Nina A. Chernyavskaya; Lela S. Dorel; Leonid A. Shuster
Admissible spaces for a first order differential equation with delayed argument

Czechoslovak Mathematical Journal, Vol. 69 (2019), No. 4, 1069–1080

Czechoslovak Mathematical Journal, 69 (144) (2019), 1069–1080

ADMISSIBLE SPACES FOR A FIRST ORDER DIFFERENTIAL EQUATION WITH DELAYED ARGUMENT

Nina A. Chernyavskaya, Be'er Sheva, Lela S. Dorel, Kfar Saba,
Leonid A. Shuster, Ramat Gan

Received January 31, 2018. Published online June 5, 2019.

Abstract. We consider the equation

$$-y'(x) + q(x)y(x - \varphi(x)) = f(x), \quad x \in \mathbb{R},$$

where φ and q ($q \geqslant 1$) are positive continuous functions for all $x \in \mathbb{R}$ and $f \in C(\mathbb{R})$. By a solution of the equation we mean any function y, continuously differentiable everywhere in $\mathbb{R}$, which satisfies the equation for all $x \in \mathbb{R}$. We show that under certain additional conditions on the functions φ and q, the above equation has a unique solution y, satisfying the inequality

$$\|y'\|_{C(\mathbb{R})} + \|qy\|_{C(\mathbb{R})} \leqslant c\|f\|_{C(\mathbb{R})},$$

where the constant $c \in (0, \infty)$ does not depend on the choice of f.

Keywords: linear differential equation; admissible pair; delayed argument

MSC 2010: 34A30, 34B05, 34B40

1. Introduction

In the present paper, we consider the equation

$$-y'(x) + q(x)y(x - \varphi(x)) = f(x), \quad x \in \mathbb{R}, \tag{1.1}$$

where $f \in C(\mathbb{R})$, and

$$0 \leqslant \varphi \in C^{\text{loc}}(\mathbb{R}), \tag{1.2}$$

$$1 \leqslant q \in C^{\text{loc}}(\mathbb{R}). \tag{1.3}$$

DOI: 10.21136/CMJ.2019.0062-18

Here the symbol $C(\mathbb{R})$ denotes the set of continuous in every point of the number axis $\mathbb{R}$ absolutely bounded functions, $C^{\mathrm{loc}}(\mathbb{R})$ denotes the set of functions continuous in every point of the number axis $\mathbb{R}$. Thus, $\mathrm{e}^{|x|} \in C^{\mathrm{loc}}(\mathbb{R})$ and $\mathrm{e}^{|x|} \notin C(\mathbb{R})$.

By a solution of (1.1) we mean any continuously differentiable function $y(x)$ satisfying (1.1) for all $x \in \mathbb{R}$. Denote by $C^{(1)}(\mathbb{R}, q)$ the set of all continuously differentiable functions $y(x)$ for $x \in \mathbb{R}$ with norm $\|y\|_{C^{(1)}(\mathbb{R},q)}$ defined by

$$\|y\|_{C^{(1)}(\mathbb{R},q)} = \|y'\|_{C(\mathbb{R})} + \|qy\|_{C(\mathbb{R})}. \tag{1.4}$$

We need the following definition (see [5]).

Definition 1.1. Let B_1, B_2 be Banach spaces and let $S\colon B_1 \to B_2$ be a linear operator. Consider the equation

$$Sy = g, \quad y \in B_1, \quad g \in B_2. \tag{1.5}$$

We say that the spaces B_1 and B_2 form a pair of spaces $\{B_1, B_2\}$ (in the sequel, a pair $\{B_1, B_2\}$) admissible for equation (1.5) (in the sequel just (1.5)) if the following assertions hold:

(i) for any $g \in B_2$ there exists a unique element $y \in B_1$ guaranteeing equality (1.5) (a solution of (1.5));

(ii) there exists a constant $c \in (0, \infty)$ such that regardless of the choice of element $g \in B_2$, the solution $y \in B_1$ of (1.5) satisfies the inequality

$$\|y\|_{B_1} \leqslant c\|g\|_{B_2}. \tag{1.6}$$

If the pair $\{B_1, B_2\}$ is admissible for (1.5), we also say that equation (1.5) is *correctly solvable* in the pair $\{B_1, B_2\}$. If $B_1 = B_2 = B$, then we say that (1.5) is correctly solvable in the space B.

