f. Let $(\vartheta^n_0, \vartheta^n_1, \dots, \vartheta^n_{24n})$ be the solution of $(\mathbb{T}_n)$. For $k = 1, \dots, 4n$ we have

$$\int_\Omega (\vartheta^n_k - \vartheta^n_{k-1})\psi_k r + \Delta t_n A(t^n_k, \vartheta^n_k, \psi_k) = \Delta t_n F(t^n_k, \psi_k) \quad \forall\, \psi_k \in \boldsymbol{H}^n_{0,k}. \tag{5.8}$$

Setting $\psi_k := \vartheta^n_k - \widetilde{\vartheta}_{\rm in}$, we obtain

$$\begin{aligned}\|\vartheta^n_k - \vartheta^n_{k-1}\|^2_{L^2_r(\Omega)} + \Delta t_n A(t^n_k, \vartheta^n_k, \vartheta^n_k) &= \int_\Omega (\vartheta^n_k - \vartheta^n_{k-1})\widetilde{\vartheta}_{\rm in} r + \|\vartheta^n_{k-1})\|^2_{L^2_r(\Omega)} \\ &- \int_\Omega \vartheta^n_k \vartheta^n_{k-1} r + \Delta t_n A(t^n_k, \vartheta^n_k, \widetilde{\vartheta}_{\rm in}) + \Delta t_n F(t^n_k, \vartheta^n_k - \widetilde{\vartheta}_{\rm in}).\end{aligned} \tag{5.9}$$

By a different manipulation we get from (5.8) using the same test function:

$$\begin{aligned}-\int_\Omega \vartheta^n_k \vartheta^n_{k-1} r &= \int_\Omega \vartheta^n_k(\widetilde{\vartheta}_{\rm in} - \vartheta^n_k) r - \int_\Omega \vartheta^n_{k-1}\widetilde{\vartheta}_{\rm in} r \\ &- \Delta t_n A(t^n_k, \vartheta^n_k, \vartheta^n_k - \widetilde{\vartheta}_{\rm in}) + \Delta t_n F(t^n_k, \vartheta^n_k - \widetilde{\vartheta}_{\rm in}).\end{aligned} \tag{5.10}$$

Inserting (5.10) into (5.9), we obtain

$$
\begin{aligned}
(5.11) \qquad & \|\vartheta_k^n\|^2_{L^2_r(\Omega)} + \|\vartheta_k^n - \vartheta_{k-1}^n\|^2_{L^2_r(\Omega)} + 2\Delta t_n A(t_k^n, \vartheta_k^n, \vartheta_k^n) \\
&\quad = \|\vartheta_{k-1}^n\|^2_{L^2_r(\Omega)} + 2\int_\Omega (\vartheta_k^n - \vartheta_{k-1}^n)\widetilde{\vartheta}_{\mathrm{in}} r \\
&\qquad + 2\Delta t_n A(t_k^n, \vartheta_k^n, \widetilde{\vartheta}_{\mathrm{in}}) + 2\Delta t_n F(t_k^n, \vartheta_k^n - \widetilde{\vartheta}_{\mathrm{in}}).
\end{aligned}
$$

Ellipticity of A and boundedness of A, F yields

$$
\begin{aligned}
A(t_k^n, \vartheta_k^n, \vartheta_k^n) &\geqslant K\|\vartheta_k^n\|_H^2, \\
A(t_k^n, \vartheta_k^n, \widetilde{\vartheta}_{\mathrm{in}}) &\leqslant C_A\|\vartheta_k^n\|_H\|\widetilde{\vartheta}_{\mathrm{in}}\|_H \leqslant \frac{K}{4}\|\vartheta_k^n\|_H^2 + \frac{C_A^2}{K}\|\widetilde{\vartheta}_{\mathrm{in}}\|_H^2, \\
F(t_k^n, \vartheta_k^n - \widetilde{\vartheta}_{\mathrm{in}}) &\leqslant C_{\mathrm{F}}(\|\vartheta_k^n\|_H + \|\widetilde{\vartheta}_{\mathrm{in}}\|_H) \leqslant \frac{K}{4}\|\vartheta_k^n\|_H^2 + \frac{C_{\mathrm{F}}^2}{K} + C_{\mathrm{F}}\|\widetilde{\vartheta}_{\mathrm{in}}\|_H,
\end{aligned}
$$

