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Wei-Feng Xuan

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AN OBSERVATION ON SPACES WITH A ZEROSET DIAGONAL

Wei-Feng Xuan, Nanjing

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Abstract. We say that a space X has the discrete countable chain condition (DCCC for short) if every discrete family of nonempty open subsets of X is countable. A space X has a zeroset diagonal if there is a continuous mapping $f\colon X^2 \to [0,1]$ with $\Delta_X = f^{-1}(0)$, where $\Delta_X = \{(x,x)\colon x \in X\}$. In this paper, we prove that every first countable DCCC space with a zeroset diagonal has cardinality at most $\mathfrak{c}$.

Keywords: first countable; discrete countable chain condition; zeroset diagonal; cardinal

MSC 2010: 54D20, 54E35

1. Introduction

All topological spaces in this paper are assumed to be Hausdorff unless otherwise stated. The cardinality of a set X is denoted by $|X|$, and $[X]^2$ will denote the set of two-element subsets of X. We write ω for the first infinite cardinal, ω_1 for the first uncountable cardinal and $\mathfrak{c}$ for the cardinality of the continuum.

In 1977, Ginsburg and Woods proved that the cardinality of a T_1-space with countable extent and a G_δ-diagonal is at most $\mathfrak{c}$ (see [5]). In the same paper, Ginsburg and Woods asked if it was true that a regular CCC-space (here CCC denotes the countable chain condition) with a G_δ-diagonal has cardinality at most $\mathfrak{c}$. This question was also posted by Arhangel'skii independently. In 1984, Shakhmatov showed that cardinalities of such spaces may not have an upper bound (see [8]). And later, Uspenskij proved that an upper bound still does not exist even assuming Fréchet property (see [9]). Regular G_δ-diagonal is a property stronger than G_δ-diagonal. Arhangel'skii asked what if "G_δ-diagonal" is replaced by "regular G_δ-diagonal".

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In 2005, Buzyakova proved that the cardinality of a CCC-space with a regular G_δ-diagonal is at most $\mathfrak{c}$ (see [3]). In 2015, Gotchev in [6] proved that the cardinality of a weakly Lindelöf space with a regular G_δ-diagonal is at most $2^{\mathfrak{c}}$.

Definition 1.1. We say that a space X has the *discrete countable chain condition* (DCCC for short) if every discrete family of nonempty open subsets of X is countable.

By Definition 1.1, it follows immediately that every CCC space is DCCC. In fact, every weakly Lindelöf space is DCCC, but the converse is not true. For example, ω_1 with the ordered topology is a first countable and countably compact (hence, DCCC) space which is not weakly Lindelöf, because the open cover $\mathcal{U} = \{[0,\alpha]\colon\ \alpha < \omega_1\}$ of ω_1 does not have a countable subfamily whose union is dense in ω_1.

Definition 1.2 ([2])**.** A space X has a *zeroset diagonal* if there is a continuous mapping $f\colon\ X^2 \to [0,1]$ with $\Delta_X = f^{-1}(0)$, where $\Delta_X = \{(x,x)\colon\ x \in X\}$.

It is well-known and easy to prove that every submetrizable space has a zeroset diagonal and every zeroset diagonal is a regular G_δ-diagonal. The converses are not true (see [1], [10]).

In this paper, we prove that every first countable DCCC space with a zeroset diagonal has cardinality at most $\mathfrak{c}$.

All notations and terminology not explained in the paper are given in [4].

2. Results

We will use the following countable version of a set-theoretic theorem due to Erdős and Radó (see [7], page 8).