In this work, we study the content of the question of admissibility of the pair $\{C^{(1)}(\mathbb{R}, q), C(\mathbb{R})\}$ for (1.1) (in the sequel, for brevity, we say "the question on (i)–(ii)" for (1.1), or "problem (i)–(ii)" for (1.1).

Such an investigation is needed because this is the first time problem (i)–(ii) is being posed for equation (1.1) (for the second order equation the same question was studied in [2]). Indeed, to the best of our knowledge, for equations with delayed argument, initial and boundary-value problems on a finite segment or on a semi-axis have been studied (see [1], [3], [4], [6]). However, the special feature of problem (i)–(ii) is that equation (1.1) is considered on the whole axis, and requirements to its solutions are imposed apart from (i)–(ii). Therefore, the main result of the paper is the statement asserting that problem (i)–(ii) makes sense, i.e. the set of equations (1.1) for which the pair $\{C^{(1)}(\mathbb{R},q), C(\mathbb{R})\}$ is admissible, is not empty. This statement follows from the following theorem which is our main result.

Theorem 1.2. *Suppose that, in addition to* (1.2), (1.3), *the following conditions are satisfied:*

(i) *there is a constant* $a \geqslant 1$ *such that for all* $x \in \mathbb{R}$,

$$a^{-1}q(x) \leqslant q(t) \leqslant aq(x) \quad \forall\, t \in [x-1, x+1]; \tag{1.7}$$

(ii) *there is* $\sigma < \infty$ *such that*

$$\sigma(a+1) < 1, \quad \text{where} \quad \sigma \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}}(\varphi(x)q^2(x)). \tag{1.8}$$

Then the pair $\{C^{(1)}(\mathbb{R},q), C(\mathbb{R})\}$ *is admissible for* (1.1). *Moreover, equation* (1.1) *is separable in* $C(\mathbb{R})$, *i.e. there is a constant* $c \in (0,\infty)$ *such that for any choice of* $f \in C(\mathbb{R})$, *the solution* $y \in C^{(1)}(\mathbb{R},q)$ *of* (1.1) *obeys the estimate*

$$\|y'\|_{C(\mathbb{R})} + \|q(x)y(x-\varphi(x))\|_{C(\mathbb{R})} \leqslant c\|f\|_{C(\mathbb{R})}. \tag{1.9}$$

The paper is organized as follows. In Section 2, we present some auxiliary assertions needed for the proof of Theorem 1.2, Section 3 contains the proof of Theorem 1.2, in Section 4, we give an example of an application of the theorem.

2. Preliminaries

In the sequel, we assume that conditions (1.2), (1.3), (1.7), (1.8) are satisfied. They are not referred to or mentioned in the statements. The symbol c stands for an absolute positive constant, the value which is not essential for the exposition, and can even change within a single chain of calculations.

Lemma 2.1. *For all* $x \in \mathbb{R}$ *the integral*

$$B(x) = \int_x^\infty \mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t, \quad x \in \mathbb{R} \tag{2.1}$$

satisfies the inequalities

$$0 < B(x) \leqslant 1 \tag{2.2}$$

and

$$\mathrm{e}^{-a} \leqslant q(x)B(x) \leqslant 4a. \tag{2.3}$$

Proof of Lemma 2.1. The fact that $B(x)$ is strictly positive is obvious (see (1.3)). We also use (1.3) to estimate $B(x)$, $x \in \mathbb{R}$, from above:

$$B(x) = \int_x^\infty \mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t = \int_x^\infty \frac{q(t)}{q(t)}\mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t \leqslant \int_x^\infty q(t)\mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t = 1.$$

Hence, (2.2) holds.