which together with (5.11) leads to the inequality

$$
\begin{aligned}
& \|\vartheta_k^n\|^2_{L^2_r(\Omega)} + \|\vartheta_k^n - \vartheta_{k-1}^n\|^2_{L^2_r(\Omega)} + K\Delta t_n\|\vartheta_k^n\|_H^2 \\
&\quad \leqslant \|\vartheta_{k-1}^n\|^2_{L^2_r(\Omega)} + 2\int_\Omega (\vartheta_k^n - \vartheta_{k-1}^n)\widetilde{\vartheta}_{\mathrm{in}} r + 2\frac{C_A^2}{K}\Delta t_n\|\widetilde{\vartheta}_{\mathrm{in}}\|_H^2 \\
&\qquad + 2C_{\mathrm{F}}\Delta t_n\|\widetilde{\vartheta}_{\mathrm{in}}\|_H + 2\frac{C_{\mathrm{F}}^2}{K}\Delta t_n.
\end{aligned}
$$

Summing the above inequality over $k = 1, \ldots, 4n$, we obtain

$$
\begin{aligned}
(5.12) \qquad & \|\vartheta_{4n}^n\|^2_{L^2_r(\Omega)} + \sum_{k=1}^{4n}\|\vartheta_k^n - \vartheta_{k-1}^n\|^2_{L^2_r(\Omega)} + K\Delta t_n\sum_{k=1}^{4n}\|\vartheta_k^n\|_H^2 \\
&\quad \leqslant \|\vartheta_0^n\|^2_{L^2_r(\Omega)} + 2\int_\Omega (\vartheta_{4n}^n - \vartheta_0^n)\widetilde{\vartheta}_{\mathrm{in}} r \\
&\qquad + 2\frac{C_A^2}{K}T_2\|\widetilde{\vartheta}_{\mathrm{in}}\|_H^2 + 2C_{\mathrm{F}}T_2\|\widetilde{\vartheta}_{\mathrm{in}}\|_H + 2\frac{C_{\mathrm{F}}^2}{K}T_2.
\end{aligned}
$$

Using the fact that $\vartheta_{4n}^n = \vartheta_0^n$ in Ω_{CP}, we can simplify (5.12) into

$$
\begin{aligned}
(5.13) \qquad & \|\vartheta_{4n}^n\|^2_{L^2_r(\Omega_{\mathrm{GM}})} + \sum_{k=1}^{4n}\|\vartheta_k^n - \vartheta_{k-1}^n\|^2_{L^2_r(\Omega)} + K\Delta t_n\sum_{k=1}^{4n}\|\vartheta_k^n\|_H^2 \\
&\quad \leqslant \|\vartheta_0^n\|^2_{L^2_r(\Omega_{\mathrm{GM}})} + 2\int_{\Omega_{\mathrm{GM}}} (\vartheta_{4n}^n - \vartheta_0^n)\widetilde{\vartheta}_{\mathrm{in}} r + 2\frac{C_A^2}{K}T_2\|\widetilde{\vartheta}_{\mathrm{in}}\|_H^2 \\
&\qquad + 2C_{\mathrm{F}}T_2\|\widetilde{\vartheta}_{\mathrm{in}}\|_H + 2\frac{C_{\mathrm{F}}^2}{K}T_2.
\end{aligned}
$$

For $k = 4n+1, \ldots, 24n$ we obtain analogously

$$
\begin{aligned}
(5.14)\quad & \|\vartheta^n_{24n}\|^2_{L^2_r(\Omega_{\mathrm{GM}})} + \sum_{k=4n+1}^{24n} \|\vartheta^n_k - \vartheta^n_{k-1}\|^2_{L^2_r(\Omega_{\mathrm{GM}})} + K\Delta t_n \sum_{k=4n+1}^{24n} \|\vartheta^n_k\|^2_{H_{\mathrm{GM}}} \\
& \leqslant \|\vartheta^n_{4n}\|^2_{L^2_r(\Omega_{\mathrm{GM}})} + 2\int_{\Omega_{\mathrm{GM}}} (\vartheta^n_{24n} - \vartheta^n_{4n})\widetilde{\vartheta}_{\mathrm{in}} r + 2\frac{C_A^2}{K}(T_4 - T_2)\|\widetilde{\vartheta}_{\mathrm{in}}\|^2_H \\
& \quad + 2C_{\mathrm{F}}(T_4 - T_2)\|\widetilde{\vartheta}_{\mathrm{in}}\|_H + 2\frac{C_{\mathrm{F}}^2}{K}(T_4 - T_2).
\end{aligned}
$$