Lemma 2.1. *Let X be a set with $|X| > \mathfrak{c}$ and suppose $[X]^2 = \bigcup\{P_n\colon\ n \in \omega\}$. Then there exist $n_0 < \omega$ and a subset S of X with $|S| > \omega$ such that $[S]^2 \subset P_{n_0}$.*

Theorem 2.2. *Every first countable DCCC space X with a zeroset diagonal has cardinality at most $\mathfrak{c}$.*

Proof. Assume the contrary, i.e. that $|X| > \mathfrak{c}$. Fix a continuous function $f\colon\ X^2 \to [0,1]$ with $\Delta_X = f^{-1}(0)$. Let $\mathcal{B}(x) = \{B_n(x)\colon\ n \in \omega\}$ be a local decreasing base for each $x \in X$. Since for any distinct $x, y \in X$ there is some $n_1 \in \omega$ such that $(x,y) \in f^{-1}((1/(n_1+2019),1])$ and since f is continuous, there are $n_2, n_3 \in \omega$ such that

$$B_{n_2}(x) \times B_{n_3}(y) \subset f^{-1}\Big(\Big(\frac{1}{n_1+2019},1\Big]\Big).$$

Let $n^* = \max\{n_1, n_2, n_3\}$. Then by our hypothesis, we can deduce that

$$B_{n^*}(x) \times B_{n^*}(y) \subset f^{-1}\Big(\Big(\frac{1}{n^* + 2019}, 1\Big]\Big).$$

Thus, the following sets P_n are well defined. For each $n \in \omega$ let

$$P_n = \Big\{\{x, y\} \in [X]^2 \colon B_n(x) \times B_n(y) \subset f^{-1}\Big(\Big(\frac{1}{n + 2019}, 1\Big]\Big)\Big\}.$$

It is clear that $[X]^2 = \bigcup\{P_n \colon n \in \omega\}$. (Note that $[X]^2$ is the set of two-element subsets of X). We can apply Lemma 2.1 to conclude that there exists an uncountable subset S of X and $n_0 \in \omega$ such that $[S]^2 \subset P_{n_0}$. It follows immediately that $\mathcal{U} = \{B_{n_0}(x) \colon x \in S\}$ is an uncountable family of nonempty open sets of X. Since X is DCCC, the family $\mathcal{U}$ must have a cluster point $x \in X$. Pick any neighbourhood O_x of x such that

$$O_x \times O_x \subset f^{-1}\Big(\Big[0, \frac{1}{n_0 + 2019}\Big)\Big).$$

Obviously, O_x meets infinitely many members of $\mathcal{U}$. Thus, there exist two distinct (at least) $y, z \in S$ such that $O_x \cap B_{n_0}(y) \neq \emptyset$ and $O_x \cap B_{n_0}(z) \neq \emptyset$. Take any $y' \in O_x \cap B_{n_0}(y)$ and $z' \in O_x \cap B_{n_0}(z)$. Hence, $f(y', z') < 1/(n_0 + 2019)$ since $y', z' \in O_x$. On the other hand, $f(y', z') > 1/(n_0 + 2019)$ since $y' \in B_{n_0}(y)$, $z' \in B_{n_0}(z)$ and $\{y, z\} \in P_{n_0}$. This gives a contradiction and we prove that $|X| \leqslant \mathfrak{c}$. $\square$

If we drop the condition "DCCC", or "zeroset diagonal" in Theorem 2.2, the conclusion is no longer true, which can be seen in the following examples.

Example 2.3. Let D be a discrete space with $|D| = 2^{\mathfrak{c}}$. It is evident that D is first countable and has a zeroset diagonal, but D is not DCCC.

Example 2.4. Let X be the subspace of $[0, 2^{\mathfrak{c}}]$, consisting of all ordinals of countable cofinality, equipped with the ordered topology. Then X is a first countable and countably compact (hence DCCC) space of cardinality $2^{\mathfrak{c}}$, but it does not have a zeroset diagonal.

We finish the paper with the following question.

Question 2.5. Is it true that every DCCC (or weakly Lindelöf) space with a zeroset diagonal has cardinality at most $\mathfrak{c}$?

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Author's address: *Wei-Feng Xuan*, School of Statistics and Mathematics, Nanjing Audit University, 86 West Yushan Road, Pukou District, Nanjing, 211815, China, e-mail: `wfxuan@nau.edu.cn`.