Now, let us prove (2.3). To get the lower estimate, we use the following inequality, which holds for all $x \in \mathbb{R}$:

$$\int_x^{x+1/q(x)} q(\xi)\,\mathrm{d}\xi = \int_x^{x+1/q(x)} \frac{q(\xi)}{q(x)}q(x)\,\mathrm{d}\xi \leqslant a\int_x^{x+1/q(x)} q(x)\,\mathrm{d}\xi = a.$$

Thus, we have:

$$\begin{aligned} B(x) &= \int_x^\infty \mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t \geqslant \int_x^{x+1/q(x)} \mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t \\ &\geqslant \int_x^{x+1/q(x)} \mathrm{e}^{-\int_x^{x+1/q(x)} q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t = \frac{1}{q(x)}\mathrm{e}^{-\int_x^{x+1/q(x)} q(\xi)\,\mathrm{d}\xi} \geqslant \frac{\mathrm{e}^{-a}}{q(x)}. \end{aligned}$$

The upper estimate in (2.3) follows from (1.3) and (1.7):

$$\begin{aligned} B(x) &= \int_x^\infty \mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t = \int_x^{x+1} \mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t + \sum_{n=1}^\infty \int_{x+n}^{x+n+1} \mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t \\ &= \int_x^{x+1} \mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t\left(1 + \sum_{n=1}^\infty \int_{x+n}^{x+n+1} \mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t \cdot \left(\int_x^{x+1} \mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t\right)^{-1}\right) \\ &\leqslant \int_x^{x+1} \mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t\left(1 + \sum_{n=1}^\infty \mathrm{e}^{-\int_x^{x+n} q(\xi)\,\mathrm{d}\xi} \cdot \mathrm{e}^{\int_x^{x+1} q(\xi)\,\mathrm{d}\xi}\right) \\ &= \int_x^{x+1} \mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t\left(1 + \sum_{n=1}^\infty \mathrm{e}^{-\int_{x+1}^{x+n} q(\xi)\,\mathrm{d}\xi}\right) \\ &\leqslant \int_x^{x+1} \mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t\left(1 + \sum_{n=1}^\infty \mathrm{e}^{-(n-1)}\right) \\ &= \frac{2\mathrm{e}-1}{\mathrm{e}-1}\int_x^{x+1} \mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t \leqslant 4\int_x^{x+1} \frac{q(x)}{q(t)} \cdot \frac{q(t)}{q(x)}\mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t \\ &\leqslant \frac{4a}{q(x)}\int_x^{x+1} q(t)\mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t \leqslant \frac{4a}{q(x)}\int_x^\infty q(t)\mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}\,\mathrm{d}t = \frac{4a}{q(x)}. \end{aligned}$$

This completes the proof of (2.3). □

Remark 2.2. To estimate inequality (2.3) from above, we could use smaller constant $(2\mathrm{e}-1)a/(\mathrm{e}-1)$. However, since using this constant would not change the results, we will not use it.

Lemma 2.3. *The equation*

$$-y'(x)+q(x)y(x)=f(x),\quad x\in\mathbb{R}, \tag{2.4}$$

has a unique solution $y\in C(\mathbb{R})$ for any $f\in C(\mathbb{R})$. This solution satisfies the inequalities:

$$\|y\|_{C(\mathbb{R})}\leqslant\|f\|_{C(\mathbb{R})}, \tag{2.5}$$

$$\|qy\|_{C(\mathbb{R})}\leqslant 4a\|f\|_{C(\mathbb{R})} \tag{2.6}$$

and

$$\|y'\|_{C(\mathbb{R})}+\|y\|_{C(\mathbb{R})}\leqslant\|y'\|_{C(\mathbb{R})}+\|qy\|_{C(\mathbb{R})}\leqslant(8a+1)\|f\|_{C(\mathbb{R})}. \tag{2.7}$$

P r o o f of Lemma 2.3. For $f\in C(\mathbb{R})$, define function $y(x)$, $x\in\mathbb{R}$ by the formula

$$y(x)=(Gf)(x):=\int_x^\infty \mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}f(t)\,\mathrm{d}t. \tag{2.8}$$

It is clear that $y(x)$ is defined for all $x\in\mathbb{R}$. Indeed,

$$|y(x)|\leqslant\int_x^\infty \mathrm{e}^{-\int_x^t q(\xi)\,\mathrm{d}\xi}|f(t)|\,\mathrm{d}t\leqslant B(x)\|f\|_{C(\mathbb{R})}\leqslant\|f\|_{C(\mathbb{R})} \tag{2.9}$$