From (5.13) and (5.14) and the fact that $\vartheta^n_{24n} = \vartheta^n_0$ in Ω_{M} it follows that

$$
\begin{aligned}
& \|\vartheta^n_{24n}\|^2_{L^2_r(\Omega_{\mathrm{G}})} + \sum_{k=1}^{4n} \|\vartheta^n_k - \vartheta^n_{k-1}\|^2_{L^2_r(\Omega)} + \sum_{k=4n+1}^{24n} \|\vartheta^n_k - \vartheta^n_{k-1}\|^2_{L^2_r(\Omega_{\mathrm{GM}})} \\
& \quad + K\Delta t_n \sum_{k=1}^{4n} \|\vartheta^n_k\|^2_H + K\Delta t_n \sum_{k=4n+1}^{24n} \|\vartheta^n_k\|^2_{H_{\mathrm{GM}}} \\
& \leqslant \|\vartheta^n_0\|^2_{L^2_r(\Omega_{\mathrm{G}})} + 2\int_{\Omega_{\mathrm{G}}} (\vartheta^n_{24n} - \vartheta^n_0)\widetilde{\vartheta}_{\mathrm{in}} r + 2\frac{C_A^2}{K} T_4 \|\widetilde{\vartheta}_{\mathrm{in}}\|^2_H \\
& \quad + 2C_{\mathrm{F}} T_4 \|\widetilde{\vartheta}_{\mathrm{in}}\|_H + 2\frac{C_{\mathrm{F}}^2}{K} T_4.
\end{aligned}
$$

Hölder's and Young's inequality then yield

$$
\begin{aligned}
& \frac{1}{2}\|\vartheta^n_{24n}\|^2_{L^2_r(\Omega_{\mathrm{G}})} + \frac{1}{6}\sum_{k=1}^{24n} \|\vartheta^n_k - \vartheta^n_{k-1}\|^2_{L^2_r(\Omega_{\mathrm{CP}})} \\
& \quad + \sum_{k=1}^{24n} \|\vartheta^n_k - \vartheta^n_{k-1}\|^2_{L^2_r(\Omega_{\mathrm{GM}})} + \frac{K}{6}\Delta t_n \sum_{k=1}^{24n} \|\vartheta^n_k\|^2_H \\
& \leqslant 2\|\vartheta^n_0\|^2_{L^2_r(\Omega_{\mathrm{G}})} + 2\|\widetilde{\vartheta}_{\mathrm{in}}\|^2_{L^2_r(\Omega_{\mathrm{G}})} + 2\frac{C_A^2}{K} T_4 \|\widetilde{\vartheta}_{\mathrm{in}}\|^2_H \\
& \quad + 2C_{\mathrm{F}} T_4 \|\widetilde{\vartheta}_{\mathrm{in}}\|_H + 2\frac{C_{\mathrm{F}}^2}{K} T_4.
\end{aligned}
$$

$\square$

Proof of Theorem 3.1. Let $(\vartheta^n_0, \vartheta^n_1, \ldots, \vartheta^n_{24n}) \in \boldsymbol{S}^n$ be the solution of $(\mathbb{T}_n)$, $n \in \mathbb{N}$. We define the piecewise constant and piecewise linear interpolations:

$$
\begin{aligned}
(5.15)\quad & \overline{\vartheta}^n(t,x) := \vartheta^n_k(x) \quad \text{for } t \in (t^n_{k-1}, t^n_k], \\
& \widehat{\vartheta}^n(t,x) := \vartheta^n_{k-1}(x) + \frac{t - t^n_k}{\Delta t_n}(\vartheta^n_k(x) - \vartheta^n_{k-1}(x)) \quad \text{for } t \in [t^n_{k-1}, t^n_k].
\end{aligned}
$$

Then $(\mathbb{T}_n)$ reads:

$$(5.16)\qquad \forall\,(\psi_1,\dots,\psi_{24n})\in \boldsymbol{S}_0^n\colon\ M\Big(\frac{\partial\widehat{\vartheta}^n}{\partial t},\overline{\psi}\Big)+\int_0^{T_4}A(\cdot,\overline{\vartheta}^n(\cdot),\overline{\psi})=\int_0^{T_4}F(\cdot,\overline{\psi}),$$

where $\overline{\psi}$ is defined analogously as in (5.15).

From Lemma 5.3 it follows that

$$(5.17)\qquad \|\widehat{\vartheta}^n(T_4)\|^2_{L^2_r(\Omega_{\rm G})}+\Delta t_n\int_0^{T_4}\Big\|\frac{\partial\widehat{\vartheta}^n}{\partial t}\Big\|^2_{L^2_r(\Omega_{\rm G})}+\int_0^{T_4}\|\overline{\vartheta}^n\|^2_H\leqslant C,$$

hence the sequence $\{\overline{\vartheta}^n\}$ is bounded in $L^2(0,T_4;H)$.