(see (2.2)). Using direct substitution, we can see that y is a particular solution of equation (2.4), which satisfies inequality (2.5) (see (2.9)). Now notice that by (1.3), the equation $-z'(x)+q(x)z(x)=0$, $x\in\mathbb{R}$, does not have any solution that is bounded in $\mathbb{R}$, except for $z=0$. Therefore $y(x)$, $x\in\mathbb{R}$, is a unique solution of (2.4) in the class $C(\mathbb{R})$. Estimate (2.6) follows from (2.8) and (2.3). Finally, by using (1.3), (2.4), (2.5), (2.6), we get (2.7):

$$\begin{aligned}\|y'\|_{C(\mathbb{R})}\leqslant\|f\|_{C(\mathbb{R})}+\|qy\|_{C(\mathbb{R})}\Rightarrow\|y'\|_{C(\mathbb{R})}+\|qy\|_{C(\mathbb{R})}\\ \leqslant\|f\|_{C(\mathbb{R})}+2\|qy\|_{C(\mathbb{R})}\leqslant(8a+1)\|f\|_{C(\mathbb{R})}.\end{aligned}$$

□

Denote

$$D(\mathbb{R}) = \{y \in C(\mathbb{R})\colon\ y \in C^{(1)}_{\mathrm{loc}}(\mathbb{R}),\ -y' + qy \in C(\mathbb{R})\}, \tag{2.10}$$

$$(Ly)(x) = -y'(x) + q(x)y(x), \quad x \in \mathbb{R},\ y \in D(\mathbb{R}). \tag{2.11}$$

Lemma 2.4. *The operator $L\colon\ D(\mathbb{R}) \to C(\mathbb{R})$ is continuously invertible. In particular, we have the following relations:*

$$L^{-1} = G, \tag{2.12}$$

$$\|G\|_{C(\mathbb{R})\to C(\mathbb{R})} \leqslant 1 \tag{2.13}$$

and

$$\|qG\|_{C(\mathbb{R})\to C(\mathbb{R})} \leqslant 4a. \tag{2.14}$$

Proof of Lemma 2.4. One proceeds exactly as in the proof of Lemma 2.3. □

Let us introduce an operator $A\colon\ C(\mathbb{R}) \to C(\mathbb{R})$ by the formula

$$(Af)(x) \overset{\text{def}}{=} q(x)\int_{x-\varphi(x)}^{x} f(t)\,\mathrm{d}t, \quad x \in \mathbb{R},\ f \in C(\mathbb{R}). \tag{2.15}$$

Lemma 2.5. *We have the inequality:*

$$\|A\|_{C(\mathbb{R})\to C(\mathbb{R})} \leqslant \sigma \leqslant \frac{1}{2}. \tag{2.16}$$

Proof of Lemma 2.5. By (1.2), (1.3), (1.7) and (1.8), and since $a \geqslant 1$, we have:

$$\sigma < \frac{1}{a+1} \leqslant \frac{1}{2} \Rightarrow \varphi(x) \leqslant \frac{\sigma}{q^2(x)} \leqslant \sigma \leqslant \frac{1}{2}, \quad x \in \mathbb{R}. \tag{2.17}$$

Now, from (1.2), (1.3), (1.7), (1.8) and (2.17) it follows that

$$\begin{aligned}\|Af\|_{C(\mathbb{R})} &= \sup_{x\in\mathbb{R}} q(x)\left|\int_{x-\varphi(x)}^{x} f(\xi)\,\mathrm{d}\xi\right| \leqslant \sup_{x\in\mathbb{R}}(q(x)\varphi(x)) \cdot \|f\|_{C(\mathbb{R})}\\ &= \sup_{x\in\mathbb{R}}\Big((\varphi(x)q^2(x))\frac{1}{q(x)}\Big) \cdot \|f\|_{C(\mathbb{R})} \leqslant \sigma\|f\|_{C(\mathbb{R})} \Rightarrow (2.16).\end{aligned}$$

□

Consider the integral operator defined on $C(\mathbb{R})$:

$$(Ky)(x)=\sum_{n=1}^{\infty}(-1)^{n+1}[GA^n(qy)](x),\quad x\in\mathbb{R} \tag{2.18}$$

(see (2.8) and (2.15)).

Lemma 2.6. *The operator $K\colon C(\mathbb{R})\to C(\mathbb{R})$ is bounded. In particular,*

$$\|K\|_{C(\mathbb{R})\to C(\mathbb{R})}\leqslant\alpha<1,\quad \alpha=\frac{a\sigma}{1-\sigma}. \tag{2.19}$$

Proof of Lemma 2.6. First note that since $\sigma(a+1)<1$ (see (1.8)), it holds that $\alpha=a\sigma/(1-\sigma)<1$. Let $y\in C(\mathbb{R})$. Using (2.13) and (2.16), we get

$$\begin{aligned}\|Ky\|_{C(\mathbb{R})}&=\Big\|\sum_{n=1}^{\infty}(-1)^{n+1}[GA^n(qy)](x)\Big\|_{C(\mathbb{R})}\\&\leqslant\|G\|_{C(\mathbb{R})\to C(\mathbb{R})}\Big(\sum_{n=1}^{\infty}\|A^{n-1}\|_{C(\mathbb{R})\to C(\mathbb{R})}\Big)\|A(qy)\|_{C(\mathbb{R})}\\&\leqslant\Big(\sum_{n=1}^{\infty}\sigma^{n-1}\Big)\|A(qy)\|_{C(\mathbb{R})}=\frac{1}{1-\sigma}\sup_{x\in\mathbb{R}}\Big(q(x)\int_{x-\varphi(x)}^{x}q(t)y(t)\,\mathrm{d}t\Big)\\&\leqslant\frac{1}{1-\sigma}\sup_{x\in\mathbb{R}}\Big(q^2(x)\int_{x-\varphi(x)}^{x}\frac{q(t)}{q(x)}\,\mathrm{d}t\Big)\|y\|_{C(\mathbb{R})}\leqslant\frac{a\sigma}{1-\sigma}\|y\|_{C(\mathbb{R})}=\alpha\|y\|_{C(\mathbb{R})}.\end{aligned}$$

Hence, we have (2.19). □

3. Proof of the main result

Here we prove Theorem 1.2. To this end we need an operator T. Let $f\in C(\mathbb{R})$, define T by the formula

$$(Ty)(x)=(G(E+A)^{-1}f)(x)+(Ky)(x),\quad x\in\mathbb{R},\ y\in C(\mathbb{R}). \tag{3.1}$$

The symbol E stands for the identity operator in $C(\mathbb{R})$. We have $T\colon C(\mathbb{R})\to C(\mathbb{R})$ (see (2.13), (2.16), (2.19)). Furthermore, by (2.19), for any functions $y_1,y_2\in C(\mathbb{R})$ we have:

$$\|Ty_2-Ty_1\|_{C(\mathbb{R})}=\|K(y_2-y_1)\|_{C(\mathbb{R})}\leqslant\alpha\|(y_2-y_1\|_{C(\mathbb{R})}, \tag{3.2}$$

and since $\alpha < 1$ (see (2.19)), we conclude that T is a contraction on the space $C(\mathbb{R})$. Now let us consider the equation

$$y(x) = (Ty)(x), \quad x \in \mathbb{R}. \tag{3.3}$$

By a solution of (3.3) we mean any continuous function $y(x)$ for $x \in \mathbb{R}$ satisfying (3.3) in every point of the number axis $\mathbb{R}$.

Definition 3.1. We say that the pair $\{C_q^{(1)}(\mathbb{R}), C(\mathbb{R})\}$ is admissible for (3.3) if:

(i) for each function $f \in C(\mathbb{R})$ (see (3.1) there exists a unique solution $y \in C_q^{(1)}(\mathbb{R})$ of (3.3));

(ii) there exists a constant $c \in (0, \infty)$ such that independently of the choice of the function $f \in C(\mathbb{R})$, the solution $y \in C_q^{(1)}(\mathbb{R})$ of (3.3) satisfies the inequality

$$\|y\|_{C_q^{(1)}(\mathbb{R})} \leqslant c\|f\|_{C(\mathbb{R})}. \tag{3.4}$$

Lemma 3.2. *The pair* $\{C^{(1)}(\mathbb{R}, q), C(\mathbb{R})\}$ *is admissible for* (1.1) *if and only if it is admissible for* (3.3).