We shall also need uniform bounds for $\{\partial\widehat{\vartheta}^n/\partial t\}$. We observe that for $k=1,\dots,2n$, the spaces $H^n_{0,k}$ are identical (we shall denote them by H_{0,T_1}). Then

$$\begin{aligned}\int_0^{T_1}\Big\|\frac{\partial\widehat{\vartheta}^n}{\partial t}\Big\|^2_{H^*_{0,T_1}}&=\int_0^{T_1}\Big(\sup_{\substack{\psi\in H_{0,T_1}\\ \|\psi\|_H=1}}\int_\Omega\frac{\partial\widehat{\vartheta}^n}{\partial t}\psi r\Big)^2\\&=\sup_{\substack{\psi\in H_{0,T_1}\\ \|\psi\|_H=1}}\int_0^{T_1}(F(\cdot,\psi)-A(\cdot,\overline{\vartheta}^n(\cdot),\psi))^2\\&\leqslant\int_0^{T_1}(C_{\rm F}+C_A\|\overline{\vartheta}^n\|_H)^2\leqslant C.\end{aligned}$$

Thus $\{\partial\widehat{\vartheta}^n/\partial t\}$ is bounded in $L^2(0,T_1;H^*_{0,T_1})$. Similarly we obtain boundedness on the intervals (T_1,T_3) and (T_3,T_4), concluding that $\{\partial\widehat{\vartheta}^n/\partial t\}$ is bounded in S_0^*.

Now we can pass to a subsequence (denoted by the same symbol) such that

$$\begin{aligned}\overline{\vartheta}^n&\rightharpoonup\vartheta &&\text{weakly in } L^2(0,T_4;H),\\ \frac{\partial\widehat{\vartheta}^n}{\partial t}&\rightharpoonup\frac{\partial\vartheta}{\partial t}&&\text{weakly in } S_0^*,\ n\to\infty,\end{aligned}$$

where $\vartheta\in L^2(0,T_4;H)$ satisfies $\vartheta-\widetilde{\vartheta}_{\rm in}\in W_{\rm per}$. Passing to the limit $n\to\infty$ in (5.16) and using the density of simple functions in S_0, we obtain that ϑ is a weak solution of $(P_h(\varkappa,\boldsymbol{w}))$.

To prove the uniqueness, we assume that ϑ_1 and ϑ_2 are two weak solutions to $(P_h(\varkappa,\boldsymbol{w}))$ and denote $\delta:=\vartheta_1-\vartheta_2$. Subtracting the integral identities (3.3) for ϑ_1 and ϑ_2 and using δ as the test function, we obtain

$$(5.18)\qquad M\Big(\frac{\partial\delta}{\partial t},\delta\Big)+\int_0^{T_4}A(\delta,\delta)=0.$$

Due to (3.2) and the fact that $\delta(0,\cdot)=0$ in $\Omega_{\rm G}$, the first term in (5.18) is nonnegative.

From the ellipticity of A and (5.18) we obtain

$$\int_0^{T_2} \|\delta\|_H^2 + \int_{T_2}^{T_4} \|\delta\|_{H_{\rm GM}}^2 \leqslant 0,$$

which implies that $\vartheta_1 = \vartheta_2$. $\square$

6. Conclusion

In this paper we studied a shape optimization problem governed by a heat transfer model for a carousel press system consisting of four parts with different dynamics and regimes of mutual interaction. One part of the system is cooled by flowing water, whose dynamics is described by the steady-state Navier-Stokes equations with nontrivial boundary conditions. We formulated an optimization problem whose aim is to govern the cooling of the glass product by means of the shape of the plunger cavity and thus help improve the design of the components of the press.

We have proved the existence of weak solutions to the Navier-Stokes equations that are bounded in a class of admissible domains. The set of admissible domains is chosen so that it is compact with respect to uniform convergence while being sufficiently rich for practical purposes of design of industrial devices. For a given admissible domain and velocity field we have proved the existence of a unique weak solution to the heat transfer problem, using a time discretization and a fixed-point argument. Further, the existence of a solution of the optimization problem was proved using direct method of calculus of variations.

In a forthcoming paper we plan to present a numerical approximation of the optimization problem.

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Authors' address: *Petr Salač*, *Jan Stebel*, Technical University of Liberec, Studentská 2, 461 17 Liberec 1, Czech Republic, e-mail: `petr.salac@tul.cz`, `jan.stebel@tul.cz`.