P r o o f of Lemma 3.2. *Necessity.* Suppose that the pair $\{C^{(1)}(\mathbb{R}, q), C(\mathbb{R})\}$ is admissible for (1.1) and let $y(x)$, $x \in \mathbb{R}$, be a solution of (1.1) from the class $C^{(1)}(\mathbb{R}, q)$. Using (2.15) and (2.16), we can write (1.1) in a different way:

$$\begin{aligned} y'(x) &= -f(x) + q(x)y(x - \varphi(x)) = -f(x) + q(x)y(x) - q(x)\int_{x-\varphi(x)}^{x} y'(\xi)\,\mathrm{d}\xi \\ &= -f(x) + q(x)y(x) - (Ay')(x), \quad x \in \mathbb{R}. \end{aligned}$$

Thus,

$$[(E + A)y'](x) = -f(x) + q(x)y(x), \quad x \in \mathbb{R}.$$

Since $\|A\|_{C(\mathbb{R})\to C(\mathbb{R})} \leqslant \sigma \leqslant \frac{1}{2}$, the operator $(E + A)$ is invertible, and

$$(E + A)^{-1} = E + \sum_{n=1}^{\infty}(-1)^n A^n.$$

So,

$$y'(x) = -[(E + A)^{-1}f](x) + q(x)y(x) + \sum_{n=1}^{\infty}(-1)^n[A^n(qy)](x), \quad x \in \mathbb{R}.$$

Now, by (2.11) we get:

$$(Ly)(x) = -y'(x) + q(x)y(x) = [(E + A)^{-1}f](x) + \sum_{n=1}^{\infty}(-1)^{n+1}[A^n(qy)](x)$$

for $x \in \mathbb{R}$. The last equality, (2.14) and (2.16) imply (3.3). Thus, y is a solution of (3.3) with $y \in C^{(1)}(\mathbb{R}, q)$.

Let us prove the uniqueness of the solution of (3.3) in the class $C^{(1)}(\mathbb{R}, q)$. Assume that for some $f \in C(\mathbb{R})$ there is another solution $y_1 \in C^{(1)}(\mathbb{R}, q)$, $y_1 \neq y$, of equation (3.3). Since $C^{(1)}(\mathbb{R}, q) \hookrightarrow C(\mathbb{R})$ (see (1.3), (1.4)), both functions y and y_1 belong to $C(\mathbb{R})$, i.e. y is not a unique solution of (3.3) in the class $C(\mathbb{R})$. This contradicts the assertion that T is a contraction acting on $C(\mathbb{R})$ (see (3.2)). Thus, y is the unique solution of (3.3) in the class $C(\mathbb{R})$, and consequently in the class $C^{(1)}(\mathbb{R}, q)$. The uniqueness is proved. Now notice that this solution of (3.3) satisfies the inequality (3.4) since it is a solution of (1.1), for which the pair $\{C^{(1)}(\mathbb{R}, q), C(\mathbb{R})\}$ is admissible. Thus, the pair $\{C^{(1)}(\mathbb{R}, q), C(\mathbb{R})\}$ is admissible for (3.3).

Sufficiency. Assume that the pair $\{C^{(1)}(\mathbb{R}, q), C(\mathbb{R})\}$ is admissible for (3.3) and let $y(x)$, $x \in \mathbb{R}$ be a solution of (3.3) from the class $C^{(1)}(\mathbb{R}, q)$. We have the equalities

$$y(x) = (Ty)(x) = (Gg)(x), \quad x \in \mathbb{R}, \tag{3.5}$$

where

$$g(x) = [(E + A)^{-1} f](x) + \sum_{n=1}^{\infty} (-1)^{n+1} A^{n-1} [A(qy)](x), \quad x \in \mathbb{R} \tag{3.6}$$

(see (3.1)). Using (2.16) and (3.4), we conclude that $g \in C(\mathbb{R})$:

$$\begin{aligned} \|g\|_{C(\mathbb{R})} &\leqslant c\|f\|_{C(\mathbb{R})} + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \|A(qy)\|_{C(\mathbb{R})} \leqslant c\|f\|_{C(\mathbb{R})} + c\|y\|_{C(\mathbb{R})} \\ &\leqslant c\|f\|_{C(\mathbb{R})}. \end{aligned} \tag{3.7}$$

Thus, $y \in D(L)$ (see Lemma 2.4). Repeating the arguments from the necessity part of the proof in reverse order, we obtain that $y \in C^{(1)}(\mathbb{R}, q)$ is a solution of (1.1). Moreover, the solution of (1.1) is unique in the class $C^{(1)}(\mathbb{R}, q)$. Indeed, otherwise the equation

$$-z'(x) + q(x) z(x - \varphi(x)) = 0, \quad x \in \mathbb{R} \tag{3.8}$$

has a solution $z \in C^{(1)}(\mathbb{R}, q)$ which does not vanish for all $x \in \mathbb{R}$; and since $C^{(1)}(\mathbb{R}, q) \hookrightarrow C(\mathbb{R})$ (see (1.3), (1.4)), $z \in C(\mathbb{R})$. Now, passing from the equation (3.8) to equation (3.3) with $f = 0$, and repeating the argument given in the necessity part of the proof, we get

$$z(x) = (Kz)(x), \quad x \in \mathbb{R}$$

(see (2.18), (3.1)).

From this and (2.19) it follows that

$$\|z\|_{C(\mathbb{R})} \leqslant \|K\|_{C(\mathbb{R})\to C(\mathbb{R})}\|z\|_{C(\mathbb{R})} \leqslant \alpha\|z\|_{C(\mathbb{R})} < \|z\|_{C(\mathbb{R})},$$

so we have reached a contradiction. Thus, (1.1) has a unique solution in the class $C^{(1)}(\mathbb{R}, q)$. This solution satisfies inequality (3.4) since it is a solution of (3.3) from the class $C^{(1)}(\mathbb{R}, q)$. The lemma is proved. □

Now we can complete the proof of the theorem. Since $T\colon C(\mathbb{R}) \to C(\mathbb{R})$ is a contraction, equation (3.3) has a unique solution $y \in C(\mathbb{R})$. Furthermore, it holds that (see (2.19), (3.1)):

$$\begin{aligned}\|y\|_{C(\mathbb{R})} &\leqslant \|G(E+A)^{-1}\|_{C(\mathbb{R})\to C(\mathbb{R})} \cdot \|f\|_{C(\mathbb{R})} + \|K\|_{C(\mathbb{R})\to C(\mathbb{R})} \cdot \|y\|_{C(\mathbb{R})} \\ &\leqslant c\|f\|_{C(\mathbb{R})} + \alpha\|y\|_{C(\mathbb{R})},\end{aligned}$$

which yields

$$\|y\|_{C(\mathbb{R})} \leqslant c\|f\|_{C(\mathbb{R})}. \tag{3.9}$$

By (3.5), (3.6), (3.7), (3.9) and (2.12), we have:

$$y(x) = (L^{-1}g)(x), \quad x \in \mathbb{R}, \quad \|g\|_{C(\mathbb{R})} \leqslant c\|f\|_{C(\mathbb{R})}. \tag{3.10}$$

Now from (3.10), Lemma 2.3 and (2.3), we get that $y \in D(L)$, and then Lemmas 2.3 and 2.4 (see (2.12), (2.8) and (2.7)) imply that $y \in C^{(1)}(\mathbb{R}, q)$, and estimates (3.4) and (2.7) coincide. Furthermore, since $y \in D(L) \cap C^{(1)}(\mathbb{R}, q)$, arguing as in the sufficiency part of the proof of Lemma 3.2, we conclude that y is a unique solution of (3.3) in the class $C^{(1)}(\mathbb{R}, q)$. Hence, the pair $\{C^{(1)}(\mathbb{R}, q), C(\mathbb{R})\}$ is admissible for (3.3). Now Theorem 1.2 follows from Lemma 3.2. □

4. Example

Let us consider equation (1.1) with

$$q(x) = 2\mathrm{e}^{|x|} + \mathrm{e}^{|x|}\sin(\mathrm{e}^{\alpha|x|}), \quad x \in \mathbb{R},\ \alpha > 0. \tag{4.1}$$

Using Theorem 1.2, we will show that if the continuous function $\varphi(x)$ on $\mathbb{R}$ satisfies the inequalities:

$$0 \leqslant \varphi(x) \leqslant \beta\mathrm{e}^{-2|x|}, \quad x \in \mathbb{R},\ \beta = 0.01, \tag{4.2}$$

then the pair $\{C^{(1)}(\mathbb{R},q), C(\mathbb{R})\}$ is admissible for (1.1) in the case where the function $q(x)$ is given by (4.1).

In order to use Theorem 1.2, we check that conditions (1.7), (1.8) are satisfied:

(1) Let $|t-x| \leqslant 1$, $x \in \mathbb{R}$. Then

$$\frac{q(t)}{q(x)} \leqslant \frac{2\mathrm{e}^{|t|}+\mathrm{e}^{|t|}}{2\mathrm{e}^{|x|}-\mathrm{e}^{|x|}} = 3\mathrm{e}^{|t|-|x|} \leqslant 3\mathrm{e}^{|t-x|} \leqslant 3\mathrm{e},$$

and

$$\frac{q(t)}{q(x)} \geqslant \frac{2\mathrm{e}^{|t|}-\mathrm{e}^{|t|}}{2\mathrm{e}^{|x|}+\mathrm{e}^{|x|}} \geqslant \frac{1}{3}\mathrm{e}^{|t|-|x|} \geqslant \frac{1}{3}\mathrm{e}^{-|t-x|} \geqslant \frac{1}{3\mathrm{e}}.$$

Thus, condition (1.7) holds with $a = 3\mathrm{e}$.

(2) Since

$$q^2(x) \leqslant (3\mathrm{e}^{|x|})^2 = 9\mathrm{e}^{2|x|}$$

(see (4.1)), we assume that the continuous function $\varphi(x)$, $x \in \mathbb{R}$, satisfies the inequalities

$$0 \leqslant \varphi(x) \leqslant \beta\mathrm{e}^{-2|x|}, \quad x \in \mathbb{R}, \quad \beta > 0. \tag{4.3}$$

Here β has to be chosen so that condition (1.8) is satisfied. Since

$$\sigma = \sup_{x\in\mathbb{R}}(\varphi(x)q^2(x)) \leqslant \sup_{x\in\mathbb{R}} \beta\mathrm{e}^{-2|x|} \cdot 9\mathrm{e}^{2|x|} = 9\beta,$$

we have the following condition for choosing the value of β:

$$\sigma(a+1) = \sigma(3\mathrm{e}+1) \leqslant 9\beta(3\mathrm{e}+1) < 1,$$

whence

$$\beta < \frac{1}{9(3\mathrm{e}+1)} := \beta_0.$$

It is easy to check that $0.01 < \beta_0$.

Thus, by Theorem 1.2, the pair $\{C^{(1)}(\mathbb{R},q), C(\mathbb{R})\}$ is admissible for equation (1.1) with $q(x)$ given by (4.1), provided the function $\varphi(x)$ satisfies condition (4.2).

Concluding remarks. We will try to solve an analogous problem for equations of the form

$$(-1)^n y^{(2n)}(x) + q(x)y(x-\varphi(x)) = f(x), \quad x \in \mathbb{R},$$
$$(-1)^n y^{(2n+1)}(x) + q(x)y(x-\varphi(x)) = f(x), \quad x \in \mathbb{R}$$

in our forthcoming works.

Acknowledgments. The authors thank Prof. B. Shklyar for productive discussions and useful remarks.

References

[1] *N. V. Azbelev, S. Yu. Kultyshev, V. Z. Tsalynk*: Functional Differential Equations and Variational Problems. R & C Dynamics, Moskva, 2006. (In Russian.)

[2] *N. Chernyavskaya, L. Shuster*: Correct solvability of the Sturm-Liouville equation with delayed argument. J. Differ. Equations *261* (2016), 3247–3267. zbl MR doi

[3] *L. È. El'sgol'ts, S. B. Norkin*: Introduction to the Theory of Differential Equations with Deviating Argument. Nauka, Moskva, 1971. (In Russian.) zbl MR

[4] *J. K. Hale*: Theory of Functional Differential Equations. Applied Mathematical Sciences 3, Springer, New York, 1977. zbl MR doi

[5] *J. L. Massera, J. J. Schäffer*: Linear Differential Equations and Function Spaces. Pure and Applied Mathematics 21, Academic Press, New York, 1966. zbl MR

[6] *A. D. Myshkis*: Linear Differential Equations with Retarded Argument. Nauka, Moskva, 1972. (In Russian.) zbl MR

Authors' addresses: Nina A. Chernyavskaya, Department of Mathematics, Ben-Gurion University of the Negev, P.O. Box 653, Be'er Sheva, 841 05, Israel, e-mail: nina@math.bgu.ac.il; Lela S. Dorel, Department of Mathematics, Beit Berl College, Kfar Saba, 449 05, Israel, e-mail: lela@post.tau.ac.il; Leonid A. Shuster, Department of Mathematics, Bar-Ilan University, Ramat Gan, 529 00, Israel, e-mail: miriam@macs.biu.ac.il.